A GROTHENDIECK TOPOS OF GENERALIZED FUNCTIONS I: BASIC THEORY

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Abstract. The main aim of the present work is to arrive at a mathematical theory close to the historically original conception of generalized functions, i.e. set theoretical functions defined on, and with values in, a suitable ring of scalars and sharing a number of fundamental properties with smooth functions, in particular with respect to composition and nonlinear operations. This is how they are still used in informal calculations in Physics. We introduce a category of generalized functions as smooth set-theoretical maps on (multidimensional) points of a ring of scalars containing infinitesimals and infinities. This category extends Schwartz distributions. The calculus of these generalized functions is closely related to classical analysis, with point values, composition, non-linear operations and the generalization of several classical theorems of calculus. Finally, we extend this category of generalized functions into a Grothendieck topos of sheaves over a concrete site. This topos hence provides a suitable framework for the study of spaces and functions with singularities. In this first paper, we present the basic theory; subsequent ones will be devoted to the resulting theory of ODE and PDE.

Contents

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1. Introduction: foundations of generalized functions as set-theoretical maps

The aim of the present work is to lay the foundations for a new approach to the theory of generalized functions, so-called *generalized smooth functions* (GSF). In developing such a theory, various objectives can be pursued, and our motivations mainly come from applications in mathematical physics and nonlinear singular differential equations, where the need for such a nonlinear theory is well-known (see, e.g., [47, 110, 52, 17, 8, 91, 81, 11, 48] for applications in mathematical physics, [106, 105, 53, 27, 50, 17] for differential equations, and references therein).

In particular, our aim is to arrive at a mathematical theory close to the historically original conception of generalized function, [20, 73, 60]: in essence, the idea of authors such as Dirac, Cauchy, Poisson, Kirchhoff, Helmholtz, Kelvin and Heaviside (who informally worked with "numbers" which also comprise infinitesimals and infinite scalars) was to view generalized functions as certain types of smooth set-theoretical maps obtained from ordinary smooth maps by introducing a dependence on suitable infinitesimal or infinite parameters. We call this idea the Cauchy-Dirac approach to generalized functions. For example, the density of a Cauchy-Lorentz distribution with an infinitesimal scale parameter was used by Cauchy to obtain classical properties which nowadays are attributed to the Dirac delta, [60]. More generally, in the GSF approach, generalized functions are seen as set-theoretical functions defined on, and attaining values in, a suitable non-Archimedean ring of scalars containing infinitesimals and infinities, as well as sharing essential properties of ordinary smooth functions. In the present work, we will develop this point of view, and prove that it generalizes the mentioned Cauchy-Dirac approach. In our view, the main benefits of this theory lie in a clarification of a number of foundational issues in the theory of generalized functions, namely:

(i) GSF include all Schwartz distributions, see Thm. 25, and Colombeau generalized functions, see [43].
(ii) They allow nonlinear operations on generalized functions, Sec. 3, and to compose them unrestrictedly, Thm. 28.
(iii) GSF are simpler than standard approaches as they allow us to treat generalized functions more closely to classical smooth functions. In particular, they

allow us to prove a number of analogues of fundamental theorems of classical analysis: e.g., mean value theorem, intermediate value theorem, extreme value theorem, Taylor's theorem, see Sec. 7, local and global inverse function theorem, [42], integrals via primitives, Sec. 6, multidimensional integrals, Sec. 8, theory of compactly supported functions, [41]. Therefore, this approach to generalized functions results in a flexible and rich framework which allows both the formalization of calculations appearing in physics and the development of new applications in mathematics and mathematical physics.

(iv) Several results of the classical theory of calculus of variations can be developed for GSF: the fundamental lemma, second variation and minimizers, higher order Euler-Lagrange equations, D'Alembert principle in differential form, a weak form of the Pontryagin maximum principle, necessary Legendre condition, Jacobi fields, conjugate points and Jacobi's theorem, Noether's theorem, see [75, 28].

(v) The closure with respect to composition leads to a solution concept of differential equations close to the classical one. In the second and third article of this series, we will introduce a non-Archimedean version of the Banach fixed point theorem that is well suited for spaces of GSF, a Picard-Lindelöf theorem for both ODE and PDE, results about the maximal set of existence, Gronwall theorem, flux properties, continuous dependence on initial conditions, full compatibility with classical smooth solutions, etc., see [77, 44].

Moreover, we think that a satisfactory theory of generalized functions as used in mathematical physics should also provide an extension to function spaces, possibly in a Cartesian closed category or, better, in a topos. The use of a Cartesian closed category as a useful framework for mathematics and mathematical physics can be motivated in several ways:

(i) It is well known that a non trivial problem of the category **Man** of smooth manifolds is the absence of closure properties with respect to interesting categorical operations such as the construction of function spaces $\mathbf{Man}(M, N)$, subspaces, equalizers, etc., see [7, 9, 14, 30, 37, 38, 58, 72, 62, 74, 82, 107, 108, 109]. The search for a Cartesian closed category embedding **Man** is the most widespread approach to solving this problem.

(ii) In physics, the necessity to use infinite-dimensional spaces frequently appears. A classical example is the space $\mathbf{Man}(M, N)$ of all smooth mappings between two smooth manifolds M and N, or some of its subspaces, e.g. the space of all the diffeomorphisms of a smooth manifold. Typically, we are interested in infinite dimensional Lie groups, because they appear, e.g., in the study of both compressible and incompressible fluids, in magnetohydrodynamics, in plasma-dynamics or in electrodynamics (see e.g. [2] and references therein). It is also well established (see e.g. [30, 37, 34]) that Cartesian closedness is a desirable condition in the calculus of variations.

(iii) The convenient setting, [72, 30], is the most advanced theory of smooth spaces extending the theory of Banach manifolds. Some applications of this notion to classical field theory can be found in [1]. In addition, several other approaches to a new notion of smooth space have been motivated by problems of physics. For example, the notion of diffeological space has been used in [107, 108, 109], starting also from a variant of [14], to study quantization of coadjoint orbits in infinite dimensional groups of diffeomorphisms. Diffeological spaces form a Cartesian closed, complete, co-complete quasi-topos, [58, 7, 64].

For these reasons, we close this work by embedding our category of generalized functions into a Grothendieck topos, see Sec. 10.

Finally, a theory of generalized functions for mathematical physics frequently appears coupled with a theory of actual infinitesimals and infinities (see e.g. [56, 64, 63, 50]). This is natural, since informal descriptions of these functions used in many calculations in physics employ a language including infinitesimals or infinities. Historically, it has turned out that approaches requiring a substantial background knowledge in mathematical logic are only reluctantly accepted by some physicists and mathematicians. Therefore, even if sometimes they appear less powerful, theories that do not need such knowledge ([35, 101, 15]) are more easily accepted. In the following section, we introduce the non-Archimedean ring of scalars in a very natural way, without requiring any notions from mathematical logic or ultrafilter set theory.

The structure of the paper is as follows: we first introduce the new ring of scalars and its natural topology in Sec. 2; in Sec. 3 we define the notion of GSF and prove that GSF are always continuous; we present the embedding of Schwartz distributions and prove the closure of GSF with respect to composition (e.g. we study and graphically represent $\delta\circ\delta$); in Sec. 5, 6, 7 we study the differential calculus, the (1-dimensional) integral calculus, and several related classical theorems. In Sec. 8, we introduce multidimensional integration, with related convergence theorems. Sheaf properties for GSF defined on different types of domains are proved in Sec. 9: they e.g. allow one to glue GSF defined on infinitesimal domains to get a global GSF defined on a finite or even on an unbounded domain. Finally, in Sec. 10 we construct the Grothendieck topos of generalized functions, including a full introduction of all the necessary preliminaries. Throughout the paper, several theorems will treat the connections of notions related to GSF to the corresponding classical notions, in case the latter can at least be formulated. Even if other papers about GSF already appeared in the literature (see [43, 42, 75]), this is the first one where all these basic results (and several others) are presented with the related proofs.

The paper is completely self-contained: only a basic knowledge of Schwartz distribution theory and the concepts of category, functor and natural transformation are needed.

2. The ring of scalars and its topologies

Exactly as real numbers can be seen as equivalence classes of sequences $(q_n)_{n\in\mathbb{N}}$ of rationals[1], it is very natural to consider a non-Archimedean extension of $\mathbb{R}$ defined by a quotient ring $\mathcal{R}/\sim$, where $\mathcal{R}\subseteq\mathbb{R}^I$. Here $\mathcal{R}$ is a subalgebra of nets $(x_\varepsilon)_{\varepsilon\in I}\in\mathbb{R}^I$ defined on a directed set $(I,\le)$, and with pointwise algebraic operations. For simplicity and for historical reasons, instead of $I=\mathbb{N}$, we consider $I:=(0,1]$, corresponding to $\varepsilon\to0^+$, $\varepsilon\in I$, but any other directed set can be used instead of I. In this work, we will denote ε-dependent nets simply by $(x_\varepsilon):=(x_\varepsilon)_{\varepsilon\in I}$, and the corresponding equivalence class simply by $[x_\varepsilon]:=[(x_\varepsilon)]_\sim\in\mathcal{R}/\sim$. We aim at constructing the quotient ring $\widetilde{\mathbb{R}}:=\mathcal{R}/\sim$ so that it contains infinitesimals and infinities. The following observation points to a natural way of achieving this goal.

[1]In the naturals $\mathbb{N}=\{0,1,2,\dots\}$ we include zero.

Let us assume that $[z_\varepsilon] = 0 \in \widetilde{\mathbb{R}}$ and $[J_\varepsilon] \in \widetilde{\mathbb{R}}$ is generated by an infinite net (J_ε), i.e. such that $\lim_{\varepsilon\to 0^+} |J_\varepsilon| = +\infty$. Then we would have

$$\begin{aligned}[z_\varepsilon]\cdot[J_\varepsilon] &= 0\cdot[J_\varepsilon] = 0\\ &= [z_\varepsilon\cdot J_\varepsilon].\end{aligned} \tag{2.1}$$

Finally, let us assume that

$$\forall [w_\varepsilon] \in \widetilde{\mathbb{R}} : \ [w_\varepsilon] = 0 \ \Rightarrow \ \lim_{\varepsilon\to 0^+} w_\varepsilon = 0. \tag{2.2}$$

Under these assumptions, (2.1) yields $\lim_{\varepsilon\to 0^+} z_\varepsilon \cdot J_\varepsilon = 0$, and hence

$$\exists \varepsilon_0 \in I \, \forall \varepsilon \in (0, \varepsilon_0] : \ |z_\varepsilon| \le \left|J_\varepsilon^{-1}\right|. \tag{2.3}$$

Consequently, *the nets* (z_ε) *representing* 0*, i.e. such that* $(z_\varepsilon) \sim 0$*, must be dominated by the reciprocals of every infinite number* $[J_\varepsilon] \in \widetilde{\mathbb{R}}$. It is not hard to prove that if every infinite net (J_ε) is in the subalgebra $\mathcal{R}$, then (2.3) implies that the equivalence relation $\sim$ must be trivial:

$$\exists \varepsilon_0 \in I \, \forall \varepsilon \in (0, \varepsilon_0] : \ z_\varepsilon = 0. \tag{2.4}$$

This situation corresponds to the Schmieden-Laugwitz model, [104].

If we do not want to have the trivial model (2.4), we can hence either negate the natural property (2.2) (this is the case of nonstandard analysis; see [18] for more details) or to restrict the class of all the nets (J_ε) generating infinite numbers in $\widetilde{\mathbb{R}}$. Since we want to start from a subalgebra $\mathcal{R} \subseteq \mathbb{R}^I$, a *first* natural idea is to consider the following class of infinite nets

$$\mathcal{I} := \left\{ (\varepsilon^{-a}) \mid a \in \mathbb{R}_{>0} \right\}. \tag{2.5}$$

and hence to consider the subalgebra $\mathcal{R} \subseteq \mathbb{R}^I$ containing nets $(b_\varepsilon) \in \mathbb{R}^I$ bounded by some $(J_\varepsilon) \in \mathcal{I}$. This idea is generalized in the following definition, where we take exactly (2.3) as the widest possible definition of $(z_\varepsilon) \sim 0$:

Definition 1. Let $\rho = (\rho_\varepsilon) \in (0,1]^I$ be a net such that $(\rho_\varepsilon) \to 0$ as $\varepsilon \to 0^+$ (in the following, such a net will be called a *gauge*). Then

(i) $\mathcal{I}(\rho) := \{(\rho_\varepsilon^{-a}) \mid a \in \mathbb{R}_{>0}\}$ is called the *asymptotic gauge* generated by ρ.

(ii) If $\mathcal{P}(\varepsilon)$ is a property of $\varepsilon \in I$, we use the notation $\forall^0 \varepsilon : \mathcal{P}(\varepsilon)$ to denote $\exists \varepsilon_0 \in I \, \forall \varepsilon \in (0, \varepsilon_0] : \mathcal{P}(\varepsilon)$. We can read $\forall^0 \varepsilon$ as *"for* ε *small"*.

(iii) We say that a net $(x_\varepsilon) \in \mathbb{R}^I$ *is* ρ*-moderate*, and we write $(x_\varepsilon) \in \mathbb{R}_\rho$ if

$$\exists (J_\varepsilon) \in \mathcal{I}(\rho) : \ x_\varepsilon = O(J_\varepsilon) \text{ as } \varepsilon \to 0^+,$$

i.e., if

$$\exists N \in \mathbb{N} \, \forall^0 \varepsilon : \ |x_\varepsilon| \le \rho_\varepsilon^{-N}.$$

(iv) Let (x_ε), $(y_\varepsilon) \in \mathbb{R}^I$. Then we say that $(x_\varepsilon) \sim_\rho (y_\varepsilon)$ if

$$\forall (J_\varepsilon) \in \mathcal{I}(\rho) : \ x_\varepsilon = y_\varepsilon + O(J_\varepsilon^{-1}) \text{ as } \varepsilon \to 0^+,$$

that is if

$$\forall n \in \mathbb{N} \, \forall^0 \varepsilon : \ |x_\varepsilon - y_\varepsilon| \le \rho_\varepsilon^n.$$

This is a congruence relation on the ring $\mathbb{R}_\rho$ of moderate nets with respect to pointwise operations, and we can hence define

$${}^\rho\widetilde{\mathbb{R}} := \mathbb{R}_\rho / \sim_\rho,$$

which we call *Robinson-Colombeau ring of generalized numbers*, [93, 15].

(v) In particular, if the gauge $\rho = (\rho_\varepsilon)$ is non-decreasing, then we say that ρ is a *monotonic gauge*. Clearly, considering a monotonic gauge narrows the class of moderate nets: e.g. if $\lim_{\varepsilon\to\frac{1}{k}} x_\varepsilon = +\infty$ for all $k \in \mathbb{N}_{>0}$, then $(x_\varepsilon) \notin \mathbb{R}_\rho$ for any monotonic gauge ρ.

In the following, ρ will always denote a net as in Def. 1, even if we will sometimes omit the dependence on the infinitesimal ρ, when this is clear from the context. We will see below that we can choose ρ e.g. depending on the class of differential equations we need to solve for the generalized functions we are going to introduce.

We can also define an order relation on ${}^\rho\widetilde{\mathbb{R}}$ by writing $[x_\varepsilon] \le [y_\varepsilon]$ if there exists $(z_\varepsilon) \in \mathbb{R}^I$ such that $(z_\varepsilon) \sim_\rho 0$ (we then say that (z_ε) is *ρ-negligible*) and $x_\varepsilon \le y_\varepsilon + z_\varepsilon$ for ε small. Equivalently, we have that $x \le y$ if and only if there exist representatives $[x_\varepsilon] = x$ and $[y_\varepsilon] = y$ such that $x_\varepsilon \le y_\varepsilon$ for all ε. The following result follows directly from the previous definitions:

Theorem 2. *${}^\rho\widetilde{\mathbb{R}}$ is a partially ordered ring. The real numbers $r \in \mathbb{R}$ are embedded in ${}^\rho\widetilde{\mathbb{R}}$ by viewing them as constant nets $[r] \in {}^\rho\widetilde{\mathbb{R}}$.*

Although the order $\le$ is not total, we still have the possibility to define the infimum $[x_\varepsilon] \wedge [y_\varepsilon] := [\min(x_\varepsilon, y_\varepsilon)]$, and analogously the supremum function $[x_\varepsilon] \vee [y_\varepsilon] := [\max(x_\varepsilon, y_\varepsilon)]$ and the absolute value $|[x_\varepsilon]| := [|x_\varepsilon|] \in {}^\rho\widetilde{\mathbb{R}}$. Henceforth, we will also use the customary notation ${}^\rho\widetilde{\mathbb{R}}^*$ for the set of invertible generalized numbers.

As in every non-Archimedean ring, we have the following

Definition 3. Let $x \in {}^\rho\widetilde{\mathbb{R}}$ be a generalized number. Then

(i) x is *infinitesimal* if $|x| \le r$ for all $r \in \mathbb{R}_{>0}$. If $x = [x_\varepsilon]$, this is equivalent to $\lim_{\varepsilon\to 0^+} x_\varepsilon = 0$. We write $x \approx y$ if $x - y$ is infinitesimal, and $D_\infty := \left\{h \in {}^\rho\widetilde{\mathbb{R}} \mid h \approx 0\right\}$ for the set of all infinitesimals.

(ii) x is *infinite* if $|x| \ge r$ for all $r \in \mathbb{R}_{>0}$. If $x = [x_\varepsilon]$, this is equivalent to $\lim_{\varepsilon\to 0^+} |x_\varepsilon| = +\infty$.

(iii) x is *finite* if $|x| \le r$ for some $r \in \mathbb{R}_{>0}$.

For example, setting $\mathrm{d}\rho := [\rho_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}$, we have that $\mathrm{d}\rho^n \in {}^\rho\widetilde{\mathbb{R}}$, $n \in \mathbb{N}_{>0}$, is an invertible infinitesimal, whose reciprocal is $\mathrm{d}\rho^{-n} = [\rho_\varepsilon^{-n}]$, which is necessarily a positive infinite number. Of course, in the ring ${}^\rho\widetilde{\mathbb{R}}$ there exist generalized numbers which are not in any of the three classes of Def. 3, like e.g. $x_\varepsilon = \frac{1}{\varepsilon}\sin\left(\frac{1}{\varepsilon}\right)$.

Definition 4.

(i) If $\mathcal{P}\{(x_\varepsilon)\}$ is a property of $(x_\varepsilon) \in \mathbb{R}^n_\rho$, then we also use the abbreviations:

$$\forall [x_\varepsilon] \in X : \ \mathcal{P}\{(x_\varepsilon)\} \quad :\Longleftrightarrow \quad \forall (x_\varepsilon) \in \mathbb{R}^n_\rho : \ [x_\varepsilon] \in X \ \Rightarrow \ \mathcal{P}\{(x_\varepsilon)\}$$
$$\exists [x_\varepsilon] \in X : \ \mathcal{P}\{(x_\varepsilon)\} \quad :\Longleftrightarrow \quad \exists (x_\varepsilon) \in \mathbb{R}^n_\rho : \ [x_\varepsilon] \in X, \ \mathcal{P}\{(x_\varepsilon)\}.$$

For example, if $X = \{x\} \subseteq {}^\rho\widetilde{\mathbb{R}}^n$, then $\forall [x_\varepsilon] = x : \ \mathcal{P}\{(x_\varepsilon)\}$ means that the property holds for all representatives of x, and $\exists [x_\varepsilon] = x : \ \mathcal{P}\{(x_\varepsilon)\}$ means that the same property holds for some representative of x.

(ii) Our notations for intervals are: $[a, b] := \{x \in {}^\rho\widetilde{\mathbb{R}} \mid a \le x \le b\}$, $[a, b]_{\mathbb{R}} := [a, b] \cap \mathbb{R}$, and analogously for segments $[x, y] := \{x + r \cdot (y - x) \mid r \in [0, 1]\} \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ and $[x, y]_{\mathbb{R}^n} = [x, y] \cap \mathbb{R}^n$.

(iii) For subsets $J, K \subseteq I$ we write $K \subseteq_0 J$ if 0 is an accumulation point of K and $K \subseteq J$ (we read it as: *K is co-final in J*). For any $J \subseteq_0 I$, the constructions introduced so far can be repeated with nets $(x_\varepsilon)_{\varepsilon\in J}$. We indicate this by using the symbol ${}^\rho\widetilde{\mathbb{R}}^n|_J$. If $K \subseteq_0 J$, $x \in {}^\rho\widetilde{\mathbb{R}}^n|_J$ and $x' \in {}^\rho\widetilde{\mathbb{R}}^n|_K$, then x' is called a *subpoint* of x, denoted as $x' \subseteq x$, if there exist representatives $(x_\varepsilon)_{\varepsilon\in J}$, $(x'_\varepsilon)_{\varepsilon\in K}$ of x, x' such that $x'_\varepsilon = x_\varepsilon$ for all $\varepsilon \in K$. In this case we write $x' = x|_K$, $\mathrm{dom}(x') := K$, and the restriction $(-)|_K : {}^\rho\widetilde{\mathbb{R}}^n \longrightarrow {}^\rho\widetilde{\mathbb{R}}^n|_K$ is a well defined operation. In general, for $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ we set $X|_J := \{x|_J \in {}^\rho\widetilde{\mathbb{R}}^n|_J \mid x \in X\}$. Finally note that

$$\left(\neg\forall^0\varepsilon:\ \mathcal{P}\{x_\varepsilon\}\right) \iff \exists J \subseteq_0 I\,\forall\varepsilon \in J:\ \neg\mathcal{P}\{x_\varepsilon\}.$$

2.1. **Topologies on ${}^\rho\widetilde{\mathbb{R}}^n$.** On the ${}^\rho\widetilde{\mathbb{R}}$-module ${}^\rho\widetilde{\mathbb{R}}^n$ we can consider the natural extension of the Euclidean norm, i.e. $|[x_\varepsilon]| := [|x_\varepsilon|] \in {}^\rho\widetilde{\mathbb{R}}$, where $[x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}^n$. Even if this generalized norm takes values in ${}^\rho\widetilde{\mathbb{R}}$, it shares some essential properties with classical norms:

$$\begin{aligned}
&|x| = x \vee (-x)\\
&|x| \ge 0\\
&|x| = 0 \Rightarrow x = 0\\
&|y\cdot x| = |y|\cdot|x|\\
&|x+y| \le |x| + |y|\\
&||x| - |y|| \le |x-y|.
\end{aligned}$$

It is therefore natural to consider on ${}^\rho\widetilde{\mathbb{R}}^n$ topologies generated by balls defined by this generalized norm and a set of radii:

Definition 5. We say that $\mathfrak{R}$ is a *set of radii* if

(i) $\mathfrak{R} \subseteq {}^\rho\widetilde{\mathbb{R}}^*_{\ge 0}$ is a non-empty subset of positive invertible generalized numbers.
(ii) For all r, $s \in \mathfrak{R}$ the infimum $r \wedge s \in \mathfrak{R}$.
(iii) $k \cdot r \in \mathfrak{R}$ for all $r \in \mathfrak{R}$ and all $k \in \mathbb{R}_{>0}$.

Moreover, if $\mathfrak{R}$ is a set of radii and x, $y \in {}^\rho\widetilde{\mathbb{R}}$, then:

(iv) We write $x <_{\mathfrak{R}} y$ if $\exists r \in \mathfrak{R}:\ r \le y - x$.
(v) $B^{\mathfrak{R}}_r(x) := \left\{ y \in {}^\rho\widetilde{\mathbb{R}}^n \mid |y-x| <_{\mathfrak{R}} r \right\}$ for any $r \in \mathfrak{R}$.
(vi) $B^{\mathrm{E}}_\rho(x) := \{y \in \mathbb{R}^n \mid |y - x| < \rho\}$, for any $\rho \in \mathbb{R}_{>0}$, denotes an ordinary Euclidean ball in $\mathbb{R}^n$.

For example, ${}^\rho\widetilde{\mathbb{R}}^*_{\ge 0}$ and $\mathbb{R}_{>0}$ are sets of radii.

Lemma 6. *Let $\mathfrak{R}$ be a set of radii and x, y, $z \in {}^\rho\widetilde{\mathbb{R}}$. Then*

(i) $\neg(x <_{\mathfrak{R}} x)$.
(ii) $x <_{\mathfrak{R}} y$ *and* $y <_{\mathfrak{R}} z$ *imply* $x <_{\mathfrak{R}} z$.
(iii) $\forall r \in \mathfrak{R}:\ 0 <_{\mathfrak{R}} r$.

Proof. (i): $x <_{\mathfrak{R}} x$ would imply $r \le 0$ for some $r \in \mathfrak{R} \subseteq {}^\rho\widetilde{\mathbb{R}}^*_{\ge 0}$. But then $r^{-1}r = 1 \le 0$.
(ii): If $r \le y - x$ and $s \le z - y$ for r, $s \in \mathfrak{R}$, then $2(r \wedge s) \le r + s \le z - x$.
(iii): In fact, we have $0 <_{\mathfrak{R}} r$ if and only if $s \le r$ for some $s \in \mathfrak{R}$. ☐

The relation $<_{\mathfrak{R}}$ has better topological properties as compared to the usual strict order relation $x \le y$ and $x \ne y$ (a relation that we will therefore never use) because of the following result:

Theorem 7. *The set of balls* $\left\{B_r^{\mathfrak{R}}(x)\mid r\in\mathfrak{R},\ x\in{}^{\rho}\widetilde{\mathbb{R}}^n\right\}$ *generated by a set of radii* $\mathfrak{R}$ *is a base for a topology on* ${}^{\rho}\widetilde{\mathbb{R}}^n$.

Proof. It suffices to consider $z\in B_r^{\mathfrak{R}}(x)\cap B_s^{\mathfrak{R}}(y)$ and to prove that $B_\nu^{\mathfrak{R}}(z)\subseteq B_r^{\mathfrak{R}}(x)\cap B_s^{\mathfrak{R}}(y)$ for some $\nu\in\mathfrak{R}$. The proof is essentially a reformulation of the classical proof in metric spaces. In fact, we have $\bar{r}\le r-|x-z|$ and $\bar{s}\le s-|y-z|$ for some $\bar{r},\ \bar{s}\in\mathfrak{R}$. Set $\nu:=\bar{r}\wedge\bar{s}\in\mathfrak{R}$. The inequality $|w-z|<_{\mathfrak{R}}\nu$ implies $\sigma\le\nu-|w-z|$ for some $\sigma\in\mathfrak{R}$. Therefore, $|w-x|\le|w-z|+|z-x|\le\nu-\sigma+r-\bar{r}$ and thereby $\sigma\le\bar{r}+\sigma-\nu\le r-|w-x|$, i.e. $|w-x|<_{\mathfrak{R}}r$. This proves that $B_\nu^{\mathfrak{R}}(z)\subseteq B_r^{\mathfrak{R}}(x)$, and the other inclusion follows analogously. □

Henceforth, we will only consider the sets of radii ${}^{\rho}\widetilde{\mathbb{R}}_{\ge0}^{*}$ and $\mathbb{R}_{>0}$. The topology generated in the former case is called *sharp topology*, whereas the latter is called *Fermat topology*. We will call *sharply open set* any open set in the sharp topology, and *large open set* any open set in the Fermat topology; clearly, the latter is coarser than the former. Let us note explicitly that taking an infinitesimal radius $r\in{}^{\rho}\widetilde{\mathbb{R}}_{\ge0}^{*}$ we can consider infinitesimal neighborhoods of $x\in{}^{\rho}\widetilde{\mathbb{R}}^n$ in the sharp topology. Of course, this is not possible in the Fermat topology. The existence of infinitesimal neighborhoods implies that the sharp topology induces the discrete topology on $\mathbb{R}$, see [40]. The necessity to consider infinitesimal neighborhoods occurs in any theory containing continuous generalized functions which have infinite derivatives. Indeed, from the mean value theorem Thm. 49(i) below, we have $f(x)-f(x_0)=f'(c)\cdot(x-x_0)$ for some $c\in[x,x_0]$. Therefore, we have $f(x)\in B_r(f(x_0))$, for a given $r\in{}^{\rho}\widetilde{\mathbb{R}}_{>0}$, if and only if $|x-x_0|\cdot|f'(c)|<r$, which yields an infinitesimal neighborhood of x_0 in case $f'(c)$ is infinite; see [40, 41] for precise statements and proofs corresponding to this intuition. By an innocuous abuse of language, we write $x<y$ instead of $x<_{{}^{\rho}\widetilde{\mathbb{R}}_{\ge0}^{*}}y$ and $x<_{\mathbb{R}}y$ instead of $x<_{\mathbb{R}_{>0}}y$. For example, ${}^{\rho}\widetilde{\mathbb{R}}_{\ge0}^{*}={}^{\rho}\widetilde{\mathbb{R}}_{>0}$. We will simply write $B_r(x)$ to denote an open ball in the sharp topology and $B_r^{\mathrm{F}}(x)$ for an open ball in the Fermat topology. Proceeding by contradiction, it is not difficult to prove that the sharp topology on ${}^{\rho}\widetilde{\mathbb{R}}^n$ is Hausdorff and that the set of all the infinitesimals D_∞ is a clopen set; moreover, as will be proved more generally in [77], this topology is also Cauchy complete.

The following result is useful in dealing with positive and invertible generalized numbers.

Lemma 8. *Let* $x\in{}^{\rho}\widetilde{\mathbb{R}}$. *Then the following are equivalent:*

(i) x *is invertible and* $x\ge0$, *i.e.* $x>0$.
(ii) *For each representative* $(x_\varepsilon)\in\mathbb{R}_\rho$ *of* x *we have* $\forall^0\varepsilon:\ x_\varepsilon>0$.
(iii) *For each representative* $(x_\varepsilon)\in\mathbb{R}_\rho$ *of* x *we have* $\exists m\in\mathbb{N}\,\forall^0\varepsilon:\ x_\varepsilon>\rho_\varepsilon^m$.
(iv) *There exists a representative* $(x_\varepsilon)\in\mathbb{R}_\rho$ *of* x *such that* $\exists m\in\mathbb{N}\,\forall^0\varepsilon:\ x_\varepsilon>\rho_\varepsilon^m$.

Proof. (i) ⇒ (ii): Since x is positive, we can find a representative $[x_\varepsilon]=x$ such that $x_\varepsilon\ge0$ for all ε. But x is also invertible, so for all ε we can also write $x_\varepsilon y_\varepsilon=1+z_\varepsilon$, where $(z_\varepsilon)\sim_\rho0$ is a negligible net. By contradiction, assume that $x_{\varepsilon_k}\le0$ for each $k\in\mathbb{N}$, where $(\varepsilon_k)_{k\in\mathbb{N}}\to0^+$. Then $x_{\varepsilon_k}=0$ and hence $x_{\varepsilon_k}y_{\varepsilon_k}=0=1+z_{\varepsilon_k}\to1$ for $k\to+\infty$, which is a contradiction.
(ii) ⇒ (iii): Assume that there exists a representative $[x_\varepsilon]=x$ such that $x_{\varepsilon_k}\le\rho_{\varepsilon_k}^k$ for each $k\in\mathbb{N}$, where $(\varepsilon_k)_{k\in\mathbb{N}}\to0^+$ monotonically. We then define a ρ-moderate net by $\hat{x}_\varepsilon:=0$ if $\varepsilon=\varepsilon_k$ and $\hat{x}_\varepsilon:=x_\varepsilon$ otherwise. For each $n\in\mathbb{N}$, if k is sufficiently

big, we have $|x_{\varepsilon_k} - \hat{x}_{\varepsilon_k}| \le \rho_{\varepsilon_k}^k \le \rho_{\varepsilon_k}^n$. This implies that $(x_\varepsilon) \sim_\rho (\hat{x}_\varepsilon)$. Therefore $(\hat{x}_\varepsilon)$ is another representative of x, which contradicts (ii) by construction.
(iii) $\Rightarrow$ (i): By assumption, $\lim_{\varepsilon\to 0^+} \rho_\varepsilon = 0^+$. This and (iii) yield that $x_\varepsilon > \rho_\varepsilon^m > 0$ for ε small, say for $\varepsilon \le \varepsilon_0$. Therefore, $0 < y_\varepsilon := x_\varepsilon^{-1} \le \rho_\varepsilon^{-m}$ for $\varepsilon \le \varepsilon_0$ (and y_ε arbitrarily defined elsewhere) is ρ-moderate and hence it is a representative of the inverse of x.

Finally, (iii) implies (iv) for logical reasons, and (iv) implies (i) because $\rho_\varepsilon > 0$. □

2.2. **Open, closed and bounded sets generated by nets.** A natural way to obtain sharply open, closed and bounded sets in ${}^\rho\widetilde{\mathbb{R}}^n$ is by using a net (A_ε) of subsets $A_\varepsilon \subseteq \mathbb{R}^n$. We have two ways of extending the membership relation $x_\varepsilon \in A_\varepsilon$ to generalized points $[x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}^n$ (cf. [87, 43]):

Definition 9. Let (A_ε) be a net of subsets of $\mathbb{R}^n$. Then

(i) $[A_\varepsilon] := \left\{[x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}^n \mid \forall^0\varepsilon : x_\varepsilon \in A_\varepsilon\right\}$ is called the *internal set* generated by the net (A_ε).

(ii) Let (x_ε) be a net of points of $\mathbb{R}^n$. Then we say that $x_\varepsilon \in_\varepsilon A_\varepsilon$, and we read it as (x_ε) *strongly belongs to* (A_ε), if
  (a) $\forall^0\varepsilon : x_\varepsilon \in A_\varepsilon$.
  (b) If $(x'_\varepsilon) \sim_\rho (x_\varepsilon)$, then also $x'_\varepsilon \in A_\varepsilon$ for ε small.

  Moreover, we set $\langle A_\varepsilon\rangle := \left\{[x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}^n \mid x_\varepsilon \in_\varepsilon A_\varepsilon\right\}$, and we call it the *strongly internal set* generated by the net (A_ε).

(iii) We say that the internal set $K = [A_\varepsilon]$ is *sharply bounded* if there exists $M \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ such that $K \subseteq B_M(0)$.

(iv) Finally, we say that the (A_ε) is a *sharply bounded net* if there exists $N \in \mathbb{R}_{>0}$ such that $\forall^0\varepsilon\, \forall x \in A_\varepsilon : |x| \le \rho_\varepsilon^{-N}$.

Therefore, $x \in [A_\varepsilon]$ if there exists a representative $[x_\varepsilon] = x$ such that $x_\varepsilon \in A_\varepsilon$ for ε small, whereas this membership is independent from the chosen representative in case of strongly internal sets. An internal set generated by a constant net $A_\varepsilon = A \subseteq \mathbb{R}^n$ will simply be denoted by $[A]$.

The following theorem (cf. [87, 43] for the case $\rho_\varepsilon = \varepsilon$) shows that internal and strongly internal sets have dual topological properties:

Theorem 10. *For $\varepsilon \in I$, let $A_\varepsilon \subseteq \mathbb{R}^n$ and let $x_\varepsilon \in \mathbb{R}^n$. Then we have*

(i) $[x_\varepsilon] \in [A_\varepsilon]$ *if and only if* $\forall q \in \mathbb{R}_{>0}\, \forall^0\varepsilon : d(x_\varepsilon, A_\varepsilon) \le \rho_\varepsilon^q$. *Therefore* $[x_\varepsilon] \in [A_\varepsilon]$ *if and only if* $[d(x_\varepsilon, A_\varepsilon)] = 0 \in {}^\rho\widetilde{\mathbb{R}}$.

(ii) $[x_\varepsilon] \in \langle A_\varepsilon\rangle$ *if and only if* $\exists q \in \mathbb{R}_{>0}\, \forall^0\varepsilon : d(x_\varepsilon, A_\varepsilon^c) > \rho_\varepsilon^q$, *where* $A_\varepsilon^c := \mathbb{R}^n \setminus A_\varepsilon$. *Therefore, if* $(d(x_\varepsilon, A_\varepsilon^c)) \in \mathbb{R}_\rho$, *then* $[x_\varepsilon] \in \langle A_\varepsilon\rangle$ *if and only if* $[d(x_\varepsilon, A_\varepsilon^c)] > 0$.

(iii) $[A_\varepsilon]$ *is sharply closed.*

(iv) $\langle A_\varepsilon\rangle$ *is sharply open.*

(v) $[A_\varepsilon] = [\mathrm{cl}\,(A_\varepsilon)]$, *where* $\mathrm{cl}\,(S)$ *is the closure of* $S \subseteq \mathbb{R}^n$.

(vi) $\langle A_\varepsilon\rangle = \langle\mathrm{int}(A_\varepsilon)\rangle$, *where* $\mathrm{int}\,(S)$ *is the interior of* $S \subseteq \mathbb{R}^n$.

Proof. (i) $\Rightarrow$: We have $x'_\varepsilon \in A_\varepsilon$ for some representative $[x'_\varepsilon] = [x_\varepsilon] \in [A_\varepsilon]$. But $d(x_\varepsilon, A_\varepsilon) \le |x_\varepsilon - x'_\varepsilon| + d(x'_\varepsilon, A_\varepsilon)$, from which the conclusion follows.
(i) $\Leftarrow$: Since the net $(\inf_{a\in A_\varepsilon} |x_\varepsilon - a|)$ is ρ-negligible, we can find a decreasing sequence $(\varepsilon_k)_{k\in\mathbb{N}} \downarrow 0$ such that $\inf_{a\in A_\varepsilon} |x_\varepsilon - a| < \rho_\varepsilon^k$ for $\varepsilon \le \varepsilon_k$. Hence, for each

$\varepsilon \in (\varepsilon_{k+1}, \varepsilon_k]_{\mathbb{R}}$ we can find $x'_\varepsilon \in A_\varepsilon$ such that $|x_\varepsilon - x'_\varepsilon| \le \rho_\varepsilon^k$. Therefore, (x'_ε) is another representative of $[x_\varepsilon]$ and $x'_\varepsilon \in A_\varepsilon$ for $\varepsilon \le \varepsilon_0$.
(ii): Let $[x_\varepsilon] \in \langle A_\varepsilon \rangle$ and suppose to the contrary that there exists a sequence $\varepsilon_k \downarrow 0$ such that $d(x_{\varepsilon_k}, A^{\mathrm{c}}_{\varepsilon_k}) \le \rho^k_{\varepsilon_k}$ for all $k \in \mathbb{N}$. For each k, pick some $x'_k \in A^{\mathrm{c}}_{\varepsilon_k}$ with $|x'_k - x_{\varepsilon_k}| < 2\rho^k_{\varepsilon_k}$ and choose $(x'_\varepsilon) \sim_\rho (x_\varepsilon)$ such that $x'_{\varepsilon_k} = x'_k$ for all k. Then $x'_{\varepsilon_k} \notin A_{\varepsilon_k}$ for all k, contradicting $x_\varepsilon \in_\varepsilon A_\varepsilon$. Conversely, let $d(x_\varepsilon, A^{\mathrm{c}}_\varepsilon) > \rho^q_\varepsilon$ for ε small. Then in particular, $x_\varepsilon \in A_\varepsilon$. Also, if $(x'_\varepsilon) \sim_\rho (x_\varepsilon)$ then $d(x'_\varepsilon, A^{\mathrm{c}}_\varepsilon) > (1/2)\rho^q_\varepsilon$ for ε small, so $x'_\varepsilon \in A_\varepsilon$. Thus, $[x_\varepsilon] \in \langle A_\varepsilon \rangle$.
(iii): Let $x = [x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}^n \setminus [A_\varepsilon]$. Then (i) yields that $d(x_{\varepsilon_k}, A_{\varepsilon_k}) > \rho^q_{\varepsilon_k}$ for some $q \in \mathbb{R}_{>0}$ and some sequence $(\varepsilon_k)_{k\in\mathbb{N}} \downarrow 0$. Set $r := \frac{1}{2}\mathrm{d}\rho^q$, then $y \in B_r(x)$ implies that for some representative $[y_\varepsilon] = y$ we have $d(y_{\varepsilon_k}, A_{\varepsilon_k}) \ge d(x_{\varepsilon_k}, A_{\varepsilon_k}) - |x_{\varepsilon_k} - y_{\varepsilon_k}| > \rho^q_{\varepsilon_k} - \frac{1}{2}\rho^q_{\varepsilon_k}$. Thereby (i) gives $y \notin [A_\varepsilon]$. This proves that ${}^\rho\widetilde{\mathbb{R}}^n \setminus [A_\varepsilon]$ is sharply open.
(iv): (ii) yields that $[x_\varepsilon] \in \langle A_\varepsilon \rangle$ if and only if $[x_\varepsilon]$ is in the interior of $\langle A_\varepsilon \rangle$ with respect to the sharp topology.
(v), (vi): Directly from (i) and (ii). □

For example, it is not hard to show that the closure in the sharp topology of a ball of center $c = [c_\varepsilon]$ and radius $r = [r_\varepsilon] > 0$ is

$$\overline{B_r(c)} = \left\{ x \in {}^\rho\widetilde{\mathbb{R}}^d \mid |x - c| \le r \right\} = \left[\overline{B^{\mathrm{E}}_{r_\varepsilon}(c_\varepsilon)} \right]. \tag{2.6}$$

In fact, it suffices to prove these equalities for $c = 0$, because the translation $x \mapsto x - c$ is sharply continuous. If (x_n) is a sequence in $\{x \mid |x| \le r\}$ that converges to x_0, then $|x_0| \le |x_0 - x_n| + |x_n| \le |x_0 - x_n| + r$. Letting $n \to +\infty$, this shows that $\{x \mid |x| \le r\}$ is closed. Conversely, if $|x| \le r$, to prove that x is an adherent point of $B_r(0)$, we need to show that

$$\forall s \in {}^\rho\widetilde{\mathbb{R}}_{>0}\, \exists \bar{x} \in B_r(0) \cap B_s(x).$$

Take $k \in \mathbb{N}$ such that $2\mathrm{d}\rho^k \le \min(r, s)$, and representatives $[x_\varepsilon] = x$ and $[r_\varepsilon] = r$ such that $|x_\varepsilon| \le r_\varepsilon$ for small ε. The point $\bar{x}_\varepsilon := x_\varepsilon$ if $|x_\varepsilon| < r_\varepsilon - \rho^k_\varepsilon$ and $\bar{x}_\varepsilon := x_\varepsilon - \frac{x_\varepsilon}{|x_\varepsilon|}\rho^k_\varepsilon$ otherwise satisfies the desired conditions. This proves the first equality in (2.6). The proof that $\overline{B_r(0)} \supseteq \left[\overline{B^{\mathrm{E}}_{r_\varepsilon}(0)} \right]$ is easy. Vice versa, if $|\bar{x}_\varepsilon| \le r_\varepsilon + z_\varepsilon$ for some representatives $[\bar{x}_\varepsilon] = x$ and $[z_\varepsilon] = 0$, then setting $x_\varepsilon := \bar{x}_\varepsilon$ if $|\bar{x}_\varepsilon| \le r_\varepsilon$ and $x_\varepsilon := \frac{\bar{x}_\varepsilon}{|\bar{x}_\varepsilon|} r_\varepsilon$ otherwise gives another representative of x that shows that $x \in \left[\overline{B^{\mathrm{E}}_{r_\varepsilon}(0)} \right]$.
From (2.6) and Thm. (10), it hence also follows that

$$B_r(c) = \langle B^{\mathrm{E}}_{r_\varepsilon}(c_\varepsilon) \rangle. \tag{2.7}$$

In a similar way, it can be shown that for every $x, y \in {}^\rho\widetilde{\mathbb{R}}$

$$y \ge x \Leftrightarrow y \in \overline{\left\{ z \in {}^\rho\widetilde{\mathbb{R}} \mid z > x \right\}}. \tag{2.8}$$

Some relations between internal and strongly internal sets that we will use below are listed in the following

Lemma 11. *Let (Ω_ε) be a net of subsets in $\mathbb{R}^n$ for all ε, and (B_ε) a sharply bounded net such that $[B_\varepsilon] \subseteq \langle \Omega_\varepsilon \rangle$. Then*
(i) $\forall^0 \varepsilon : \ B_\varepsilon \subseteq \Omega_\varepsilon$.
(ii) *If each B_ε is closed, then* $\exists S \in \mathbb{N}\, \forall [x_\varepsilon] \in [B_\varepsilon]\, \forall^0 \varepsilon : \ d(x_\varepsilon, \Omega^{\mathrm{c}}_\varepsilon) \ge \rho^S_\varepsilon$.

(iii) *If* $r=[r_\varepsilon]\in{}^\rho\widetilde{\mathbb{R}}_{>0}$, $b=[b_\varepsilon]\in{}^\rho\widetilde{\mathbb{R}}^n$ *and* $B_r(b)\subseteq\text{int}\left([B_\varepsilon]\right)$, *then* $\forall^0\varepsilon:\ B^{\text{E}}_{r_\varepsilon}(b_\varepsilon)\subseteq B_\varepsilon$.

(iv) *If* $x\in\langle A_\varepsilon\rangle\subseteq[B_\varepsilon]$, *then* $\forall[x_\varepsilon]=x\,\forall^0\varepsilon:\ x_\varepsilon\in B_\varepsilon$.

(v) *If* (C_ε) *is also sharply bounded and* $[B_\varepsilon]\subseteq\text{int}\left([C_\varepsilon]\right)$, *then there exists* $S\in\mathbb{N}$ *such that:*

(a) $\forall^0\varepsilon:\ B^{\text{E}}_{\rho_\varepsilon^S}(B_\varepsilon)\subseteq C_\varepsilon$

(b) $B_{\text{d}\rho^S}(B)\subseteq C$,

where, in general

$$B^d_r(B)=\{x\mid d(x,B)<r\}=\bigcup_{b\in B}B^d_r(b)$$

(in ${}^\rho\widetilde{\mathbb{R}}^n$, *we also set* $d(x,B):=[d(x_\varepsilon,B_\varepsilon)]\in{}^\rho\widetilde{\mathbb{R}}$*).*

(vi) *If each* B_ε *is closed, then there exists a sharply bounded net* (B^+_ε) *of closed sets such that* $[B^+_\varepsilon]\subseteq\langle\Omega_\varepsilon\rangle$ *is a sharp neighborhood of* $[B_\varepsilon]$.

Proof. To prove (i), let us assume, by contradiction, that we can find sequences $(\varepsilon_k)_k$ and $(x_k)_k$ such that $\varepsilon_k\downarrow0$ and $x_k\in B_{\varepsilon_k}\setminus\Omega_{\varepsilon_k}$. Defining $x_\varepsilon:=x_k$ for $\varepsilon=\varepsilon_k$, and $x_\varepsilon\in B_\varepsilon$ otherwise, we have that $x:=[x_\varepsilon]$ is moderate since (B_ε) is sharply bounded. Hence $x\in[B_\varepsilon]$, but $x_{\varepsilon_k}\notin\Omega_{\varepsilon_k}$ by construction, hence $x\notin\langle\Omega_\varepsilon\rangle$ by Def. 9, which is impossible because $[B_\varepsilon]\subseteq\langle\Omega_\varepsilon\rangle$.
(ii): Assume that (i) holds for all $\varepsilon\le\varepsilon_0$. If B_ε is closed, then $B_\varepsilon\Subset\mathbb{R}^n$ because (B_ε) is sharply bounded. We can therefore find a point $\bar{x}_\varepsilon\in B_\varepsilon$ such that $d(\bar{x}_\varepsilon,\Omega^c_\varepsilon)=d(B_\varepsilon,\Omega^c_\varepsilon)$. At $\bar{x}:=[\bar{x}_\varepsilon]\in[B_\varepsilon]$ property (ii) of Thm. 10 yields the existence of some $S\in\mathbb{N}$ such that $d(\bar{x}_\varepsilon,\Omega^c_\varepsilon)\ge\rho_\varepsilon^S$ for ε small. From this the conclusion follows because $d(x_\varepsilon,\Omega^c_\varepsilon)\ge d(B_\varepsilon,\Omega^c_\varepsilon)=d(\bar{x}_\varepsilon,\Omega^c_\varepsilon)$ if $x_\varepsilon\in B_\varepsilon$ for ε small. If $[x'_\varepsilon]=[x_\varepsilon]$ is any other representative, then claim (ii) still holds because $d(x_\varepsilon,x'_\varepsilon)$ is negligible.
(iii): By contradiction, assume that for some $J\subseteq_0 I$ we can find $x_\varepsilon\in B^{\text{E}}_{r_\varepsilon}(b_\varepsilon)\setminus B_\varepsilon$ for all $\varepsilon\in J$. Therefore, $x:=\left[(x_\varepsilon)_{\varepsilon\in J}\right]\in\overline{B_r(B)}|_J$. But the assumption $B_r(b)\subseteq\text{int}\left([B_\varepsilon]\right)$ yields $\overline{B_r(b)}\subseteq[B_\varepsilon]=:B$ and hence $x\in B|_J$, which is impossible.
(iv): Directly from the previous result and Thm. 10(iv).
(v): We prove by contradiction that there exists $S\in\mathbb{N}$ satisfying (a); we will then show that this S also works for (b). So, assume that for all $s\in\mathbb{N}$ there exists $J_s\subseteq_0 I$ and $x_{s\varepsilon}\in B^{\text{E}}_{\rho^s_\varepsilon}(B_\varepsilon)\setminus C_\varepsilon$ for all $\varepsilon\in J_s$. We can hence find $\varepsilon_s\in J_s$ such that $\varepsilon_s<\frac{1}{s}$ and $x_{s\varepsilon_s}\in B^{\text{E}}_{\rho^s_{\varepsilon_s}}(B_{\varepsilon_s})\setminus C_{\varepsilon_s}$. Choosing recursively these ε_s, we can assume that $\varepsilon_{s+1}<\varepsilon_s$, so that $(\varepsilon_s)_s\downarrow0$. Set $J:=\{\varepsilon_s\in J_s\mid s\in\mathbb{N}_{>0}\}\subseteq_0 I$. For each $\varepsilon\in J$, we can set $x_\varepsilon:=x_{s\varepsilon_s}$ for the unique $s\in\mathbb{N}_{>0}$ such that $\varepsilon=\varepsilon_s$, so that $x\in{}^\rho\widetilde{\mathbb{R}}|_J$. For all $\varepsilon=\varepsilon_p\in J$, if $\varepsilon<\varepsilon_s$, then $p>s$ because $(\varepsilon_s)_s$ is strictly decreasing. Thereby

$$x_\varepsilon=x_{\varepsilon_p}\in B^{\text{E}}_{\rho^p_{\varepsilon_p}}(B_{\varepsilon_p})\setminus C_{\varepsilon_p}\subseteq B^{\text{E}}_{\rho^s_\varepsilon}(B_\varepsilon)\setminus C_\varepsilon$$

because $p>s$ and $\varepsilon=\varepsilon_p$. This proves that $(d(x_\varepsilon,B_\varepsilon))_{\varepsilon\in J}\sim_\rho0$ and hence that $x:=\left[(x_\varepsilon)_{\varepsilon\in J}\right]\in B|_J\subseteq\text{int}\left(C\right)|_J$. Therefore, $B_r(x)\subseteq\text{int}\left(C|_J\right)$ for some $r\in{}^\rho\widetilde{\mathbb{R}}_{>0}|_J$. Using (iii), we get $B^{\text{E}}_{r_\varepsilon}(x_\varepsilon)\subseteq C_\varepsilon$ for $\varepsilon\in J$ sufficiently small, and hence $x_\varepsilon\in C_\varepsilon$, a contradiction. Now assume that $S\in\mathbb{N}$ satisfies (a). Then for all $x\in B_{\text{d}\rho^S}(B)$, we have $x\in B_{\text{d}\rho^S}(b)$ for some $b=[b_\varepsilon]\in B$. Therefore, for all $[x_\varepsilon]=x$ and ε small, we have $x_\varepsilon\in B^{\text{E}}_{r_\varepsilon}(b_\varepsilon)\subseteq C_\varepsilon$ using (a).

(vi): To prove this property, it suffices to consider an $M \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$ such that $[B_\varepsilon] \in B_M(0)$ and to define

$$B_\varepsilon^+ := \overline{\left\{x \in B^{\mathrm{E}}_{M_\varepsilon}(0) \mid d(x, \Omega_\varepsilon^c) \geq \rho_\varepsilon^{S+1}\right\}} \Subset \mathbb{R}^n,$$

where $S \in \mathbb{N}$ comes from (ii). □

Let $X = \langle A_\varepsilon \rangle$ be a strongly internal set, $x, y \in X$ and both $K, K^c \subseteq_0 I$. Set $e_K := [1_{K\varepsilon}] \in {}^{\rho}\widetilde{\mathbb{R}}$, where $1_{K\varepsilon} := 1$ if $\varepsilon \in K$ and $1_{K\varepsilon} := 0$ otherwise. Then $z := x \cdot e_K + y \cdot e_{K^c} \in X$ and $z|_K \subseteq x$, $z|_{K^c} \subseteq y$ (we then say that X is closed with respect to *interleaving*; this property holds also for internal sets, see [87]). The same property does not hold if $x \in B_r(c) \setminus B_s(d)$ and $y \in B_s(d) \setminus B_r(c)$, so that $B_r(c) \cup B_s(d)$ is sharply open but is not strongly internal. The same kind of example can be repeated e.g. considering arbitrary unions of pairwise disjoint balls.

To obtain large open sets starting from a net of subsets $A_\varepsilon \subseteq \mathbb{R}^n$, we can consider the analogue of $\langle A_\varepsilon \rangle$ but using the radii of the Fermat topology:

Definition 12. Let (A_ε) be a net of subsets of $\mathbb{R}^n$ and let (x_ε), (x'_ε) be nets of points of $\mathbb{R}^n$. Then

(i) We write $(x_\varepsilon) \sim_{\mathrm{F}} (x'_\varepsilon)$ to denote the property $|x - x'| < r$ for all $r \in \mathbb{R}_{>0}$, i.e., $\lim_{\varepsilon \to 0^+} |x_\varepsilon - x'_\varepsilon| = 0$.

(ii) We say that $x_\varepsilon \in_{\mathrm{F}} A_\varepsilon$, and we read it as (x_ε) *strongly belongs to* (A_ε) *in the Fermat topology*, if
 (a) $\forall^0 \varepsilon : \ x_\varepsilon \in A_\varepsilon$.
 (b) If $(x'_\varepsilon) \sim_{\mathrm{F}} (x_\varepsilon)$, then also $x'_\varepsilon \in A_\varepsilon$ for ε small.
 Moreover, we set $\langle A_\varepsilon \rangle_{\mathrm{F}} := \left\{[x_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}^n \mid x_\varepsilon \in_{\mathrm{F}} A_\varepsilon\right\}$, and we call it the *strongly internal set* generated by the net (A_ε) *in the Fermat topology*.

The following result can be proved simply by generalizing the proof of Thm. 10.

Theorem 13. *For $\varepsilon \in I$, let $A_\varepsilon \subseteq \mathbb{R}^n$ and let $x_\varepsilon \in \mathbb{R}^n$. Then we have*

(i) $[x_\varepsilon] \in \langle A_\varepsilon \rangle_{\mathrm{F}}$ *if and only if* $\exists r \in \mathbb{R}_{>0}\, \forall^0 \varepsilon : \ d(x_\varepsilon, A_\varepsilon^c) > r$.

(ii) $\langle A_\varepsilon \rangle_{\mathrm{F}}$ *is a Fermat open set.*

Sharply bounded internal sets (which are always sharply closed by Thm. 10 (iii)) serve as compact sets for our generalized functions. We will show this by proving for them an extreme value theorem (see Thm. 51); for a deeper study of this type of sets in the case $\rho = (\varepsilon)$ see [41]; in the same particular setting, the notion of sharp topology was introduced in [10, 97]; see also [80, 50] for an analogue of Lem. 8; see [87] for the study of internal sets, and see [43] for strongly internal sets.

3. Generalized functions as smooth set-theoretical maps

3.0.1. *Definition and sharp continuity.* Using the ring ${}^{\rho}\widetilde{\mathbb{R}}$, it is easy to consider a Gaussian with an infinitesimal standard deviation. If we denote this probability density by $f(x, \sigma)$, and if we set $\sigma = [\sigma_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$, where $\sigma \approx 0$, we obtain the net of smooth functions $(f(-, \sigma_\varepsilon))_{\varepsilon \in I}$. This is the basic idea we are going to develop in the following definitions. We will first introduce the notion of a net (f_ε) defining a generalized smooth function of the type $X \longrightarrow Y$, where $X \subseteq {}^{\rho}\widetilde{\mathbb{R}}^n$ and $Y \subseteq {}^{\rho}\widetilde{\mathbb{R}}^d$. This is a net of smooth functions $f_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R}^d)$ that induces well-defined maps of the form $[\partial^\alpha f_\varepsilon(-)] : \langle \Omega_\varepsilon \rangle \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}^d$, for every multi-index $\alpha \in \mathbb{N}^n$.

Definition 14. Let $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ and $Y \subseteq {}^\rho\widetilde{\mathbb{R}}^d$ be arbitrary subsets of generalized points. Let (Ω_ε) be a net of open subsets of $\mathbb{R}^n$, and (f_ε) be a net of smooth functions, with $f_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R}^d)$. Then we say that

$$(f_\varepsilon) \textit{ defines a generalized smooth function} : X \longrightarrow Y$$

if:

(i) $X \subseteq \langle \Omega_\varepsilon \rangle$ and $[f_\varepsilon(x_\varepsilon)] \in Y$ for all $[x_\varepsilon] \in X$.
(ii) $\forall [x_\varepsilon] \in X \, \forall \alpha \in \mathbb{N}^n : \ (\partial^\alpha f_\varepsilon(x_\varepsilon)) \in \mathbb{R}^d_\rho$.

We recall that the notation

$$\forall [x_\varepsilon] \in X : \ \mathcal{P}\{(x_\varepsilon)\}$$

means

$$\forall (x_\varepsilon) \in \mathbb{R}^n_\rho : \ [x_\varepsilon] \in X \ \Rightarrow \ \mathcal{P}\{(x_\varepsilon)\},$$

i.e. for all representatives (x_ε) generating a point $[x_\varepsilon] \in X$, the property $\mathcal{P}\{(x_\varepsilon)\}$ holds.

A generalized smooth function (or map, in this paper these terms are used as synonymous) is simply a function of the form $f = [f_\varepsilon(-)]|_X$:

Definition 15. Let $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ and $Y \subseteq {}^\rho\widetilde{\mathbb{R}}^d$ be arbitrary subsets of generalized points. Then we say that

$$f : X \longrightarrow Y \text{ is a } \textit{generalized smooth function}$$

if $f \in \mathbf{Set}(X, Y)$ and there exists a net $f_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R}^d)$ defining a generalized smooth map of type $X \longrightarrow Y$, in the sense of Def. 14, such that

$$\forall [x_\varepsilon] \in X : \ f([x_\varepsilon]) = [f_\varepsilon(x_\varepsilon)] \,. \tag{3.1}$$

We will also say that f *is defined by the net of smooth functions* (f_ε) or that the net (f_ε) *represents* f. The set of all these generalized smooth functions (GSF) will be denoted by ${}^\rho\mathcal{GC}^\infty(X, Y)$ or simply by $\mathcal{GC}^\infty(X, Y)$.

Let us note explicitly that definitions 14 and 15 state minimal logical conditions to obtain a set-theoretical map from X into Y and defined by a net of smooth functions. In particular, the following Thm. 16 states that in equality (3.1) we have independence from the representatives for all derivatives $[x_\varepsilon] \in X \mapsto [\partial^\alpha f_\varepsilon(x_\varepsilon)] \in {}^\rho\widetilde{\mathbb{R}}^d$, $\alpha \in \mathbb{N}^n$.

Theorem 16. *Let $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ and $Y \subseteq {}^\rho\widetilde{\mathbb{R}}^d$ be arbitrary subsets of generalized points. Let (Ω_ε) be a net of open subsets of $\mathbb{R}^n$, and (f_ε) be a net of smooth functions, with $f_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R}^d)$. Assume that (f_ε) defines a generalized smooth map of the type $X \longrightarrow Y$, then*

$$\forall \alpha \in \mathbb{N}^n \, \forall (x_\varepsilon), (x'_\varepsilon) \in \mathbb{R}^n_\rho : \ [x_\varepsilon] = [x'_\varepsilon] \in X \ \Rightarrow \ (\partial^\alpha f_\varepsilon(x_\varepsilon)) \sim_\rho (\partial^\alpha f_\varepsilon(x'_\varepsilon))$$

Proof. Let $\alpha \in \mathbb{N}^n$ and (x_ε), (x'_ε) be two representatives of the same point $x = [x_\varepsilon] = [x'_\varepsilon] \in X \subseteq \langle \Omega_\varepsilon \rangle$. Thm. 10 (ii) yields

$$d(x_\varepsilon, \Omega^c_\varepsilon) > \rho^q_\varepsilon \tag{3.2}$$

for some $q \in \mathbb{R}_{>0}$ and ε small. Thus $B^{\mathrm{E}}_{\rho^q_\varepsilon}(x_\varepsilon) \subseteq \Omega_\varepsilon$ for these values of ε. Choose $r \in \mathbb{R}_{>0}$ sufficiently big so that

$$2\rho^r_\varepsilon < \rho^q_\varepsilon \tag{3.3}$$

for ε small. Since $(x_\varepsilon) \sim_\rho (x'_\varepsilon)$ we have that

$$x'_\varepsilon \in B^{\mathrm{E}}_{\rho^r_\varepsilon}(x_\varepsilon) \tag{3.4}$$

for ε small, and the entire segment $[x_\varepsilon, x'_\varepsilon]$ connecting x_ε and x'_ε lies in $B^{\mathrm{E}}_{\rho^r_\varepsilon}(x_\varepsilon)$. Suppose that (3.2), (3.3) and (3.4) hold for $\varepsilon \in (0, \varepsilon_0]$. Fix $i \in \{1, \dots, d\}$ and set $\mu_\varepsilon(t) := \partial^\alpha f^i_\varepsilon(x_\varepsilon + t(x'_\varepsilon - x_\varepsilon))$ for $t \in [0,1]_{\mathbb{R}}$ and $\varepsilon \in (0, \varepsilon_0]$. By the classical mean value theorem $\partial^\alpha f^i_\varepsilon(x'_\varepsilon) - \partial^\alpha f^i_\varepsilon(x_\varepsilon) = \mu_\varepsilon(1) - \mu_\varepsilon(0) = \mu'_\varepsilon(\theta_\varepsilon)$ for some $\theta_\varepsilon \in (0,1)$, and hence for all $\varepsilon \in (0, \varepsilon_0]$ we get

$$\partial^\alpha f^i_\varepsilon(x'_\varepsilon) - \partial^\alpha f^i_\varepsilon(x_\varepsilon) = \sum_{k=1}^n \partial^{\alpha+e_k} f^i_\varepsilon(\xi_\varepsilon) \cdot (x'^k_\varepsilon - x^k_\varepsilon), \tag{3.5}$$

where $\xi_\varepsilon := x_\varepsilon + \theta_\varepsilon(x'_\varepsilon - x_\varepsilon)$ and $e_k := (0, \overset{k-1}{\ldots}, 0, 1, 0, \dots, 0) \in \mathbb{N}^n$. The generalized point $[\xi_\varepsilon] = [x_\varepsilon] \in X$ since $(x'_\varepsilon) \sim_\rho (x_\varepsilon)$. Therefore by Def. 14 (ii) we get that every derivative $\left(\partial^{\alpha+e_k} f^i_\varepsilon(\xi_\varepsilon)\right)$ is ρ-moderate. From this and the equivalence $(x'_\varepsilon) \sim_\rho (x_\varepsilon)$, equation (3.5) yields the conclusion $(\partial^\alpha f_\varepsilon(x'_\varepsilon)) \sim_\rho (\partial^\alpha f_\varepsilon(x_\varepsilon))$. $\square$

Note that taking arbitrary subsets $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ in Def. 14, we can also consider GSF defined on closed sets, like the set of all infinitesimals, or like a closed interval $[a, b] \subseteq {}^\rho\widetilde{\mathbb{R}}$. We can also consider GSF defined at infinite generalized points. A simple case is the exponential map

$$e^{(-)} : [x_\varepsilon] \in \left\{ x \in {}^\rho\widetilde{\mathbb{R}} \mid \exists z \in {}^\rho\widetilde{\mathbb{R}}_{>0} : \ x \le \log z \right\} \mapsto [e^{x_\varepsilon}] \in {}^\rho\widetilde{\mathbb{R}}. \tag{3.6}$$

The domain of this map depends on the infinitesimal net ρ. For instance, if $\rho = (\varepsilon)$ then all its points are bounded by generalized numbers of the form $[-N \log \varepsilon]$, $N \in \mathbb{N}$; whereas if $\rho = \left(e^{-\frac{1}{\varepsilon}}\right)$, all points are bounded by $[N\varepsilon^{-1}]$, $N \in \mathbb{N}$. Another possibility for the exponential function is to consider two gauges $\rho \ge \sigma$ and the subring of ${}^\sigma\widetilde{\mathbb{R}}$ defined by

$${}^\sigma_\rho\widetilde{\mathbb{R}} := \{x \in {}^\sigma\widetilde{\mathbb{R}} \mid \exists N \in \mathbb{N} : \ |x| \le \mathrm{d}\rho^{-N}\},$$

where here we have set $\mathrm{d}\rho := [\rho_\varepsilon]_{\sim_\sigma} \in {}^\sigma\widetilde{\mathbb{R}}$. If we have

$$\forall N \in \mathbb{N}\, \exists M \in \mathbb{N} : \ \mathrm{d}\rho^{-N} \le -M \log \mathrm{d}\sigma, \tag{3.7}$$

then $e^{(-)} : [x_\varepsilon] \in {}^\sigma_\rho\widetilde{\mathbb{R}} \mapsto [e^{x_\varepsilon}] \in {}^\sigma\widetilde{\mathbb{R}}$ is well defined. For example, if $\sigma_\varepsilon := \exp\left(-\rho_\varepsilon^{1/\varepsilon}\right)$, then $\sigma \le \rho$ and (3.7) holds for $M = 1$. Note that the natural ring morphism $[x_\varepsilon]_{\sim_\sigma} \in {}^\sigma_\rho\widetilde{\mathbb{R}} \mapsto [x_\varepsilon]_{\sim_\rho} \in {}^\rho\widetilde{\mathbb{R}}$ is surjective but generally not injective.

A first regularity property of GSF is the continuity with respect to the sharp topology, as proved in the following

Theorem 17. *Let $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$, $Y \subseteq {}^\rho\widetilde{\mathbb{R}}^d$ and $f_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R}^d)$ be a net of smooth functions that defines a GSF of the type $X \longrightarrow Y$. Then*

(i) $\forall [x_\varepsilon] \in X\, \forall \alpha \in \mathbb{N}^n\, \exists q \in \mathbb{R}_{>0}\, \forall^0 \varepsilon : \ \sup_{y \in B^{\mathrm{E}}_{\rho^q_\varepsilon}(x_\varepsilon)} |\partial^\alpha f_\varepsilon(y)| \le \rho_\varepsilon^{-q}$.

(ii) *For all $\alpha \in \mathbb{N}^n$, the GSF $g : [x_\varepsilon] \in X \mapsto [\partial^\alpha f_\varepsilon(x_\varepsilon)] \in \widetilde{\mathbb{R}}^d$ is locally Lipschitz in the sharp topology, i.e. each $x \in X$ possesses a sharp neighborhood U such that $|g(x) - g(y)| \le L|x - y|$ for all x, $y \in U$ and some $L \in {}^\rho\widetilde{\mathbb{R}}$.*

(iii) *Each $f \in {}^\rho\mathcal{GC}^\infty(X, Y)$ is continuous with respect to the sharp topologies induced on X, Y.*

(iv) *Assume that the GSF f is locally Lipschitz in the Fermat topology and that its Lipschitz constants are always finite: $L\in\mathbb{R}$. Then f is continuous in the Fermat topology.*

Proof. We first prove (i) by contradiction, assuming that for some $[x_\varepsilon]\in X$ and some α there exists $(\varepsilon_k)_k\downarrow 0$ and a sequence $(y_k)_k$ of points in $\mathbb{R}^n$ such that $|x_{\varepsilon_k}-y_k|<\rho_{\varepsilon_k}^k$ but $|\partial^\alpha f_\varepsilon(y_k)|>\rho_{\varepsilon_k}^{-k}$. Define $x'_\varepsilon:=y_k$ for $\varepsilon=\varepsilon_k$ and $x'_\varepsilon:=x_\varepsilon$ otherwise. Then $(x'_\varepsilon)\sim_\rho(x_\varepsilon)$ but $(\partial^\alpha f_\varepsilon(x'_\varepsilon))$ is not ρ-moderate, which contradicts Def. 14 (ii).

To prove (ii), we apply (i) to each derivative $\partial^{\alpha+e_k}f_\varepsilon$ to obtain

$$\forall k=1,\dots,n\,\exists q_k\in\mathbb{R}_{>0}\exists\varepsilon_k\in I\,\forall\varepsilon\in(0,\varepsilon_k]:\ \sup_{y\in B^{\mathrm{E}}_{\rho_\varepsilon^{q_k}}(x_\varepsilon)}|\partial^{\alpha+e_k}f_\varepsilon(y)|\le\rho_\varepsilon^{-q_k}. \tag{3.8}$$

Set $q:=\max_{k=1,\dots,n}q_k$, so that for $y,\,z\in B_{\mathrm{d}\rho^q}(x)$ we get

$$\exists\varepsilon_0\,\forall\varepsilon\in(0,\varepsilon_0]:\ [y_\varepsilon,z_\varepsilon]\subseteq B^{\mathrm{E}}_{\rho_\varepsilon^{q}}(x_\varepsilon). \tag{3.9}$$

For any $i\in\{1,\dots,d\}$ and ε small we can write

$$|\partial^\alpha f^i_\varepsilon(y_\varepsilon)-\partial^\alpha f^i_\varepsilon(z_\varepsilon)|=\left|\sum_{k=1}^n\partial^{\alpha+e_k}f^i_\varepsilon(\zeta_\varepsilon)\cdot(y^k_\varepsilon-z^k_\varepsilon)\right|$$

where $\zeta_\varepsilon:=y_\varepsilon+\sigma_\varepsilon(z_\varepsilon-y_\varepsilon)$ for some $\sigma_\varepsilon\in(0,1)$. Therefore $\zeta_\varepsilon\in B^{\mathrm{E}}_{\rho_\varepsilon^{q}}(x_\varepsilon)\subseteq B^{\mathrm{E}}_{\rho_\varepsilon^{q_k}}(x_\varepsilon)$ and (3.8) implies

$$|\partial^\alpha f_\varepsilon(y_\varepsilon)-\partial^\alpha f_\varepsilon(z_\varepsilon)|\le d\sqrt{n}\rho_\varepsilon^{-q}|y_\varepsilon-z_\varepsilon|.$$

Property (iii) follows upon setting $\alpha=0$ in (ii). Property (iv) follows directly from the definition of locally Lipschitz function in the Fermat topology. In fact, we have that $L|x-y|<r\in\mathbb{R}$ if $y\in B^{\mathrm{F}}_{r/L}(x)$, which is an open ball in the Fermat topology because $L\in\mathbb{R}$. □

In the following result, we show that the dependence of the domains Ω_ε on ε can be avoided since it does not lead to a larger class of generalized functions. In spite of this possibility, we preferred to formulate Def. 14 using ε-dependent domains because the extension of $f_\varepsilon\in\mathcal{C}^\infty(\Omega_\varepsilon,\mathbb{R}^d)$ to the whole of $\mathbb{R}^n$ is not unique and hence introduces extrinsic elements.

Lemma 18. *Let $X\subseteq{}^\rho\widetilde{\mathbb{R}}^n$ and $Y\subseteq{}^\rho\widetilde{\mathbb{R}}^d$ be arbitrary subsets of generalized points. Then $f:X\longrightarrow Y$ is a GSF if and only if there exists a net $v_\varepsilon\in\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R}^d)$ defining a generalized smooth map of type $X\longrightarrow Y$ such that $f=[v_\varepsilon(-)]|_X$.*

Proof. The stated condition is clearly sufficient. Conversely, assume that $f: X\longrightarrow Y$ is defined by the net $f_\varepsilon\in\mathcal{C}^\infty(\Omega_\varepsilon,\mathbb{R}^d)$. For every $\varepsilon\in I$ let $\Omega'_\varepsilon:=\left\{x\in\Omega_\varepsilon\mid d(x,\Omega^{\mathrm{c}}_\varepsilon)>\rho_\varepsilon^{\frac{1}{\varepsilon}}\right\}$, $\Omega''_\varepsilon:=\left\{x\in\Omega_\varepsilon\mid d(x,\Omega^{\mathrm{c}}_\varepsilon)>\rho_{\varepsilon/2}^{\frac{2}{\varepsilon}}\right\}$ and choose $\chi_\varepsilon\in\mathcal{C}^\infty(\Omega_\varepsilon)$ with $\mathrm{supp}(\chi_\varepsilon)\subseteq\Omega''_\varepsilon$ and $\chi_\varepsilon=1$ in a neighborhood of Ω'_ε. Set $f_\varepsilon:=0$ on $\mathbb{R}^n\setminus\Omega_\varepsilon$ and $v_\varepsilon:=\chi_\varepsilon\cdot f_\varepsilon$, so that $v_\varepsilon\in\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R}^d)$. If $x=[x_\varepsilon]\in X\subseteq\langle\Omega_\varepsilon\rangle$, then $x_\varepsilon\in\Omega'_\varepsilon\subseteq\Omega_\varepsilon$ for ε small by Thm. 10, so for all $\alpha\in\mathbb{N}^n$ we get $\partial^\alpha v_\varepsilon(x_\varepsilon)=\partial^\alpha f_\varepsilon(x_\varepsilon)$. Therefore, $(v_\varepsilon)_\varepsilon$ defines a GSF of the type $X\longrightarrow Y$ and clearly $f=[f_\varepsilon(-)]|_X=[v_\varepsilon(-)]|_X$. □

Consider a GSF $f:X\longrightarrow Y$. We want to show that for a large class of domains X, the function f is uniquely determined by its values on particularly well behaved points $x\in X$. These domains and these points are introduced in the following

Definition 19.

(i) Let $x \in {}^\rho\widetilde{\mathbb{R}}^n$. Then we say that the point x is *near-standard* if there exists a representative (x_ε) of x such that $\exists \lim_{\varepsilon\to 0^+} x_\varepsilon =: x^\circ \in \mathbb{R}^n$ (x° is called the standard part of x). Clearly, this limit does not depend on the representative of x.

(ii) If $\Omega \subseteq \mathbb{R}^n$, then $\Omega^\bullet := \left\{ x \in {}^\rho\widetilde{\mathbb{R}}^n \mid \exists x^\circ \in \Omega \right\}$.

(iii) We say that $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ *contains its converging subpoints* if for all $J \subseteq_0 I$ and all $x' \in X|_J$ which is near standard or infinite, there exists some $x \in X$ with $x' \subseteq x$ and such that $\lim_{\varepsilon\to 0, \varepsilon\in J} x'_\varepsilon = \lim_{\varepsilon\to 0} x_\varepsilon$.

Theorem 20. *Let $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$, $Y \subseteq {}^\rho\widetilde{\mathbb{R}}^d$, and let $f : X \longrightarrow Y$ be a GSF. If X contains its converging subpoints and if $f(x) = 0$ for all near-standard and for all infinite points $x \in X$. Then $f = 0$.*

Proof. In fact, suppose that f vanishes on every near-standard and every infinite point belonging to X, but that $f(x) \neq 0$ for some $x \in X$. Let (x_ε) be a representative of x. Then there exist $m \in \mathbb{N}$ and $(\varepsilon_k)_k \downarrow 0$ such that $|f_{\varepsilon_k}(x_{\varepsilon_k})| > \rho_{\varepsilon_k}^m$, where (f_ε) is a net that defines f. If $(x_{\varepsilon_k})_k$ is a bounded sequence, we can extract from it a convergent subsequence $(x_{\varepsilon_{k_l}})_l$. Setting $J := \{k_l \mid l \in \mathbb{N}\}$, $x' = x|_J$ is a subpoint of x and by assumption there exists some $y \in X$ that satisfies $y|_J = x'$ and additionally is near-standard, with the same limit as x'. By construction, $f(y) \neq 0$, a contradiction. If, on the other hand, the sequence $(x_{\varepsilon_k})_k$ is unbounded, then we can extract a subsequence with $\lim_{l\to+\infty} |x_{\varepsilon_{k_l}}| = +\infty$, and can then proceed as above to construct an infinite point $y \in X$ at which $f(y) \neq 0$. □

Analogously, we can prove the following:

Theorem 21. *Let $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ and $Y \subseteq {}^\rho\widetilde{\mathbb{R}}^d$. Let (Ω_ε) be a net of open subsets of $\mathbb{R}^n$, and (f_ε) be a net of smooth functions, with $f_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R}^d)$. Assume that X contains its converging subpoints. Then (f_ε) defines a GSF of the type $X \longrightarrow Y$ if and only if*

(i) $X \subseteq \langle\Omega_\varepsilon\rangle$ *and* $[f_\varepsilon(x_\varepsilon)] \in Y$ *for all* $[x_\varepsilon] \in X$.

(ii) $\forall \alpha \in \mathbb{N}^n :\ (\partial^\alpha f_\varepsilon(x_\varepsilon)) \in \mathbb{R}_\rho^d$ *for all near-standard and for all infinite points* $[x_\varepsilon] \in X$.

For example, if Ω is an open subset of $\mathbb{R}^n$, and we define the set of compactly supported generalized points by

$$\mathrm{c}(\Omega) := \{[x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}^n \mid \exists K \Subset \Omega\ \forall^0\varepsilon :\ x_\varepsilon \in K\} \subseteq \langle\Omega\rangle,$$

then $\mathrm{c}(\Omega)$ contains its converging subpoints. Internal and strongly internal sets generated by a constant sequence $A \subseteq \mathbb{R}^n$, i.e. $[A]$ and $\langle A\rangle$, provide further examples of a subset containing its converging subpoints. Moreover, an arbitrary union $\bigcup_{j\in J} X_j$ of sets, with each X_j containing its converging subpoints, still contains its converging subpoints.

The subset $\mathrm{c}(\Omega)$ is the natural domain for embedded distributions, as shown in the following section.

4. Embedding of Schwartz distributions

Introduction. Among the re-occurring themes of this work are the choices which the solution of a given problem within our framework may depend upon. For instance, (3.6) shows that the domain of a GSF depends on the infinitesimal net ρ. It is also easy to show that the trivial Cauchy problem

$$\begin{cases} x'(t) - [\varepsilon^{-1}] \cdot x(t) = 0 \\ x(0) = 1 \end{cases}$$

has no solution (in a finite interval) if $\rho = (\varepsilon)$, but it has the unique solution $x(t) = \left[e^{\frac{1}{\varepsilon}t}\right] \in {}^{\bar{\rho}}\mathcal{GC}^\infty(\mathbb{R}, \mathbb{R})$ for all $t \in \mathbb{R}$ if we consider another gauge $\bar{\rho} := (e^{-1/\varepsilon})$. Therefore, the choice of the infinitesimal net ρ is closely tied to the possibility of solving a given class of differential equations. This illustrates the dependence of the theory on the infinitesimal net ρ.

Further choices concern the embedding of Schwartz distributions. Since we need to associate a net of smooth functions (f_ε) to a given distribution $T \in \mathcal{D}'(\Omega)$, this embedding is naturally built upon a regularization process. In our approach, this regularization will depend on an infinite number $b \in {}^\rho\widetilde{\mathbb{R}}$, and the choice of b depends on what properties we need from the embedding. For example, if δ is the (embedding of the) one-dimensional Dirac delta, then we have the property

$$\delta(0) = b, \tag{4.1}$$

We can also choose the embedding so as to get the property

$$H(0) = \frac{1}{2}, \tag{4.2}$$

where H is the (embedding of the) Heaviside step function. Equalities like these are used in diverse applications (see, e.g., [17, 84] and references therein). In fact, we are going to construct a family of structures of the type $(\mathcal{G}, \partial, \iota)$, where $(\mathcal{G}, \partial)$ is a a sheaf of differential algebra and $\iota : \mathcal{D}' \longrightarrow \mathcal{G}$ is an embedding. The particular structure we need to consider depends on the problem we have to solve. Of course, one may be more interested in having an intrinsic embedding of distributions. This can be done by following the ideas of the full Colombeau algebra (see e.g. [50, 46, 45, 51]). Nevertheless, this choice decreases the simplicity of the present approach and is incompatible with properties like (4.1) and (4.2).

The embedding. If $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $r \in \mathbb{R}_{>0}$ and $x \in \mathbb{R}^n$, we use the notations $r \odot \varphi$ for the function $x \in \mathbb{R}^n \mapsto \frac{1}{r^n} \cdot \varphi\left(\frac{x}{r}\right) \in \mathbb{R}$ and $x \oplus \varphi$ for the function $y \in \mathbb{R}^n \mapsto \varphi(y - x) \in \mathbb{R}$. These notations highlight that $\odot$ is a free action of the multiplicative group $(\mathbb{R}_{>0}, \cdot, 1)$ on $\mathcal{D}(\mathbb{R}^n)$ and $\oplus$ is a free action of the additive group $(\mathbb{R}_{>0}, +, 0)$ on $\mathcal{D}(\mathbb{R}^n)$. We also have the distributive property $r \odot (x \oplus \varphi) = rx \oplus r \odot \varphi$. Our embedding procedure will ultimately rely on convolution with suitable mollifiers. To construct these, we need some technical preparations.

Lemma 22. *For any $n \in \mathbb{N}_{>0}$ there exists some $\mu_n \in \mathcal{S}(\mathbb{R})$ with the following properties:*

(i) $\int \mu_n(x)\,\mathrm{d}x = 1$.
(ii) $\int_0^\infty x^{\frac{j}{n}} \mu_n(x)\,\mathrm{d}x = 0$ *for all* $j \in \mathbb{N}_{>0}$.
(iii) $\mu_n(0) = 1$.
(iv) μ_n *is even.*
(v) $\mu_n(k) = 0$ *for all* $k \in \mathbb{Z} \setminus \{0\}$.

Proof. Consider the Fréchet space

$$F := \{\mu \in \mathcal{S}(\mathbb{R}) \mid \mu \text{ even}, \ \forall k \in \mathbb{Z} \setminus \{0\} : \mu(k) = 0\}$$

and define the continuous linear functionals $f_m : F \to \mathbb{R}$, where $f_0(\mu) := \mu(0)$, $f_1(\mu) := \int \mu(x)\, dx$, and $f_m(\mu) := \int_0^\infty x^{\frac{m-1}{n}} \mu(x)\, dx$ $(m \geq 2)$. Our objective then is to implement conditions (i)–(iii) by showing the solvability of the system

$$f_0(\mu) = 1,\ f_1(\mu) = 1,\ f_m(\mu) = 0\ (m \geq 2) \tag{4.3}$$

in F. To this end, we employ a classical result of M. Eidelheit ([22, Satz 2]). First, the family $(f_m)_{m\in\mathbb{N}}$ is linearly independent. Next, the topology of $F \subseteq \mathcal{S}(\mathbb{R})$ is generated by the family of norms $p_k(\mu) = \sup_{l+m\leq k} \sup_{x\in\mathbb{R}}(1+|x|)^l|\mu^{(m)}(x)|$, $k \in \mathbb{N}$. Suppose now that $\lambda_1, \ldots, \lambda_i$ are nonzero numbers and that the *order* of the linear functional $\sum_{m=0}^{i} \lambda_m f_m$ is less or equal l. Here, the order of an element f of $\mathcal{S}'(\mathbb{R})$ is defined to be the smallest k such that $|f(\mu)| \leq C p_k(\mu)$ for some $C > 0$ and all $\mu \in \mathcal{S}(\mathbb{R})$. Let $i_l := nl + 1$, then certainly $i \leq i_l$. Hence both conditions of [22, Satz 2] are satisfied and we may conclude that (4.3) has a solution μ_n in F. □

Remark 23. In addition to conditions (i)-(v) from Lemma 22 we may require that μ_n satisfy finitely many additional properties expressible by linearly independent functionals as in the above proof (again by [22, Satz 2]). In particular, we may prescribe the values for μ_n or its derivatives at finitely many further points.

Finally, we note that any element of $\mathcal{S}(\mathbb{R})$ satisfying condition (ii) from Lemma 22 must change sign infinitely often.

We call *Colombeau mollifier* (for a fixed dimension n) any function μ that satisfies the properties of the previous lemma. Concerning embeddings of Schwartz distributions, the idea is classically to regularize distributions using a mollifier. The use of a Colombeau mollifier allows us, on the one hand, to identify the distribution $\varphi \in \mathcal{D}(\Omega) \mapsto \int f\varphi$ with the GSF $f \in \mathcal{C}^\infty(\Omega) \subseteq {}^\rho\mathcal{GC}^\infty(\Omega, \mathbb{R})$ (thanks to property (ii)); on the other hand, it allows us to explicitly calculate compositions such as $\delta \circ \delta$, $H \circ \delta$, $\delta \circ H$ (see below).
It is worth noting that the condition (ii) of null moments is well known in the study of convergence of numerical solutions of singular differential equations, see e.g. [57, 24, 115] and references therein.

Next we show that the assignment $U \mapsto {}^\rho\mathcal{GC}^\infty(\mathrm{c}(U), {}^\rho\widetilde{\mathbb{R}})$ ($U \subseteq \Omega$ open) is a fine sheaf on Ω. In fact, for $V \subseteq U$, the natural restriction map ${}^\rho\mathcal{GC}^\infty(\mathrm{c}(U), {}^\rho\widetilde{\mathbb{R}}) \to \mathcal{GC}^\infty(\mathrm{c}(V), {}^\rho\widetilde{\mathbb{R}})$ can also be written, in terms of defining nets, as $f = [f_\varepsilon] \mapsto [f_\varepsilon|_V]$. Also, $\mathrm{c}(U) \cap \mathrm{c}(V) = \mathrm{c}(U \cap V)$.

Suppose that Ω_j $(j \in J)$ is an open covering of Ω and that for each $j \in J$ we are given $f^j = [f^j_\varepsilon] \in {}^\rho\mathcal{GC}^\infty(\mathrm{c}(\Omega_j), {}^\rho\widetilde{\mathbb{R}})$ such that $f^j|_{\mathrm{c}(\Omega_j\cap\Omega_k)} = f^k|_{\mathrm{c}(\Omega_j\cap\Omega_k)}$ for all j, $k \in J$. Then letting χ_j $(j \in J)$ be a partition of unity subordinate to Ω_j $(j \in J)$, the GSF defined by the net

$$f_\varepsilon := \sum_{j\in J} \chi_j \cdot f^j_\varepsilon \in \mathcal{C}^\infty(\Omega)$$

is the unique element of ${}^\rho\mathcal{GC}^\infty(\mathrm{c}(\Omega), {}^\rho\widetilde{\mathbb{R}})$ with $f|_{\mathrm{c}(\Omega_j)} = f^j$ for all $j \in J$. In particular, we may define a corresponding notion of standard support for each $f \in {}^\rho\mathcal{GC}^\infty(\mathrm{c}(\Omega), {}^\rho\widetilde{\mathbb{R}})$ by

$$\mathrm{stsupp}(f) := \left(\bigcup\{\Omega' \subseteq \Omega \mid \Omega' \text{ open},\ f|_{\Omega'} = 0\}\right)^{\mathrm{c}}.$$

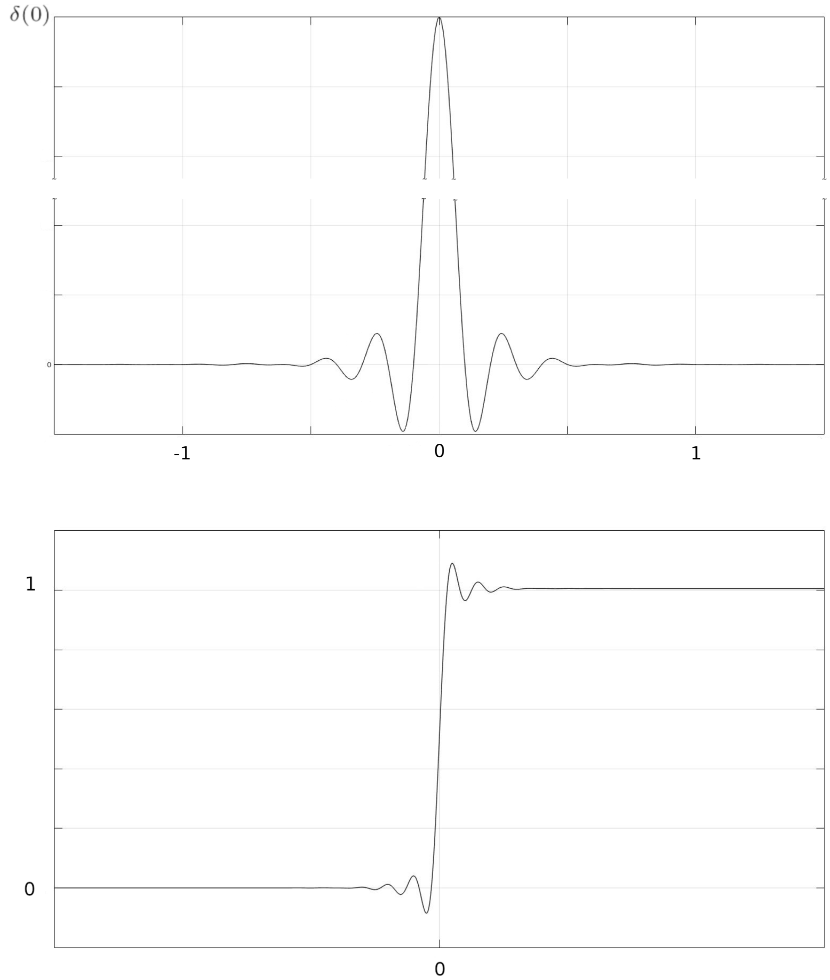

FIGURE 4.1. A representation of Dirac delta and Heaviside function. A Colombeau mollifier has a representation similar to Dirac delta (but with finite values).

The adjective *standard* underscores that this set is made only of standard points; a better notion of support for GSF is defined as $\operatorname{supp}(f) = \overline{\{x \in X \mid |f(x)| > 0\}}$ and studied in [41].

As a final preparation for the embedding of $\mathcal{D}'(\Omega)$ into ${}^{\rho}\mathcal{GC}^{\infty}(\mathrm{c}(\Omega), {}^{\rho}\widetilde{\mathbb{R}})$ we need to construct suitable n-dimensional mollifiers from a Colombeau mollifier μ as given by Lemma 22. To this end, let ω_n denote the surface area of S^{n-1} and set

$$c_n := \begin{cases} \frac{2n}{\omega_n} & \text{for } n > 1 \\ 1 & \text{for } n = 1. \end{cases}$$

Then let $\tilde{\mu} : \mathbb{R}^n \to \mathbb{R}$, $\tilde{\mu}(x) := c_n \mu(|x|^n)$. Since μ is even, $\tilde{\mu}$ is smooth. Moreover, by Lemma 22, it has unit integral and all its higher moments $\int x^\alpha \tilde{\mu}(x)\,dx$ vanish ($|\alpha| \ge 1$). With this notation we have:

Lemma 24. *Let $\chi \in \mathcal{D}(\mathbb{R}^n)$, $\chi = 1$ on $\overline{B_1^{\mathrm{E}}(0)}$, and $\chi = 0$ on $\mathbb{R}^n \setminus B_2^{\mathrm{E}}(0)$. Also, let $b = [b_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}$ be an infinite positive number, i.e. $\lim_{\varepsilon\to 0^+} b_\varepsilon = +\infty$. Now set*

$$\mu_\varepsilon^b(x) := (b_\varepsilon^{-1} \odot \tilde{\mu})(x)\chi(x|\log(b_\varepsilon)|) = b_\varepsilon^n \tilde{\mu}(b_\varepsilon x)\chi(x|\log(b_\varepsilon)|). \tag{4.4}$$

(i) $\forall\varepsilon :\ \mu_\varepsilon^b \in \mathcal{C}^\infty(\mathbb{R}^n)$, *stsupp*$(\mu_\varepsilon^b) \subseteq B^{\mathrm{E}}_{2|\log(b_\varepsilon)|^{-1}}(0)$.

(ii) $\forall\alpha \in \mathbb{N}^n\, \exists N \in \mathbb{N} :\ \sup_{x\in\mathbb{R}^n} |\partial^\alpha \mu_\varepsilon^b(x)| = O(b_\varepsilon^N)$ $(\varepsilon \to 0)$.

(iii) $\forall\alpha \in \mathbb{N}^n\, \forall q \in \mathbb{N} :\ \sup_{x\in\mathbb{R}^n} |\partial^\alpha(\mu_\varepsilon^b - b_\varepsilon^n \tilde{\mu}(b_\varepsilon\, .))(x)| = O(b_\varepsilon^{-q})$ $(\varepsilon \to 0)$.

(iv) $\forall q \in \mathbb{N} :\ \int \mu_\varepsilon^b(x)\,\mathrm{d}x = 1 + O(b_\varepsilon^{-q})$ $(\varepsilon \to 0)$.

(v) $\forall q \in \mathbb{N}\, \forall\alpha \in \mathbb{N}^n :\ |\alpha| > 0 \ \Rightarrow\ \int x^\alpha \mu_\varepsilon^b(x)\,\mathrm{d}x = O(b_\varepsilon^{-q})$ $(\varepsilon \to 0)$.

Proof. All the claimed properties have been proved for the special case $b_\varepsilon = \varepsilon^{-1}$ in [19, Sec. 3], and the arguments employed there carry over in a straightforward way to the present setting. □

Theorem 25. *Let $(\emptyset \ne)\Omega \subseteq \mathbb{R}^n$ be an open set and let μ_ε^b as in Lemma 24. Set $\Omega_\varepsilon := \left\{x \in \Omega \mid d(x, \Omega^c) \ge \varepsilon,\ |x| \le \frac{1}{\varepsilon}\right\}$ and fix some $\chi \in \mathcal{D}(\mathbb{R}^n)$, $\chi = 1$ on $\overline{B_1^{\mathrm{E}}(0)}$, $0 \le \chi \le 1$ and $\chi = 0$ on $\mathbb{R}^n \setminus B_2^{\mathrm{E}}(0)$. Also, take $\kappa_\varepsilon \in \mathcal{D}(\Omega)$ such that $\kappa_\varepsilon = 1$ on a neighborhood L_ε of Ω_ε. Then the map*

$$\iota_\Omega^b : T \in \mathcal{D}'(\Omega) \mapsto \left[\left((\kappa_\varepsilon \cdot T) * \mu_\varepsilon^b\right)(-)\right] \in {}^\rho\mathcal{GC}^\infty(\mathrm{c}(\Omega), {}^\rho\widetilde{\mathbb{R}}). \tag{4.5}$$

satisfies:

(i) $\iota^b : \mathcal{D}' \longrightarrow {}^\rho\mathcal{GC}^\infty(\mathrm{c}(-), {}^\rho\widetilde{\mathbb{R}})$ *is a sheaf-morphism of real vector spaces: If $\Omega' \subseteq \Omega$ is another open set and $T \in \mathcal{D}'(\Omega)$, then $\iota_\Omega^b(T)|_{\mathrm{c}(\Omega')} = \iota_{\Omega'}^b(T|_{\Omega'})$.*

(ii) *ι^b preserves supports, hence is in fact a sheaf-monomorphism.*

(iii) *Any $f \in \mathcal{C}^\infty(\Omega)$ can naturally be considered an element of ${}^\rho\mathcal{GC}^\infty(\mathrm{c}(\Omega), {}^\rho\widetilde{\mathbb{R}})$ via $[x_\varepsilon] \mapsto [f(x_\varepsilon)]$. Moreover, $\forall q \in \mathbb{N}_{>0}\ \forall x \in \mathrm{c}(\Omega) :\ \left|\iota_\Omega^b(f)(x) - f(x)\right| \le b^{-q}$.*

(iv) *If $f \in \mathcal{C}^\infty(\Omega)$ and if $b \ge \mathrm{d}\rho^{-a}$ for some $a \in \mathbb{R}_{>0}$, then $\iota_\Omega^b(f) = f$. In particular, ι^b then provides a multiplicative sheaf-monomorphism $\mathcal{C}^\infty(-) \hookrightarrow {}^\rho\mathcal{GC}^\infty(c(-), \mathbb{R})$.*

(v) *For any $T \in \mathcal{D}'(\Omega)$ and any $\alpha \in \mathbb{N}^n$, $\iota_\Omega^b(\partial^\alpha T) = \partial^\alpha \iota_\Omega^b(T)$.*

(vi) *Let $b \ge \mathrm{d}\rho^{-a}$ for some $a \in \mathbb{R}_{>0}$. Then for any $\varphi \in \mathcal{D}(\Omega)$ and any $T \in \mathcal{D}'(\Omega)$,*

$$\left[\int_\Omega \iota_\Omega^b(T)_\varepsilon(x) \cdot \varphi(x)\,\mathrm{d}x\right] = \langle T, \varphi\rangle \quad \text{in } {}^\rho\widetilde{\mathbb{R}}.$$

(vii) *$\iota_{\mathbb{R}^n}^b(\delta)(0) = c_n b^n$ and if $b \ge \mathrm{d}\rho^{-a}$ for some $a \in \mathbb{R}_{>0}$, then $\iota_{\mathbb{R}}^b(H)(0) = \frac{1}{2}$.*

(viii) *The embedding ι^b does not depend on the particular choice of (κ_ε) and (if $b \ge \mathrm{d}\rho^{-a}$ for some $a \in \mathbb{R}_{>0}$) χ as above.*

(ix) *ι^b does not depend on the representative (b_ε) of b employed in (4.4).*

Proof. We follow ideas from [50, Sec. 1.2] and [19]. Let $T \in \mathcal{D}'(\Omega)$ and let $[x_\varepsilon] \in \mathrm{c}(\Omega)$. Then there exists some $K \Subset \Omega$ such that $x_\varepsilon \in K$ for ε small. Also, we may assume that $K + B^{\mathrm{E}}_{2|\log(b_\varepsilon)|^{-1}}(0) \subseteq L \subseteq \Omega_\varepsilon$ for these ε, where $L \Subset \Omega$. Then by (i) of Lemma 24, for ε small we have

$$\iota_\Omega^b(T)_\varepsilon(x_\varepsilon) = (\kappa_\varepsilon \cdot T) * \mu_\varepsilon^b(x_\varepsilon) = T * \mu_\varepsilon^b(x_\varepsilon) = \langle T, \mu_\varepsilon^b(x_\varepsilon - \,.)\rangle. \tag{4.6}$$

Since $T \in \mathcal{D}'(\Omega)$, we have a seminorm estimate of the form

$$\forall \varphi \in \mathcal{D}_L(\Omega): \ |\langle T, \varphi\rangle| \le C \max_{|\beta|\le m} \sup_{x\in L} |\partial^\beta \varphi(x)|.$$

Together with Lemma 24 (i) and (ii) this implies that $(\partial^\alpha \iota^b_\Omega(T)_\varepsilon(x_\varepsilon)) \in \mathbb{R}^n_\rho$ for each α. Hence ι^b_Ω indeed maps $\mathcal{D}'(\Omega)$ into ${}^\rho\mathcal{GC}^\infty(\mathrm{c}(\Omega), {}^\rho\widetilde{\mathbb{R}})$.

To show (i), let $\Omega' \subseteq \Omega$ be open and let $[x_\varepsilon] \in c(\Omega')$. Then using the notations introduced before (4.6), we may suppose that $L \subseteq \Omega'_\varepsilon$, and so for ε small we have $\mu^b_\varepsilon \in \mathcal{D}(\Omega')$. Therefore, (4.6) implies for such ε:

$$\iota^b_\Omega(T)_\varepsilon(x_\varepsilon) = \langle T, \mu^b_\varepsilon(x_\varepsilon - \ .)\rangle = \langle T|_{\Omega'}, \mu^b_\varepsilon(x_\varepsilon - \ .)\rangle = \iota^b_{\Omega'}(T|_{\Omega'})_\varepsilon(x_\varepsilon).$$

Next we show (ii). Suppose first that $T|_{\Omega'} = 0$ for some open subset Ω' of Ω. Let $[x_\varepsilon] \in \mathrm{c}(\Omega')$ and pick $K \Subset \Omega'$ such that $x_\varepsilon \in K$ for ε small. As above, for ε small we have $\mathrm{stsupp}(\mu^b_\varepsilon(x_\varepsilon - \ .)) \subseteq \Omega'$, as well as $\iota^b_\Omega(T)_\varepsilon(x_\varepsilon) = \langle T, \mu^b_\varepsilon(x_\varepsilon - \ .)\rangle$, which therefore vanishes. Hence $\iota^b_\Omega(T)|_{\Omega'} = 0$, and thereby $\mathrm{stsupp}(\iota^b_\Omega(T)) \subseteq \mathrm{stsupp}(T)$.

Conversely, let $\Omega' \subseteq \Omega$ such that $\iota^b_\Omega(T)|_{\Omega'} = 0$ and let $\varphi \in \mathcal{D}(\Omega')$. Since $(\kappa_\varepsilon T) * \mu^b_\varepsilon \to T$ in $\mathcal{D}'(\Omega)$, in order to show $\langle T, \varphi\rangle = 0$ it suffices to demonstrate that $(\kappa_\varepsilon T) * \mu^b_\varepsilon \to 0$ as $\varepsilon \to 0$, uniformly on compact subsets of Ω'. Suppose this were not the case, then we could find some $L \Subset \Omega'$, some $c > 0$ and sequences $\varepsilon_k \downarrow 0$ and $x_k \in L$ such that $|(\kappa_\varepsilon T) * \mu^b_{\varepsilon_k}(x_k)| \ge c$ for all k. Fixing any $z \in \Omega'$ and setting $x_\varepsilon := x_k$ for $\varepsilon = \varepsilon_k$ and $x_\varepsilon = z$ otherwise then defines an element $[x_\varepsilon] \in \mathrm{c}(\Omega')$ with $\iota^b_\Omega(T)([x_\varepsilon]) \neq 0$, a contradiction.

Consequently, ι^b_Ω induces an injective sheaf morphism (again denoted by) ι^b : $\mathcal{D}'(-) \longrightarrow {}^\rho\mathcal{GC}^\infty(\mathrm{c}(-), {}^\rho\widetilde{\mathbb{R}})$.

(iii): If $f \in \mathcal{C}^\infty(\Omega)$ then any derivative of f is uniformly (in ε) bounded on any (x_ε) (for $[x_\varepsilon] \in c(\Omega)$). Thus $f \in \mathcal{GC}^\infty(\mathrm{c}(\Omega), {}^\rho\widetilde{\mathbb{R}})$. Now let $[x_\varepsilon] \in \mathrm{c}(\Omega)$ and suppose first that f has compact support. By Lemma 24 (iv), for any $x \in \Omega$, $f(x) = \int f(x)\mu^b_\varepsilon(y)\,dy + n_\varepsilon$, where $n_\varepsilon = O(b_\varepsilon^{-q})$ for every $q > 0$. Thus for ε small and any $q \in \mathbb{N}$ we have by Taylor expansion

$$\begin{aligned}(\iota^b_\Omega(f)_\varepsilon - f)(x_\varepsilon) &= \int (f(x_\varepsilon - y) - f(x_\varepsilon))\mu^b_\varepsilon(y)\,\mathrm{d}y + n_\varepsilon \\ &= \sum_{k=1}^{q-1} \int \frac{1}{k!}((-y\cdot D)^k f)(x_\varepsilon)\mu^b_\varepsilon(y)\,\mathrm{d}y \\ &+ \frac{b_\varepsilon^{-q}}{q!}\int ((-y\cdot D)^q f)(x_\varepsilon - \theta_\varepsilon b_\varepsilon^{-1}y)\tilde{\mu}(y)\chi(b_\varepsilon^{-1}y|\log(b_\varepsilon)|)\,\mathrm{d}y + n_\varepsilon,\end{aligned} \tag{4.7}$$

where $\theta_\varepsilon \in (0,1)$. Here, the first sum is $O(b_\varepsilon^{-q})$ by Lemma 24 (v), as is the second term since f is compactly supported, χ is globally bounded, and $\tilde{\mu} \in \mathcal{S}(\mathbb{R}^n)$. If f is not compactly supported, pick $L \Subset \Omega$ such that $x_\varepsilon \in L$ for ε small and let $\varphi \in \mathcal{D}(\Omega)$ equal 1 in a neighborhood of L. Then $f(x) = (\varphi f)(x)$ and (ii) implies that $\iota^b_\Omega(f)(x) = \iota^b_\Omega(\varphi f)(x)$, so the general case follows as well.

(iv): It suffices to observe that, by our assumption on b, (iii) implies that $\iota^b_\Omega(f)([x_\varepsilon]) = [f(x_\varepsilon)] = f(x)$ for any $f \in \mathcal{C}^\infty(\Omega)$ and any $x = [x_\varepsilon] \in \mathrm{c}(\Omega)$.

(v): As in the proof of (iv) we may assume that T has compact support. Then for ε small we have

$$\iota^b_\Omega(\partial^\alpha T)_\varepsilon = \partial^\alpha T * \mu^b_\varepsilon = \partial^\alpha(T * \mu^b_\varepsilon) = \partial^\alpha \iota^b_\Omega(T)_\varepsilon.$$

(vi): Pick $\zeta \in \mathcal{D}(\Omega)$ such that $\zeta \equiv 1$ on a neighborhood of $\mathrm{stsupp}(\varphi)$. Then

$$\mathrm{stsupp}(\iota^b_\Omega(T) - \iota^b_\Omega(\zeta T)) \cap \mathrm{stsupp}\varphi = \mathrm{stsupp}(T - \zeta T) \cap \mathrm{stsupp}\varphi = \emptyset,$$

so we may replace T by ζT, i.e., we may assume without loss of generality that $T \in \mathcal{E}'(\Omega)$. By the representation theorem of distribution theory T then is a finite sum of terms of the form $\partial^\alpha f$ with $f \in \mathcal{C}^0(\Omega)$ compactly supported in Ω, so it will suffice to treat the case $T = \partial^\alpha f$. For any $\varphi \in \mathcal{D}(\Omega)$ we have

$$\begin{aligned}\int (\iota^b_\Omega(\partial^\alpha f)_\varepsilon - \partial^\alpha f)(x)\varphi(x)\,\mathrm{d}x &= \int\int (\partial^\alpha f(x-y) - \partial^\alpha f(x))\mu^b_\varepsilon(y)\varphi(x)\;\mathrm{d}y\mathrm{d}x + n_\varepsilon \\ &= \int \partial^\alpha f(x) \int \mu^b_\varepsilon(y)(\varphi(x+y) - \varphi(x))\;\mathrm{d}y\mathrm{d}x + n_\varepsilon.\end{aligned}$$

with $n_\varepsilon = O(b_\varepsilon^{-q})$ for any $q \in \mathbb{N}$ by Lemma 24 (v). As in the proof of (iii) it follows that also the integral term in the above equality is of order $O(b_\varepsilon^{-q})$, giving the claim due to our assumption on b.

(vii): The first claim is immediate from $\iota^b_{\mathbb{R}}(\delta)_\varepsilon(0) = \mu^b_\varepsilon(0)$. To show the second, note first that

$$\iota^b_{\mathbb{R}}(H)_\varepsilon(0) - \int H(y)\mu^b_\varepsilon(-y)\,\mathrm{d}y = \int H(y)(\kappa_\varepsilon(y) - 1)\mu^b_\varepsilon(-y)\,\mathrm{d}y = 0$$

for ε small by the support properties of κ_ε and χ. Furthermore, since $\int_0^\infty \mu(y)\,dy = 1/2$, we obtain

$$\begin{aligned}\Big|\int H(y)\mu^b_\varepsilon(-y)\,\mathrm{d}y - \frac{1}{2}\Big| &= \Big|\int_0^\infty \mu(y)(\chi(b_\varepsilon^{-1}|\log(b_\varepsilon)|)y) - 1)\,\mathrm{d}y\Big| \\ &\le \int_{b_\varepsilon|\log(b_\varepsilon)|^{-1}}^\infty |\mu(y)|\,\mathrm{d}y = O(b_\varepsilon^{-q})\end{aligned}$$

for any $q \in \mathbb{N}$, so the claim follows.

(viii): We first note that any two choices for either (κ_ε) or χ provide sheaf morphisms as in (i), (ii), hence it suffices to check that the resulting embeddings coincide on compactly supported distributions. For any such T we have $\kappa_\varepsilon T = T$ for ε small, so independence from the choice of (κ_ε) follows.

Now suppose that two different χ's have been chosen and denote the corresponding functions from (4.4) by μ^b_ε and $\bar\mu^b_\varepsilon$, and the resulting embeddings by ι^b and $\bar\iota^b$, respectively. Since $T \in \mathcal{E}'(\Omega)$, it satisfies a seminorm estimate of the form

$$\forall \varphi \in \mathcal{C}^\infty(\Omega) : \ |\langle T, \varphi\rangle| \le C \max_{|\beta|\le m} \sup_{x\in L} |\partial^\beta \varphi(x)|. \tag{4.8}$$

for some $L \Subset \Omega$. Together with Lemma 24 (iii) this implies that, for any $[x_\varepsilon] \in c(\Omega)$ and ε small, we have

$$|(\iota^b_\Omega(T)_\varepsilon - \bar\iota^b_\Omega(T)_\varepsilon)(x_\varepsilon)| = |\langle T, (\mu^b_\varepsilon - \bar\mu^b_\varepsilon)(x_\varepsilon - \ .)\rangle| = O(b_\varepsilon^{-q})$$

for any $q \in \mathbb{N}$.

(ix): Let (c_ε) be another representative of b, so $(c_\varepsilon) \sim_\rho (b_\varepsilon)$. As in the proof of (viii) it then suffices to show that $\iota^b(T) = \iota^c(T)$ for any $T \in \mathcal{E}'(\Omega)$. Given $x = [x_\varepsilon] \in c(\Omega)$, let $K \Subset \Omega$ be such that $x_\varepsilon \in K$ for ε small. Then by (4.8) and (4.5),

$$|(\iota^b(T) - \iota^c(T))(x_\varepsilon)| \le C \max_{|\beta|\le m} \sup_{x\in K-L} |\partial^\beta(\mu^b_\varepsilon - \mu^c_\varepsilon)(x)|.$$

Inserting from (4.4) it follows by a straightforward estimate that the right hand side here is of order $O(\rho_\varepsilon^q)$ for any $q \in \mathbb{N}$, proving the claim. □

Whenever we use the notation ι^b for an embedding, we assume that $b \in {}^\rho\widetilde{\mathbb{R}}$ satisfies the overall assumptions of Thm. 25 and of (iv) in that Theorem, and that ι^b has been defined as in (4.5) using a Colombeau mollifier μ for the given dimension. Note in particular that by Theorem 25 (ix) we are justified in using the short hand notation ι^b for the embedding defined via any representative (b_ε) of b.

Remark 26.

(i) In Def. 1, we introduced the asymptotic gauge $\mathcal{I}(\rho)$, and the entire construction depends on the fixed infinitesimal net ρ only through this set $\mathcal{I}(\rho)$. A more general definition of asymptotic gauge is possible (see [45]). Anyhow, [45, Sec. 4.3] shows that an embedding of Schwartz distribution having certain minimal properties necessarily requires that the asymptotic gauge be generated by a single net, as is the case for $\mathcal{I}(\rho)$.

(ii) Let δ, $H \in {}^\rho\mathcal{GC}^\infty({}^\rho\widetilde{\mathbb{R}}, {}^\rho\widetilde{\mathbb{R}})$ be the corresponding ι^b-embeddings of the Dirac delta and of the Heaviside function. Then $\delta(x) = b \cdot \mu(b \cdot x)$ and $\delta(x) = 0$ if x is near-standard and $x^\circ \neq 0$ or if x is infinite because $\mu \in \mathcal{S}(\mathbb{R})$. Also, by construction of μ^b_ε, δ can be represented like in the first diagram of Fig. 4.1. E.g., $\delta(k/b) = 0$ for each $k \in \mathbb{Z} \setminus \{0\}$, and each $\frac{k}{b}$ is a nonzero infinitesimal. Similar properties can be stated e.g. for $\delta^2(x) = b^2 \cdot \mu(b \cdot x)^2$.

(iii) Analogously, we have $H(x) = 1$ if x is near-standard and $x^\circ > 0$ or if $x > 0$ is infinite; $H(x) = 0$ if x is near-standard and $x^\circ < 0$ or if $x < 0$ is infinite.

(iv) Let $\mathrm{vp}(\frac{1}{x}) \in \mathcal{D}'(\mathbb{R})$ be the Cauchy principal value, so that $\iota^b_{\mathbb{R}}(\mathrm{vp}(\frac{1}{x}))(x) = [(\mathrm{vp}(\frac{1}{x}) * \mu^b_\varepsilon)(x_\varepsilon)] = [\langle \mathrm{vp}(\frac{1}{x})(y), \mu^b_\varepsilon(x_\varepsilon - y)\rangle]$ and $\mu^b_\varepsilon(x) = b_\varepsilon \mu(b_\varepsilon x)\chi(x|\log b_\varepsilon|)$. If $x = [x_\varepsilon]$ is far from the origin, in the sense that $|x| \geq r$ for some $r \in \mathbb{R}_{>0}$, then $\iota^b_{\mathbb{R}}(\mathrm{vp}(\frac{1}{x}))(x) = [\int_{-R_\varepsilon}^{R_\varepsilon} \frac{\mu^b_\varepsilon(x_\varepsilon - y)}{y}\,\mathrm{d}y]$, where $R_\varepsilon := x_\varepsilon + 2|\log b_\varepsilon|^{-1}$. Proceeding as above for the smooth function $x \mapsto \frac{1}{x}$ in a neighborhood of x_ε not containing the origin (i.e. for ε small), we can prove that $\iota^b_{\mathbb{R}}(\mathrm{vp}(\frac{1}{x}))(x) = \frac{1}{x}$. The behavior of the GSF $\iota^b_{\mathbb{R}}(\mathrm{vp}(\frac{1}{x}))$ in an infinitesimal neighborhood of the origin depends on the Colombeau mollifier μ. For example, if in Lem. 22 we add the linear condition $\int \frac{\mu_n(x)}{x}\,\mathrm{d}x = 0$, then also $\iota^b_{\mathbb{R}}(\mathrm{vp}(\frac{1}{x}))(0) = 0$.

(v) In [76], S. Łojasiewicz introduced the notion of a point value for distributions. He defined that $T \in \mathcal{D}'(\Omega)$ has the point value $c \in \mathbb{C}$ in $x_0 \in \Omega$ if

$$\lim_{\varepsilon \to 0} \langle T(x_0 + \varepsilon x), \varphi(x)\rangle = c \int \varphi(x)\,\mathrm{d}x \qquad \forall \varphi \in \mathcal{D}(\Omega). \tag{4.9}$$

Not every distribution has point values in arbitrary points — in fact, if it does, it already has to be a function of first Baire class ([76]). Conversely, a continuous function f clearly has point value $f(x)$ in any point x in its domain.

We show that if T has point value c at $x_0 \in \Omega$ then $\iota^b_\Omega(T)_\varepsilon(x_0) \to c$ as $\varepsilon \to 0$. In fact, since $\mathcal{S}'(\mathbb{R}^n)$ is a normal space of distributions that is invariant under translations, by [103, Prop. 7] this follows if for any sequence $\varepsilon_k \downarrow 0$, the functions $g_k := \mu^b_{\varepsilon_k}$ satisfy the following conditions:

(a) $\int g_k(x)\,\mathrm{d}x \to 1$, and $\forall \eta > 0$: $\int_{|x| \geq \eta} g_k(x)\,\mathrm{d}x \to 0$ as $k \to \infty$.

(b) For each $\alpha \in \mathcal{D}(\mathbb{R}^n)$ that is 1 on a neighborhood of 0, $(1 - \alpha)g_k \to 0$ in $\mathcal{S}'(\mathbb{R}^n)$ for $k \to \infty$.

(c) For each $\alpha \in \mathbb{N}^n$ there exists some $M_\alpha > 0$ such that, for any $\eta > 0$: $\int_{|x| \leq \eta} |x|^{|\alpha|} |\partial^\alpha g_k(x)|\,\mathrm{d}x \leq M_\alpha$.

Indeed, all these properties follow readily from (4.4).

(vi) Colombeau's special (or simplified) algebra $\mathcal{G}$ ([15, 16, 84, 50]) is defined, for $\Omega \subseteq \mathbb{R}^n$ open, as the quotient $\mathcal{G}^s(\Omega) := \mathcal{E}_M(\Omega)/\mathcal{N}^s(\Omega)$ of *moderate nets* modulo *negligible nets*, where

$$\mathcal{E}_M(\Omega) := \{(u_\varepsilon) \in \mathcal{C}^\infty(\Omega)^I \mid \forall K \Subset \Omega \, \forall \alpha \in \mathbb{N}^n \, \exists N \in \mathbb{N} : \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{-N})\}$$

and

$$\mathcal{N}^s(\Omega) := \{(u_\varepsilon) \in \mathcal{C}^\infty(\Omega)^I \mid \forall K \Subset \Omega \, \forall \alpha \in \mathbb{N}^n \, \forall m \in \mathbb{N} : \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^m)\}.$$

It follows from [43, Th. 37] that $\mathcal{G}^s(\Omega)$ can be identified with the algebra ${}^\rho\mathcal{GC}^\infty(\mathrm{c}(\Omega), {}^\rho\widetilde{\mathbb{R}})$ in the special case of $\rho(\varepsilon) = \varepsilon$. In this setting, Theorem 25 gives an alternative proof of the well known facts that the Colombeau algebra contains $\mathcal{C}^\infty(\Omega)$ as a faithful subalgebra, $\mathcal{D}'(\Omega)$ as a linear subspace and that the embedding is a sheaf morphism that commutes with partial derivatives. An alternative point of view is that Colombeau generalized functions correspond to those generalized smooth functions that are defined on compactly supported generalized points. As was already mentioned, more general domains are both useful in applications and are indeed a necessary requirement for obtaining a construction that is closed with respect to composition of generalized functions.

We close this section by considering the following natural problem: let us define two embeddings ι^b_Ω, ι^c_Ω as in (4.5), but using two different infinite positive numbers b, $c \in {}^\rho\widetilde{\mathbb{R}}$, so that for all $T \in \mathcal{E}'(\Omega)$ we have

$$\iota^b_\Omega(T) := \left[T * \mu^b_\varepsilon\right],$$
$$\iota^c_\Omega(T) := \left[T * \mu^c_\varepsilon\right].$$

The following result characterizes equality of such embeddings.

Theorem 27. *Let b, $c \in {}^\rho\widetilde{\mathbb{R}}$ be infinite positive numbers and let μ be a Colombeau mollifier for dimension n. Let $\Omega \subseteq \mathbb{R}^n$ be open. Then $\iota^b_\Omega = \iota^c_\Omega$ if and only if $b = c$ in ${}^\rho\widetilde{\mathbb{R}}$, i.e. if and only if they are equal as Robinson-Colombeau generalized number.*

Proof. By Theorem 25 (ix), ι^b is well-defined, i.e., does not depend on the representative of $b \in {}^\rho\widetilde{\mathbb{R}}$. Conversely, suppose that $\iota^b_\Omega = \iota^c_\Omega$ and fix any $x_0 \in \Omega$. Then in particular $\iota^b_\Omega(\delta_{x_0}) = \iota^c_\Omega(\delta_{x_0})$ in ${}^\rho\mathcal{GC}^\infty(\mathrm{c}(\Omega), {}^\rho\widetilde{\mathbb{R}})$. Due to (4.4), (4.5), an evaluation of these GSF at x_0 implies

$$\forall m \in \mathbb{N} : \ |(b^n_\varepsilon - c^n_\varepsilon)c_n| = O(\rho^m_\varepsilon),$$

so $b = c$ in ${}^\rho\widetilde{\mathbb{R}}$. □

4.0.1. *Closure with respect to composition.* In contrast to the case of distributions, there is no problem in considering the composition of two GSF. This property opens new interesting possibilities, e.g. in considering differential equations $y' = f(y, t)$, where y and f are GSF. For instance, there is no problem in studying $y' = \delta(y)$ (see [77]).

Theorem 28. *Subsets $S \subseteq {}^\rho\widetilde{\mathbb{R}}^s$ with the trace of the sharp topology, and generalized smooth maps as arrows form a subcategory of the category of topological spaces. We will call this category ${}^\rho\mathcal{GC}^\infty$, the* category of GSF.

Proof. From Thm. 17 (iii) we already know that every GSF is continuous; we have hence to prove that these arrows are closed with respect to identity and composition in order to obtain a concrete subcategory of topological spaces and continuous maps. If $T \subseteq {}^\rho\widetilde{\mathbb{R}}^t$ is an arbitrary object, then $f_\varepsilon(x) := x$ is the net of smooth functions that globally defines the identity 1_T on T. It is immediate that 1_T is generalized smooth.

To prove that arrows of ${}^\rho\mathcal{GC}^\infty$ are closed with respect to composition, let $S \subseteq {}^\rho\widetilde{\mathbb{R}}^s, T \subseteq {}^\rho\widetilde{\mathbb{R}}^t, R \subseteq {}^\rho\widetilde{\mathbb{R}}^r$ and $f : S \longrightarrow T$, $g : T \longrightarrow R$ be GSF, then $f(x) = [f_\varepsilon(x_\varepsilon)] \in T$ and $g(y) = [g_\varepsilon(y_\varepsilon)] \in R$ for every $x \in S$ and $y \in T$, where $f_\varepsilon \in \mathcal{C}^\infty(\Omega'_\varepsilon, \mathbb{R}^t)$ and $g_\varepsilon \in \mathcal{C}^\infty(\Omega''_\varepsilon, \mathbb{R}^r)$ are suitable nets of smooth functions as in Def. 14, and where Ω'_ε is open in $\mathbb{R}^s$ and Ω''_ε is open in $\mathbb{R}^t$. Of course, the idea is to consider $g_\varepsilon \circ f_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R}^r)$, where $\Omega_\varepsilon := f_\varepsilon^{-1}(\Omega''_\varepsilon)$ (let us note that, even in the case where Ω''_ε does not depend by ε, generally speaking Ω_ε still depends on ε).

Take $x \in S$, so that $f(x) = [f_\varepsilon(x_\varepsilon)] \in T \subseteq \langle \Omega''_\varepsilon \rangle$ and hence $f_\varepsilon(x_\varepsilon) \in_\varepsilon \Omega''_\varepsilon$ and $x_\varepsilon \in \Omega_\varepsilon$ for ε small. If we take another representative $(x'_\varepsilon) \sim_\rho (x_\varepsilon)$ we have $f(x') = f(x)$ since f is well-defined and, proceeding as before, we still have that $x'_\varepsilon \in \Omega_\varepsilon$ for ε sufficiently small. This proves that $S \subseteq \langle \Omega_\varepsilon \rangle$. Moreover, since $[f_\varepsilon(x_\varepsilon)] \in T$, we also have that $[g_\varepsilon(f_\varepsilon(x_\varepsilon))] \in R$ and $g(f(x)) = [g_\varepsilon(f_\varepsilon(x_\varepsilon))]$. It remains to show that the net $(g_\varepsilon \circ f_\varepsilon)$ defines a GSF (Def. 14) of the type $S \longrightarrow R$. To this end, let us consider any $[x_\varepsilon] \in S$ and any $\gamma \in \mathbb{N}^s$. We can write

$$\partial^\gamma (g_\varepsilon \circ f_\varepsilon)(x_\varepsilon) = p\left[\partial^{\alpha_1} f_\varepsilon(x_\varepsilon), \ldots, \partial^{\alpha_A} f_\varepsilon(x_\varepsilon), \partial^{\beta_1} g_\varepsilon(f_\varepsilon(x_\varepsilon)), \ldots, \partial^{\beta_B} g_\varepsilon(f_\varepsilon(x_\varepsilon))\right], \tag{4.10}$$

where p is a suitable polynomial (from the Faà di Bruno formula) not depending on x_ε. Every term $\partial^{\alpha_i} f_\varepsilon(x_\varepsilon)$ and $\partial^{\beta_j} g_\varepsilon(f_\varepsilon(x_\varepsilon))$ is ρ-moderate by (ii) of Def. 14. Since moderateness is preserved by polynomial operations, it follows that also $\partial^\gamma (g_\varepsilon \circ f_\varepsilon)(x_\varepsilon)$ is ρ-moderate. □

For instance, we can think of the Dirac delta as a map of the form $\delta : {}^\rho\widetilde{\mathbb{R}} \longrightarrow {}^\rho\widetilde{\mathbb{R}}$, and therefore the composition e^δ is defined in $\{x \in {}^\rho\widetilde{\mathbb{R}} \mid \exists z \in {}^\rho\widetilde{\mathbb{R}}_{>0} : \ \delta(x) \le \log z\}$, which of course does not contain $x = 0$ but only suitable non zero infinitesimals. On the other hand, $\delta \circ \delta : {}^\rho\widetilde{\mathbb{R}} \longrightarrow {}^\rho\widetilde{\mathbb{R}}$. Moreover, from the inclusion of ordinary smooth functions (Thm. 25) and the closure with respect to composition, it directly follows that every ${}^\rho\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}})$ is an algebra with pointwise operations for every subset $U \subseteq {}^\rho\widetilde{\mathbb{R}}^n$. For an open subset $\Omega \subseteq \mathbb{R}^n$, the algebra ${}^\rho\mathcal{GC}^\infty(\mathrm{c}(\Omega), {}^\rho\widetilde{\mathbb{R}})$ contains the space $\mathcal{D}'(\Omega)$ of Schwartz distributions.

A natural way to define a GSF is to follow the original idea of classical authors (see [60, 73, 20]) to fix an infinitesimal or infinite parameter in a suitable ordinary smooth function. We will call this type of GSF of *Cauchy-Dirac type*; the next theorem specifies this notion and states that GSF are of Cauchy-Dirac type whenever the generating net (f_ε) is smooth in ε.

Corollary 29. *Let $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^d$, $P \subseteq \mathbb{R}^m$ be open sets and $\varphi \in \mathcal{C}^\infty(P \times X, Y)$ be an ordinary smooth function. Let $p \in [P]$, and define $f_\varepsilon := \varphi(p_\varepsilon, -) \in \mathcal{C}^\infty(X, Y)$, then $[f_\varepsilon(-)] : [X] \longrightarrow [Y]$ is a GSF. In particular, if $f : [X] \longrightarrow [Y]$ is a GSF*

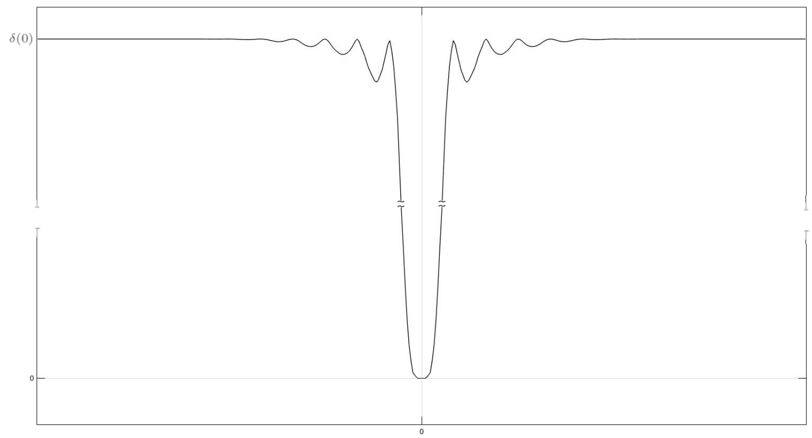

FIGURE 4.2. A representation of $\delta\circ\delta$

defined by (f_ε) and the net (f_ε) is smooth in ε, i.e. if

$$\exists\varphi\in\mathcal{C}^\infty((0,1)\times X,Y):\ f_\varepsilon=\varphi(\varepsilon,-)\quad\forall\varepsilon\in(0,1),$$

and if $[\varepsilon]\in{}^\rho\widetilde{\mathbb{R}}$, then the GSF f is of Cauchy-Dirac type because $f(x)=\varphi([\varepsilon],x)$ for all $x\in[X]$. Finally, Cauchy-Dirac GSF are closed with respect to composition.

Proof. In fact, the map $x\in[X]\mapsto(p,x)\in[P]\times[X]$ is trivially generalized smooth and hence from the inclusion of smooth functions (Theorem 25) and the closure with respect to composition (Theorem 28) the conclusions follow. □

Example 30. The composition $\delta\circ\delta\in{}^\rho\mathcal{GC}^\infty({}^\rho\widetilde{\mathbb{R}},{}^\rho\widetilde{\mathbb{R}})$ is given by $(\delta\circ\delta)(x)=b\mu\left(b^2\mu(bx)\right)$ and is an even function. If x is near-standard and $x^\circ\ne0$, or x is infinite, then $(\delta\circ\delta)(x)=b$. Since $(\delta\circ\delta)(0)=0$, by the intermediate value theorem (see Cor. 48 below), we have that $\delta\circ\delta$ attains any value in the interval $[0,b]\subseteq{}^\rho\widetilde{\mathbb{R}}$. If $0\le x\le\frac{1}{2b}$, then (for a μ as in Fig. 4) x is infinitesimal and $(\delta\circ\delta)(x)=0$ because $\delta(x)\ge b\mu\left(\frac{1}{k}\right)$ is an infinite number. If $x=\frac{k}{b}$ for some $k\in\mathbb{N}_{>0}$, then x is still infinitesimal but $(\delta\circ\delta)(x)=b$ because $\mu(bx)=0$. A representation of $\delta\circ\delta$ is given in Fig. 4.2. Analogously, one can deal with $H\circ\delta$ and $\delta\circ H$.

The theory of GSF originates from the theory of Colombeau quotient algebras. In this well-developed approach, strong analytic tools, including microlocal analysis, and an elaborate theory of pseudodifferential and Fourier integral operators have been developed over the past few years (cf. [15, 16, 84, 50, 54, 31, 32] and references therein). In these quotient algebras, each generalized function generates a unique GSF defined on a subset of ${}^{(\varepsilon)}\widetilde{\mathbb{R}}$. On the other hand, Colombeau generalized functions are in general not closed with respect to composition because they cannot be defined on arbitrary domains $X\subseteq{}^\rho\widetilde{\mathbb{R}}^n$. We refer to [43] for details about the links between Colombeau algebras and GSF, and to [112, 113, 114] for a treatment of Colombeau algebras in the framework of nonstandard analysis.

5. Differential calculus and the Fermat-Reyes theorem

In this section we show how the derivatives of a GSF can be calculated using a form of incremental ratio. The idea is to prove the Fermat-Reyes theorem for GSF (see [38, 40, 62]). Essentially, this theorem shows the existence and uniqueness of another GSF serving as incremental ratio. This is the first of a long list of results demonstrating the close similarities between ordinary smooth functions and GSF.

We recall that the *thickening* of an open set $\Omega\subseteq\mathbb{R}^n$ *along* $v\in\mathbb{R}^n$ is $\text{th}_v(\Omega):=\{(x,h)\in\mathbb{R}^{n+1}\mid[x,x+hv]_{\mathbb{R}^n}\subseteq\Omega\}$, and serves as a natural domain of a partial incremental ratio along v of any function defined on Ω. In order to prove the Fermat-Reyes theorem, it is simpler to define what *a* thickening of $U\subseteq{}^\rho\widetilde{\mathbb{R}}^n$ along $v\in{}^\rho\widetilde{\mathbb{R}}^n$ is.

Definition 31. Let $U\subseteq{}^\rho\widetilde{\mathbb{R}}^n$ and let $v\in{}^\rho\widetilde{\mathbb{R}}^n$. Then we say that $T\subseteq{}^\rho\widetilde{\mathbb{R}}^{n+1}$ is *a (sharp) thickening of U along v* if

(i) $\forall x\in U:\ (x,0)\in T$

(ii) For all $(x,h)\in T$ there exist $a,\,b\in{}^\rho\widetilde{\mathbb{R}}_{>0}$, with $b<a$, such that:
  (a) $|h\cdot v|<b$
  (b) $B_a(x)\subseteq U$
  (c) $B_a(x)\times B_b(0)\subseteq T$.

Finally, we will say that T is *a large thickening of U along v* if the radii a, b in (ii) are real: $a,\,b\in\mathbb{R}_{>0}$.

Remark 32.

(i) Conditions (i) and (ii) imply that necessarily U is a sharply open set, whereas U is a large open set if T is a large thickening.

(ii) Let $(x,h)\in T$ and let the radii a, b be as in (ii). Then for all $s\in[0,1]$ we have $|x+shv-x|\le|hv|<b<a$. Therefore $[x,x+hv]\subseteq B_a(x)\subseteq U$. This gives a connection with the classical definition of thickening and shows that if $f:U\longrightarrow{}^\rho\widetilde{\mathbb{R}}$, we can consider the difference $f(x+hv)-f(x)$.

(iii) Condition (ii) of Def. 31 yields that T is a sharply open subset of ${}^\rho\widetilde{\mathbb{R}}^{n+1}$; it is a large open subset in case T is a large thickening.

(iv) If T and $\bar{T}$ are two (large) thickenings of U along v, then also $T\cap\bar{T}$ is a (large) thickening of the same type. Finally, thickenings are also closed with respect to arbitrary non empty unions.

In the present setting, the Fermat-Reyes theorem is the following.

Theorem 33. *Let $U\subseteq{}^\rho\widetilde{\mathbb{R}}^n$ be a sharply open set, let $v=[v_\varepsilon]\in{}^\rho\widetilde{\mathbb{R}}^n$, and let $f\in{}^\rho\mathcal{GC}^\infty(U,{}^\rho\widetilde{\mathbb{R}})$ be a generalized smooth map generated by the net of smooth functions $f_\varepsilon\in\mathcal{C}^\infty(\Omega_\varepsilon,\mathbb{R})$. Then*

(i) If S is a thickening of U along v such that $S\subseteq\langle\text{th}_{v_\varepsilon}(\Omega_\varepsilon)\rangle$, then there exists a thickening $T\subseteq S$ of U along v and a generalized smooth map $r\in{}^\rho\mathcal{GC}^\infty(T,{}^\rho\widetilde{\mathbb{R}})$, called the generalized incremental ratio *of f* along v, *such that*

$$f(x+hv)=f(x)+h\cdot r(x,h)\qquad\forall(x,h)\in T.$$

Moreover $r(x,0)=\left[\frac{\partial f_\varepsilon}{\partial v_\varepsilon}(x_\varepsilon)\right]$ for every $x\in U$, and we can thus define $\frac{\partial f}{\partial v}(x):=r(x,0)$, so that $\frac{\partial f}{\partial v}\in{}^\rho\mathcal{GC}^\infty(U,{}^\rho\widetilde{\mathbb{R}})$.

(ii) Any two generalized incremental ratios of f coincide on the intersection of their domains.

If U is a large open set and S is a large thickening of U along v, then an analogous statement holds for a large thickening T of U along v.

Note that this result allows us to consider the partial derivative of f with respect to an arbitrary generalized vector $v \in {}^{\rho}\widetilde{\mathbb{R}}^n$ which can be, e.g., near-standard or infinite.

Before proving the theorem, it is essential to show that GSF are uniquely determined by invertible elements.

Lemma 34. *Let $U \subseteq {}^{\rho}\widetilde{\mathbb{R}}^n$ be an open set in the sharp topology, and let $f \in {}^{\rho}\mathcal{GC}^{\infty}(U, {}^{\rho}\widetilde{\mathbb{R}}^d)$ be a GSF. Then $f(x) = 0$ for every $x \in U$ if and only if $f(x) = 0$ for all $x \in U$ such that $|x|$ is invertible.*

Proof. Using Lem. 8, it is straightforward to prove that the group of invertible elements is dense in ${}^{\rho}\widetilde{\mathbb{R}}$ with respect to the sharp topology. This implies that the set of points in U whose every coordinate is invertible is dense in U. Clearly for any such point y, $|y|$ is invertible. Thus given any point $x \in U$ there exists a sequence (x_k) in U converging to x in the sharp topology and such that $|x_k|$ is invertible for each k. Since f is continuous with respect to the sharp topology (Thm. 17 (iii)), this yields $0 = f(x_k) \to f(x)$. □

To show the existence of thickenings, we also need the following result

Lemma 35. *Let (Ω_ε) be a net of open sets of $\mathbb{R}^n$, and let $v = [v_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}^n$. Then*

(i) if $x \in \langle\Omega_\varepsilon\rangle$ then $(x, 0) \in \langle \mathrm{th}_{v_\varepsilon}(\Omega_\varepsilon)\rangle$.
(ii) If $(x, h) \in \langle \mathrm{th}_{v_\varepsilon}(\Omega_\varepsilon)\rangle$ then $x + thv \in \langle\Omega_\varepsilon\rangle$ for all $t \in [0, 1]$.
(iii) If $U \subseteq \langle\Omega_\varepsilon\rangle$ is sharply open, there exists a sharp thickening T of U along v such that $T \subseteq \langle \mathrm{th}_{v_\varepsilon}(\Omega_\varepsilon)\rangle$.

The same properties hold if we consider the strongly internal sets $\langle\Omega_\varepsilon\rangle_{\mathrm{F}}$ and $\langle \mathrm{th}_{v_\varepsilon}(\Omega_\varepsilon)\rangle_{\mathrm{F}}$ in the Fermat topology. In this case in (iii), U has to be supposed large open and the resulting thickening is large as well.

Proof. If $x \in \langle\Omega_\varepsilon\rangle$, then $x_\varepsilon \in \Omega_\varepsilon$ for ε small, and we also have $(x_\varepsilon, 0) \in \mathrm{th}_{v_\varepsilon}(\Omega_\varepsilon)$ for the same ε. Now take $(x'_\varepsilon, z_\varepsilon) \sim_\rho (x_\varepsilon, 0)$, so that $x = [x'_\varepsilon] \in \langle\Omega_\varepsilon\rangle$ and hence $B_r(x) \subseteq \langle\Omega_\varepsilon\rangle$ for some $r \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$ such that $r_\varepsilon < d(x'_\varepsilon, \Omega^c_\varepsilon)$ for all $\varepsilon \in I$ (see Thm. 10). The net $(z_\varepsilon) \sim_\rho 0$, so we also have $|z_\varepsilon v_\varepsilon| < r_\varepsilon$ for ε small. Thus for ε sufficiently small we obtain both $x'_\varepsilon \in \Omega_\varepsilon$ and $|z_\varepsilon v_\varepsilon| < r_\varepsilon$, so that for all $s \in [0, 1]_{\mathbb{R}}$ we have that $|x'_\varepsilon + s z_\varepsilon v_\varepsilon - x'_\varepsilon| \le |z_\varepsilon v_\varepsilon| < r_\varepsilon < d(x'_\varepsilon, \Omega^c_\varepsilon)$. Hence $x'_\varepsilon + s z_\varepsilon v_\varepsilon \in \Omega_\varepsilon$, i.e. $(x'_\varepsilon, z_\varepsilon) \in \mathrm{th}_{v_\varepsilon}(\Omega_\varepsilon)$ for ε sufficiently small. This shows that $(x, 0) \in \langle \mathrm{th}_{v_\varepsilon}(\Omega_\varepsilon)\rangle$, implying (i).

To prove (ii), assume that $(x, h) \in \langle \mathrm{th}_{v_\varepsilon}(\Omega_\varepsilon)\rangle$ and $t \in [0, 1]$. Therefore, $0 \le t_\varepsilon \le 1$ for ε small and some representative (t_ε) of t. Since $(x_\varepsilon, h_\varepsilon) \in_\varepsilon \mathrm{th}_{v_\varepsilon}(\Omega_\varepsilon)$, we have that $x_\varepsilon + t_\varepsilon h_\varepsilon v_\varepsilon \in \Omega_\varepsilon$ for ε small. If we take another representative $(y_\varepsilon) \sim_\rho (x_\varepsilon + t_\varepsilon h_\varepsilon v_\varepsilon)$, then we can define $x'_\varepsilon := y_\varepsilon - t_\varepsilon h_\varepsilon v_\varepsilon$ so that $(x_\varepsilon, h_\varepsilon) \sim_\rho (x'_\varepsilon, h_\varepsilon)$. From $(x_\varepsilon, h_\varepsilon) \in_\varepsilon \mathrm{th}_{v_\varepsilon}(\Omega_\varepsilon)$ we thus get that also $(x'_\varepsilon, h_\varepsilon) \in \mathrm{th}_{v_\varepsilon}(\Omega_\varepsilon)$ for ε small. Therefore $x'_\varepsilon + t_\varepsilon h_\varepsilon v_\varepsilon = y_\varepsilon \in \Omega_\varepsilon$ for ε small. This shows that $x + thv \in \langle\Omega_\varepsilon\rangle$.

Finally, in order to prove (iii), we assume that $U \subseteq \langle\Omega_\varepsilon\rangle$ is a sharply open subset. For all $x \in U \subseteq \langle\Omega_\varepsilon\rangle$, we have $(x, 0) \in \langle \mathrm{th}_{v_\varepsilon}(\Omega_\varepsilon)\rangle$ from (i), and hence Thm. 10 (iv) yields the existence of $c_x \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$ such that $B_{c_x}(x, 0) \subseteq \langle \mathrm{th}_{v_\varepsilon}(\Omega_\varepsilon)\rangle$. Since U is a neighborhood of x, there exists $a_x \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$, $a_x < c_x$, such that $B_{a_x}(x) \subseteq U$. Choose $b_x \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$ such that $b_x < a_x$ and $a_x + b_x < c_x$. Because $v \in {}^{\rho}\widetilde{\mathbb{R}}^n$ is ρ-moderate, we have $|v| < \mathrm{d}\rho^{-N}$ for some $N \in \mathbb{N}$. Take $d_x \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ such that $d_x < b_x \cdot \mathrm{d}\rho^N$ and define

$$T := \bigcup_{x \in U} B_{a_x}(x) \times B_{d_x}(0).$$

If $|h| < d_x$, then $|h \cdot v| < |v| \cdot d_x < b_x$ and hence T is a sharp thickening of U along v. We finally note that $(x', h) \in B_{a_x}(x) \times B_{d_x}(0)$ implies $|(x', h) - (x, 0)| \leq |x' - x| + |h| < a_x + d_x < a_x + b_x < c_x$, so that $(x', h) \in B_{c_x}(x, 0) \subseteq \langle \mathrm{th}_{v_\varepsilon}(\Omega_\varepsilon) \rangle$. Therefore $T \subseteq \langle \mathrm{th}_{v_\varepsilon}(\Omega_\varepsilon) \rangle$.

Considering $\sim_\mathrm{F}$ instead of $\sim_\rho$ and radii in $\mathbb{R}_{>0}$, in the same way we can prove the analogous properties for strongly internal sets in the Fermat topology. □

We can now prove the Fermat-Reyes theorem for GSF.

Proof of Theorem 33. Since U is sharply open, for any point $x \in U$ we can find a ball $B_{R_x}(x) \subseteq U$, $R_x \in {}^\rho\widetilde{\mathbb{R}}_{>0}$. Define $a_x := \frac{R_x}{2}$ and $b_x := \frac{a_x}{2}$. Because $v \in {}^\rho\widetilde{\mathbb{R}}^n$ is ρ-moderate, we have $|v| < \mathrm{d}\rho^{-N}$ for some $N \in \mathbb{N}$. Take $d_x \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ such that $d_x < b_x \cdot \mathrm{d}\rho^N$ and set $T := \bigcup_{x \in U} B_{a_x}(x) \times B_{d_x}(0)$. Since for all $x \in U$ the pair $(x, 0)$ is an interior point of the given thickening S, we can assume to have chosen a_x and d_x so that $T \subseteq S \subseteq \langle \mathrm{th}_{v_\varepsilon}(\Omega_\varepsilon) \rangle$. In case U is large open, we can proceed as above to obtain a_x, $d_x \in \mathbb{R}_{>0}$, so that T would then be a large thickening.
Let us consider the net of smooth function $r_\varepsilon \in \mathcal{C}^\infty(\mathrm{th}_{v_\varepsilon}(\Omega_\varepsilon))$ defined by $r_\varepsilon(y, h) := \int_0^1 \frac{\partial f_\varepsilon}{\partial v_\varepsilon}(y + thv_\varepsilon)\mathrm{d}t$ for all $\varepsilon \in I$. We calculate the partial derivative $\partial^\alpha r_\varepsilon(y_\varepsilon, h_\varepsilon)$ for $\alpha \in \mathbb{N}^{n+1}$ and an arbitrary point $(y, h) \in T$. For simplicity, set $\hat{\alpha} := (\alpha_1, \ldots, \alpha_n)$, and $v_\varepsilon =: (v_{1\varepsilon}, \ldots, v_{n\varepsilon}) \in \mathbb{R}^n$.

$$\partial^\alpha r_\varepsilon(y_\varepsilon, h_\varepsilon) = \int_0^1 \frac{\partial^{|\alpha|}}{\partial h^{\alpha_{n+1}} \partial y^{\hat{\alpha}}} \left[\frac{\partial f_\varepsilon}{\partial v_\varepsilon}(y_\varepsilon + th_\varepsilon v_\varepsilon) \right] \mathrm{d}t \tag{5.1}$$

Applying the chain rule and the mean value theorem for integrals, (5.1) can be written as a sum of terms of the form $\partial^\beta f(y_\varepsilon + t_\varepsilon h_\varepsilon v_\varepsilon) v_\varepsilon^\gamma t_\varepsilon^m$, for suitable multi-indices β, γ, and $m \in \mathbb{N}$. Here, $t_\varepsilon \in [0, 1]_\mathbb{R}$ for all $\varepsilon \in I$. From $(y, h) \in T$, we get $y \in B_{a_x}(x)$ and $h \in B_{d_x}(0)$ for some $x \in U$. This gives $|y + thv - x| \leq |y - x| + |hv| < a_x + b_x < R_x$, so that $y + thv \in B_{R_x}(x) \subseteq U$. From Def. 14 (ii) we hence have that $\left(\partial^\beta f_\varepsilon(y_\varepsilon + t_\varepsilon h_\varepsilon v_\varepsilon)\right)$ is ρ-moderate. Since moderateness is preserved by polynomials, and $t_\varepsilon \in [0, 1]_\mathbb{R}$ is moderate, from (5.1) we obtain that $(\partial^\alpha r_\varepsilon(y_\varepsilon, h_\varepsilon))$ is moderate. This proves that $r := [r_\varepsilon(-, -)]|_T : T \longrightarrow {}^\rho\widetilde{\mathbb{R}}$ is a GSF. We have

$$\begin{aligned} h \cdot r(x, h) &= \left[h_\varepsilon \cdot \int_0^1 \frac{\partial f_\varepsilon}{\partial v_\varepsilon}(x_\varepsilon + th_\varepsilon v_\varepsilon)\, \mathrm{d}t \right] \\ &= \left[\int_0^{h_\varepsilon} \frac{d}{ds} \left\{ f_\varepsilon(x_\varepsilon + sv_\varepsilon) \right\}(s)\, \mathrm{d}s \right] \\ &= [f_\varepsilon(x_\varepsilon + h_\varepsilon v_\varepsilon)] - [f_\varepsilon(x_\varepsilon)] = f(x + hv) - f(x). \end{aligned}$$

Of course $r(x, 0) = \left[\frac{\partial f_\varepsilon}{\partial v_\varepsilon}(x_\varepsilon) \right]$, and this concludes the existence part.
To prove uniqueness, consider $(x, h) \in T \cap \bar{T}$, where T and $\bar{T}$ are two thickenings (along v) of the incremental ratios r, $\bar{r}$. Define $R(k) := r(x, k)$ and $\bar{R}(k) := \bar{r}(x, k)$ for $k \in B_b(0)$, where $(x, h) \in B_a(x) \times B_b(0) \subseteq T \cap \bar{T}$ by the definition of thickening.

Since $k \in B_b(0) \mapsto (x,k) \in T \cap \bar{T}$ is a GSF, both R and $\bar{R}$ are still generalized smooth maps by the closure with respect to composition. Moreover

$$k \cdot R(k) = k \cdot r(x,k) = f(x+kv) - f(x), \tag{5.2}$$

and analogously $k \cdot \bar{R}(k) = f(x+kv) - f(x) = k \cdot R(k)$. Therefore $R(k) = \bar{R}(k)$ for every $k \in B_b(0)$ which is invertible, and Lemma 34 yields $R = \bar{R}$. Since $h \in B_b(0)$ we get $R(h) = r(x,h) = \bar{R}(h) = \bar{r}(x,h)$. □

We will use the notation $\frac{\partial f}{\partial v}[-,-]_T \in {}^\rho\mathcal{GC}^\infty(T, {}^\rho\widetilde{\mathbb{R}})$ (or simply $\frac{\partial f}{\partial v}[-,-]$ in case the domain is clear from the context) for the generalized smooth incremental ratio of a function $f \in {}^\rho\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}})$ defined on the thickening T, so as to distinguish it from the derivative $\frac{\partial f}{\partial v} \in {}^\rho\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}})$. Since any partial derivative of a GSF is still a GSF, higher order derivatives $\frac{\partial^\alpha f}{\partial v^\alpha} \in {}^\rho\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}})$ are simply defined recursively.

As follows from Thm. 33 (i) and Thm. 25 (v), the concept of derivative defined using the Fermat-Reyes theorem is compatible with the classical derivative of Schwartz distributions via the embeddings ι^b from Thm. 25. The following result follows from the analogous properties for the nets of smooth functions defining f and g.

Theorem 36. *Let $U \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ be an open subset in the sharp topology, let $v \in {}^\rho\widetilde{\mathbb{R}}^n$ and $f,\, g : U \longrightarrow {}^\rho\widetilde{\mathbb{R}}$ be generalized smooth maps. Then*

(i) $\frac{\partial(f+g)}{\partial v} = \frac{\partial f}{\partial v} + \frac{\partial g}{\partial v}$
(ii) $\frac{\partial(r\cdot f)}{\partial v} = r \cdot \frac{\partial f}{\partial v} \quad \forall r \in {}^\rho\widetilde{\mathbb{R}}$
(iii) $\frac{\partial(f\cdot g)}{\partial v} = \frac{\partial f}{\partial v} \cdot g + f \cdot \frac{\partial g}{\partial v}$
(iv) *For each $x \in U$, the map $\mathrm{d}f(x).v := \frac{\partial f}{\partial v}(x) \in {}^\rho\widetilde{\mathbb{R}}$ is ${}^\rho\widetilde{\mathbb{R}}$-linear in $v \in {}^\rho\widetilde{\mathbb{R}}^n$.*

Using the Fermat-Reyes theorem, it is also possible to give intrinsic proofs (i.e. without using nets of smooth functions that define a given GSF), as exemplified in the following

Theorem 37. *Let $U \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ and $V \subseteq {}^\rho\widetilde{\mathbb{R}}^d$ be open subsets in the sharp topology and $g \in {}^\rho\mathcal{GC}^\infty(V,U)$, $f \in {}^\rho\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}})$ be generalized smooth maps. Then for all $x \in V$ and all $v \in {}^\rho\widetilde{\mathbb{R}}^d$*

$$\frac{\partial\,(f\circ g)}{\partial v}(x) = \mathrm{d}f\,(g(x))\,.\frac{\partial g}{\partial v}(x)$$
$$\mathrm{d}(f\circ g)\,(x) = \mathrm{d}f\,(g(x)) \circ \mathrm{d}g(x).$$

Proof. For h small (in the sharp topology), we can write

$$f\,[g(x+hv)] = f\left[g(x) + h\frac{\partial g}{\partial v}[x,h]\right]. \tag{5.3}$$

Set $u(x,h) := \frac{\partial g}{\partial v}[x,h] \in {}^\rho\widetilde{\mathbb{R}}^n$. Then (5.3) yields

$$f\,[g(x+hv)] = f(g(x)) + h \cdot \frac{\partial f}{\partial u(x,h)}\,[g(x),h]\,.$$

Therefore, the uniqueness of the smooth incremental ratio of $f\circ g$ in the direction v implies

$$\frac{\partial\,(f\circ g)}{\partial v}[x,h] = \frac{\partial f}{\partial u(x,h)}\,[g(x),h]\,.$$

For $h=0$, we get

$$\frac{\partial\left(f\circ g\right)}{\partial v}(x)=\frac{\partial f}{\partial u(x,0)}\left(g(x)\right)=\mathrm{d}f(g(x)).u(x,0)=\mathrm{d}f(g(x)).\frac{\partial g}{\partial v}(x),$$

which is our conclusion. □

6. Integral calculus using primitives

In this section, we inquire existence and uniqueness of primitives F of a GSF $f\in{}^{\rho}\mathcal{GC}^{\infty}([a,b],{}^{\rho}\widetilde{\mathbb{R}})$ (see also [121] for an analogous approach). To this end, we shall have to introduce the derivative $F'(x)$ at boundary points $x\in[a,b]$, i.e. such that $x-a$ or $b-x$ is not invertible. Let us note explicitly, in fact, that the Fermat-Reyes Theorem 33 is stated only for sharply open domains. We shall therefore require the following result.

Lemma 38. *Let a, $b\in{}^{\rho}\widetilde{\mathbb{R}}$ be such that $a<b$. Then the interior $\mathrm{int}([a,b])$ in the sharp topology is dense in $[a,b]$.*

Proof. Take representatives of a, b and $x\in[a,b]$ such that $a_{\varepsilon}<b_{\varepsilon}$ and $a_{\varepsilon}\le x_{\varepsilon}\le b_{\varepsilon}$ for ε small. Thm. 10 (ii) yields $\mathrm{int}\left([a,b]\right)=\langle(a_{\varepsilon},b_{\varepsilon})\rangle$. To prove the conclusion, it suffices to define

$$y_{k\varepsilon}:=\begin{cases}x_{\varepsilon} & \text{if }a_{\varepsilon}+\rho_{\varepsilon}^{k}\le x_{\varepsilon}\le b_{\varepsilon}-\rho_{\varepsilon}^{k}\\ a_{\varepsilon}+\rho_{\varepsilon}^{k} & \text{if }x_{\varepsilon}<a_{\varepsilon}+\rho_{\varepsilon}^{k}\\ b_{\varepsilon}-\rho_{\varepsilon}^{k} & \text{if }x_{\varepsilon}>b_{\varepsilon}-\rho_{\varepsilon}^{k}\end{cases}$$

for any $k\in\mathbb{N}$ and $\varepsilon\in I$. We have $d(y_{k\varepsilon},(a_{\varepsilon},b_{\varepsilon})^{c})\ge\rho_{\varepsilon}^{k}$, so that $y_{k}\in\langle(a_{\varepsilon},b_{\varepsilon})\rangle$. Moreover, $|y_{k\varepsilon}-x_{\varepsilon}|<\rho_{\varepsilon}^{k}$ for all ε, and from this the desired limit condition follows. □

The following result shows that every GSF can have at most one primitive GSF up to an additive constant.

Theorem 39. *Let $X\subseteq{}^{\rho}\widetilde{\mathbb{R}}$ and let $f\in{}^{\rho}\mathcal{GC}^{\infty}(X,{}^{\rho}\widetilde{\mathbb{R}})$ be a generalized smooth function. Let a, $b\in{}^{\rho}\widetilde{\mathbb{R}}$, with $a<b$, such that $(a,b)\subseteq X$. If $f'(x)=0$ for all $x\in\mathrm{int}(a,b)$, then f is constant on (a,b). An analogous statement holds if we take any other type of interval (closed or half closed) instead of (a,b).*

Proof. By Lemma 18, we can assume that f is defined by a net of smooth functions $f_{\varepsilon}\in\mathcal{C}^{\infty}(\mathbb{R},\mathbb{R})$. From the Fermat-Reyes Theorem 33, we know that $f'(x)=[f'_{\varepsilon}(x_{\varepsilon})]$ for every interior point $x=[x_{\varepsilon}]\in X$. For all x, $y\in\mathrm{int}(a,b)\subseteq X$, we can write

$$\begin{aligned}f(x)-f(y)=[f_{\varepsilon}(x_{\varepsilon})-f_{\varepsilon}(y_{\varepsilon})]&=\left[(y_{\varepsilon}-x_{\varepsilon})\cdot\int_{0}^{1}f'_{\varepsilon}(x_{\varepsilon}+s(y_{\varepsilon}-x_{\varepsilon}))\,\mathrm{d}s\right]\\&=(y-x)\cdot[f'_{\varepsilon}(x_{\varepsilon}+s_{\varepsilon}(y_{\varepsilon}-x_{\varepsilon}))]=(y-x)\cdot f'(x+s(y-x)),\end{aligned}\tag{6.1}$$

where $s_{\varepsilon}\in[0,1]_{\mathbb{R}}$ is provided by the integral mean value theorem and $s:=[s_{\varepsilon}]\in[0,1]$. Since x, $y\in\mathrm{int}(a,b)$, we have $x+s(y-x)\in\mathrm{int}(a,b)$ and hence $f'(x+s(y-x))=0$. Thereby, (6.1) yields $f(x)=f(y)$ as claimed. For a different type of interval, it suffices to consider Lemma 38 and sharp continuity of GSF (Thm. 17). □

Remark 40. From the Fermat-Reyes Thm. 33 and from Thm. 39, it follows that the function $i(x) := 1$ if $x \approx 0$ and $i(x) := 0$ otherwise cannot be a GSF on any large neighborhood of $x = 0$. This example stems from the property that different standard real numbers can always be separated by infinitesimal balls.

At interior points $x \in [a, b]$ in the sharp topology, the definition of derivative $f^{(k)}(x)$ follows from the Fermat-Reyes Theorem 33. At boundary points, we have the following

Theorem 41. *Let $a, b \in {}^\rho\widetilde{\mathbb{R}}$ with $a < b$, and $f \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}})$ be a generalized smooth function. Then for all $x \in [a, b]$, the following limit exists in the sharp topology*

$$\lim_{\substack{y\to x\\ y\in\mathrm{int}([a,b])}} f^{(k)}(y) =: f^{(k)}(x).$$

Moreover, if the net $f_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R})$ defines f and $x = [x_\varepsilon]$, then $f^{(k)}(x) = [f_\varepsilon^{(k)}(x_\varepsilon)]$ and hence $f^{(k)} \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}})$.

Proof. We have

$$\lim_{\substack{y\to x\\ y\in\mathrm{int}([a,b])}} f^{(k)}(y) = \lim_{\substack{y\to x\\ y\in\mathrm{int}([a,b])}} \left[f_\varepsilon^{(k)}(y_\varepsilon)\right] = [f_\varepsilon^{(k)}(x_\varepsilon)],$$

where the last equality follows due to the sharp continuity of $[f_\varepsilon^{(k)}(-)]$ at every point $x \in [a, b] \subseteq \langle\Omega_\varepsilon\rangle$ (Thm. 17 (iii) and Lem. 38). □

We can now prove existence and uniqueness of primitives of GSF:

Theorem 42. *Let $a, b, c \in {}^\rho\widetilde{\mathbb{R}}$, with $a < b$ and $c \in [a, b]$. Let $f \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}})$ be a generalized smooth function. Then, there exists one and only one generalized smooth function $F \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}})$ such that $F(c) = 0$ and $F'(x) = f(x)$ for all $x \in [a, b]$. Moreover, if f is defined by the net $f_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ and $c = [c_\varepsilon]$, then $F(x) = \left[\int_{c_\varepsilon}^{x_\varepsilon} f_\varepsilon(s)\,\mathrm{d}s\right]$ for all $x = [x_\varepsilon] \in [a, b]$.*

Proof. Fix representatives (a_ε), (b_ε) and (c_ε) of a, b, c such that

$$a_\varepsilon \le c_\varepsilon \le b_\varepsilon \tag{6.2}$$

for ε small. By Lemma 18, we can assume that f is generated by a net $f_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$. Set

$$F_\varepsilon(x) := \int_{c_\varepsilon}^{x} f_\varepsilon(s)\,\mathrm{d}s \quad \forall x \in \mathbb{R}. \tag{6.3}$$

We want to prove that the net (F_ε) defines a GSF of type $[a, b] \longrightarrow {}^\rho\widetilde{\mathbb{R}}$, and therefore we take $x \in [a, b]$ and $\alpha \in \mathbb{N}$. Choose a representative (x_ε) of x such that

$$a_\varepsilon \le x_\varepsilon \le b_\varepsilon \tag{6.4}$$

for ε small. If $\alpha > 0$, then $F_\varepsilon^{(\alpha)}(x_\varepsilon) = f_\varepsilon^{(\alpha-1)}(x_\varepsilon)$ and hence moderateness is clear since $x \in [a, b]$. For $\alpha = 0$ we have $F_\varepsilon(x_\varepsilon) = f_\varepsilon(\sigma_\varepsilon) \cdot (x_\varepsilon - c_\varepsilon)$, where

$$\sigma_\varepsilon \in [c_\varepsilon, x_\varepsilon] \cup [x_\varepsilon, c_\varepsilon] \quad \forall \varepsilon \in I \tag{6.5}$$

is obtained by the integral mean value theorem. For ε small, we have both (6.2) and (6.4), so that these inequalities and (6.5) yield $\sigma \in [a, b] \subseteq U$. Therefore $(f_\varepsilon(\sigma_\varepsilon))$ and $(F_\varepsilon(x_\varepsilon))$ are moderate. This proves condition Def. 14 (ii) for the net (F_ε), and

we can hence set $F(x) := [F_\varepsilon(x_\varepsilon)] \in {}^\rho\widetilde{\mathbb{R}}$ for all $x = [x_\varepsilon] \in [a, b]$.
If $y \in \text{int}([a, b])$, we can apply our differential calculus to the generalized smooth map $F|_{\text{int}([a,b])} = [F_\varepsilon(-)]|_{\text{int}([a,b])}$, obtaining $F'(y) = [f_\varepsilon(y_\varepsilon)] = f(y)$. From this, if $x \in [a, b]$, we get

$$F'(x) = \lim_{\substack{y\to x\\ y\in\text{int}([a,b])}} F'(y) = \lim_{\substack{y\to x\\ y\in\text{int}([a,b])}} f(y) = f(x)$$

because f is sharply continuous at $x \in [a, b] \subseteq U$. The uniqueness part follows from Theorem 39. □

Definition 43. Under the assumptions of Theorem 42, we denote by $\int_c^{(-)} f := \int_c^{(-)} f(s)\,\mathrm{d}s \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}})$ the unique generalized smooth function such that:

(i) $\int_c^c f = 0$

(ii) $\left(\int_c^{(-)} f\right)'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_c^x f(s)\,\mathrm{d}s = f(x)$ for all $x \in [a, b]$.

In Sec. 8, we develop a generalization of this concept of integration to GSF in several variables and to more general domains of integration $M \subseteq {}^\rho\widetilde{\mathbb{R}}^d$.

Example 44.

(i) Since ${}^\rho\widetilde{\mathbb{R}}$ contains both infinitesimal and infinite numbers, our notion of definite integral also includes "improper integrals". Let e.g. $f(x) = \frac{1}{x}$ for $x \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ and $a = 1$, $b = \mathrm{d}\rho^{-q}$, $q > 0$. Then

$$\int_a^b f(s)\,\mathrm{d}s = \left[\int_1^{\rho_\varepsilon^{-q}} \frac{1}{s}\,\mathrm{d}s\right] = [\log \rho_\varepsilon^{-q}] - \log 1 = -q \log \mathrm{d}\rho, \tag{6.6}$$

which is, of course, a positive infinite generalized number. This apparently trivial result is closely tied to the possibility to define GSF on arbitrary domains, like $F \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}})$ in Thm. 42 where b is an infinite number as in (6.6), which is one of the key properties allowing one to get the closure with respect to composition.

(ii) If p, $q \in {}^\rho\widetilde{\mathbb{R}}$, $p < 0 < q$ and both p and q are not infinitesimal, then $\int_p^q \delta(t)\,\mathrm{d}t \approx 1$. If $p \le -r$ and $q \ge s$ where r, $s \in \mathbb{R}_{>0}$, then $\int_p^q \delta(t)\,\mathrm{d}t = 1$.

Theorem 45. *Let $f \in {}^\rho\mathcal{GC}^\infty(X, {}^\rho\widetilde{\mathbb{R}})$ and $g \in {}^\rho\mathcal{GC}^\infty(Y, {}^\rho\widetilde{\mathbb{R}})$ be generalized smooth functions defined on arbitrary domains in ${}^\rho\widetilde{\mathbb{R}}$. Let a, $b \in {}^\rho\widetilde{\mathbb{R}}$ with $a < b$ and $[a, b] \subseteq X \cap Y$. Then*

(i) $\int_a^b (f + g) = \int_a^b f + \int_a^b g$

(ii) $\int_a^b \lambda f = \lambda \int_a^b f \quad \forall \lambda \in {}^\rho\widetilde{\mathbb{R}}$

(iii) $\int_a^b f = \int_a^c f + \int_c^b f$ *for all* $c \in [a, b]$

(iv) $\int_a^b f = -\int_b^a f$

(v) $\int_a^b f' = f(b) - f(a)$

(vi) $\int_a^b f' \cdot g = [f \cdot g]_a^b - \int_a^b f \cdot g'$

(vii) *If* $f(x) \le g(x)$ *for all* $x \in [a, b]$*, then* $\int_a^b f \le \int_a^b g$.

Proof. This follows directly from (6.3) and the usual rules of the integral calculus, or from Def. 43 and Thm. 33 for property (vii). □

Theorem 46. *Let $f \in {}^\rho\mathcal{GC}^\infty(T, {}^\rho\widetilde{\mathbb{R}})$ and $\varphi \in {}^\rho\mathcal{GC}^\infty(S, T)$ be generalized smooth functions defined on arbitrary domains in ${}^\rho\widetilde{\mathbb{R}}$. Let a, $b \in {}^\rho\widetilde{\mathbb{R}}$, with $a < b$, such that $[a, b] \subseteq S$, $\varphi(a) < \varphi(b)$ and $[\varphi(a), \varphi(b)] \subseteq T$. Finally, assume that $\varphi([a, b]) \subseteq [\varphi(a), \varphi(b)]$. Then*

$$\int_{\varphi(a)}^{\varphi(b)} f(t)\,\mathrm{d}t = \int_a^b f[\varphi(s)] \cdot \varphi'(s)\,\mathrm{d}s.$$

Proof. Define

$$F(x) := \int_{\varphi(a)}^{x} f(t)\,\mathrm{d}t \quad \forall x \in [\varphi(a), \varphi(b)]$$
$$H(y) := \int_{\varphi(a)}^{\varphi(y)} f(t)\,\mathrm{d}t \quad \forall y \in [a, b]$$
$$G(y) := \int_a^y f[\varphi(s)] \cdot \varphi'(s)\,\mathrm{d}s \quad \forall y \in [a, b],$$

Each one of these functions is generalized smooth by Def. 43 of the integral or by Thm. 28, because it can be written as a composition of generalized smooth maps. We have $H(a) = G(a) = 0$, $H(y) = F[\varphi(y)]$ for every $y \in [a, b]$ and, by the chain rule (Prop. 37) $H'(y) = F'[\varphi(y)] \cdot \varphi'(y) = f[\varphi(y)] \cdot \varphi'(y) = G'(y)$, the last two equalities following by Def. 43 of the integral. From the uniqueness Theorem 39, the conclusion $H = G$ follows. □

Remark 47. (Relation to distributional primitives) Let $a, b \in \mathbb{R}$, $a < b$, and set $\Omega = (a, b) \subseteq \mathbb{R}$. By [105, Ch. II, §4] there exists a sequentially continuous operator $R : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ assigning to any $T \in \mathcal{D}'(\Omega)$ a primitive $R(T)$, i.e., $R(T)' = T$ in $\mathcal{D}'(\Omega)$. Now let $\iota := \iota^b_\Omega : \mathcal{D}'(\Omega) \to {}^\rho\mathcal{GC}^\infty(\mathrm{c}(\Omega), {}^\rho\widetilde{\mathbb{R}})$ be an embedding as in Theorem 25, and fix any $c \in {}^\rho\widetilde{\mathbb{R}}$ with $a \le c \le b$. Then

$$\iota(R(T))' = \iota(R(T)') = \iota(T) = \Big(\int_c^{(-)} \iota(T)\Big)'.$$

Therefore, Theorem 45(v) implies that

$$\int_r^s \iota(T) = \iota(R(T))(s) - \iota(R(T))(r)$$

for all s, $t \in {}^\rho\widetilde{\mathbb{R}}$ with $a \le s$, $t \le b$.

7. Some classical theorems for generalized smooth functions

It is natural to expect that several classical theorems of differential and integral calculus can be extended from the ordinary smooth case to the generalized smooth framework. Once again, we underscore that these faithful generalizations are possible because we do not have a priori limitations in the evaluation $f(x)$ for GSF. For example, one does not have similar results in Colombeau theory, where an arbitrary generalized function can be evaluated only at compactly supported points.
We start from the intermediate value theorem.

Corollary 48. *Let $f \in {}^\rho\mathcal{GC}^\infty(X, {}^\rho\widetilde{\mathbb{R}})$ be a generalized smooth function defined on the subset $X \subseteq {}^\rho\widetilde{\mathbb{R}}$. Let a, $b \in {}^\rho\widetilde{\mathbb{R}}$, with $a < b$, such that $[a, b] \subseteq X$. Assume that $f(a) < f(b)$. Then*

$$\forall y \in {}^\rho\widetilde{\mathbb{R}} : \ f(a) \le y \le f(b) \ \Rightarrow \ \exists c \in [a, b] : \ y = f(c).$$

Proof. Let f be defined by the net $f_\varepsilon \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$. For small ε and for suitable representatives (a_ε), (b_ε), (y_ε), we have

$$a_\varepsilon < b_\varepsilon \quad , \quad f_\varepsilon(a_\varepsilon) \le y_\varepsilon \le f_\varepsilon(b_\varepsilon).$$

By the classical intermediate value theorem we get some $c_\varepsilon \in [a_\varepsilon, b_\varepsilon]$ such that $f_\varepsilon(c_\varepsilon) = y_\varepsilon$. Therefore $c := [c_\varepsilon] \in [a,b] \subseteq X$ and hence $f(c) = [f_\varepsilon(c_\varepsilon)] = [y_\varepsilon] = y$. □

Using this theorem we can conclude that no GSF can assume only a finite number of values which are comparable with respect to the relation $<$ on any nontrivial interval $[a,b] \subseteq X$, unless it is constant. For example, this provides an alternative way of seeing that the function i of Rem. 40 cannot be a generalized smooth map. We note that the solution $c \in [a,b]$ of the previous generalized smooth equation $y = f(x)$ need not even be continuous in ε. Indeed, let us consider the net of smooth functions depicted in Figure 7.1, where it is understood that, as ε approaches 0, the two waves at the extremes oscillate around the dashed rectilinear positions shown in the figure. Set $f(x) = \left[\int_0^1 f_\varepsilon(s)\,\mathrm{d}s - f_\varepsilon(x_\varepsilon)\right] \in {}^\rho\widetilde{\mathbb{R}}$ for $x \in [0,1] \subseteq {}^\rho\widetilde{\mathbb{R}}$, and analyze the generalized smooth equation $f(x) = 0$. Let $\varepsilon_k = \frac{1}{k}$ be the "times" where the two waves of the net (f_ε) are rectilinear. At these times the solution $f(x_{\varepsilon_k}) = 0$ can be any point $x_{\varepsilon_k} \in [b,c]$. Assume that for $\varepsilon \in \left[\frac{1}{k}, \frac{1}{k} + \delta_k\right]$ only the wave on the left is rectilinear and for $\varepsilon \in \left[\frac{1}{k} - \delta_k, \frac{1}{k}\right]$ only the wave on the right is rectilinear (where $\delta_k \downarrow 0$ is sufficiently small). Therefore, in the first case, any solution must be of the form $x_\varepsilon \in [c,1]$ and in the second case $x_\varepsilon \in [0,b]$. Thus any solution must jump at every time ε_k and the height of the jump must be at least $c - b$.

This example allows us to draw the following general conclusion: if we consider generalized numbers as solutions of smooth equations, then we are forced to work on a non-totally ordered ring of scalars derived from discontinuous (in ε) representatives. To put it differently: if we choose a ring of scalars with a total order or continuous representatives, we will not be able to solve every smooth equation, and the given ring can be considered, in some sense, incomplete. Of course, this does not mean that the study of better behaved (non-totally ordered) subrings of ${}^\rho\widetilde{\mathbb{R}}$, useful for special purposes, is not interesting.

Theorem 49. *Let $f \in {}^\rho\mathcal{GC}^\infty(X, {}^\rho\widetilde{\mathbb{R}}^d)$ be a generalized smooth function defined in the sharply open set $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$. Let a, $b \in {}^\rho\widetilde{\mathbb{R}}^n$ such that $[a,b] \subseteq X$. Then*

(i) If $n = d = 1$, then $\exists c \in [a,b] :\ f(b) - f(a) = (b-a)\cdot f'(c)$.
(ii) If $n = d = 1$, then $\exists c \in [a,b] :\ \int_a^b f(t)\,\mathrm{d}t = (b-a)\cdot f(c)$.
(iii) If $d = 1$, then $\exists c \in [a,b] :\ f(b) - f(a) = \nabla f(c)\cdot(b-a)$.
(iv) Let $h := b - a$. Then $f(a+h) - f(a) = \int_0^1 \mathrm{d}f(a + t\cdot h).h\,\mathrm{d}t$.

Proof. Using the usual notations, for small ε we have $a_\varepsilon < b_\varepsilon$ and

$$\exists c_\varepsilon \in [a_\varepsilon, b_\varepsilon] :\ f_\varepsilon(b_\varepsilon) - f_\varepsilon(a_\varepsilon) = (b_\varepsilon - a_\varepsilon)\cdot f'_\varepsilon(c_\varepsilon) \tag{7.1}$$

$$\exists c_\varepsilon \in [a_\varepsilon, b_\varepsilon] :\ \int_{a_\varepsilon}^{b_\varepsilon} f_\varepsilon = (b_\varepsilon - a_\varepsilon)\cdot f_\varepsilon(c_\varepsilon), \tag{7.2}$$

from which the conclusions (i) and (ii) follow directly. The several variables and vector valued cases (iii), (iv) follow as usual by reduction to the one-variable and scalar valued case. □

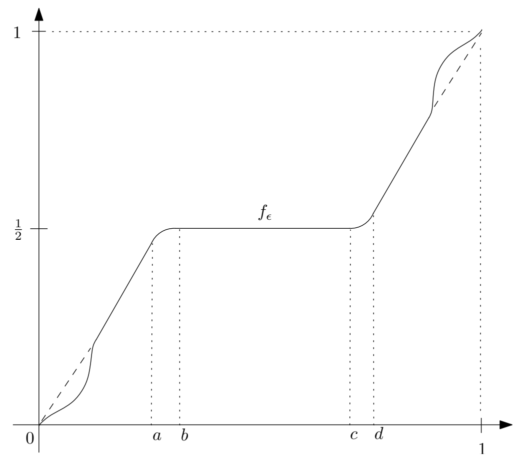

FIGURE 7.1. A net (f_ε) defining a discontinuous solution of a smooth equation.

Internal sets generated by a sharply bounded net of compact sets serve as a substitute for compact subsets for GSF, as can be seen from the following extreme value theorem:

Lemma 50. *Let $\emptyset \neq K = [K_\varepsilon] \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ be an internal set generated by a sharply bounded net (K_ε) of compact sets $K_\varepsilon \Subset \mathbb{R}^n$ Assume that $\alpha : K \longrightarrow {}^\rho\widetilde{\mathbb{R}}$ is a well-defined map given by $\alpha(x) = [\alpha_\varepsilon(x_\varepsilon)]$ for all $x \in K$, where $\alpha_\varepsilon : K_\varepsilon \longrightarrow \mathbb{R}$ are continuous maps (e.g. $\alpha(x) = |x|$). Then*

$$\exists m, M \in K\, \forall x \in K : \ \alpha(m) \leq \alpha(x) \leq \alpha(M).$$

Proof. Since $K \neq \emptyset$, for ε sufficiently small, let us say for $\varepsilon \in (0, \varepsilon_0]$, K_ε is non empty and, by our assumptions, it is also compact. Since each α_ε is continuous, for all $\varepsilon \in (0, \varepsilon_0]$ we have

$$\exists m_\varepsilon, M_\varepsilon \in K_\varepsilon\, \forall x \in K_\varepsilon : \ \alpha_\varepsilon(m_\varepsilon) \leq \alpha_\varepsilon(x) \leq \alpha_\varepsilon(M_\varepsilon).$$

Since the net (K_ε) is sharply bounded, both the nets (m_ε) and (M_ε) are moderate. Therefore $m = [m_\varepsilon]$, $M = [M_\varepsilon] \in K$. Take any $x \in [K_\varepsilon]$, then there exists a representative (x_ε) such that $x_\varepsilon \in K_\varepsilon$ for ε small. Therefore $\alpha(m) = [\alpha_\varepsilon(m_\varepsilon)] \leq [\alpha_\varepsilon(x_\varepsilon)] = \alpha(x) \leq \alpha(M)$. $\square$

Corollary 51. *Let $f \in {}^\rho\mathcal{GC}^\infty(X, {}^\rho\widetilde{\mathbb{R}})$ be a generalized smooth function defined in the subset $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$. Let $\emptyset \neq K = [K_\varepsilon] \subseteq X$ be an internal set generated by a sharply bounded net (K_ε) of compact sets $K_\varepsilon \Subset \mathbb{R}^n$. Then*

$$\exists m, M \in K\, \forall x \in K : \ f(m) \leq f(x) \leq f(M). \tag{7.3}$$

These results motivate the following

Definition 52. A subset K of ${}^\rho\widetilde{\mathbb{R}}^n$ is called *functionally compact*, denoted by $K \Subset_{\mathrm{f}} {}^\rho\widetilde{\mathbb{R}}^n$, if there exists a net (K_ε) such that

(i) $K = [K_\varepsilon] \subseteq {}^\rho\widetilde{\mathbb{R}}^n$
(ii) (K_ε) is sharply bounded
(iii) $\forall \varepsilon \in I : \ K_\varepsilon \Subset \mathbb{R}^n$

If, in addition, $K \subseteq U \subseteq {}^{\rho}\widetilde{\mathbb{R}}^n$ then we write $K \Subset_{\mathrm{f}} U$. Finally, we write $[K_\varepsilon] \Subset_{\mathrm{f}} U$ if (ii), (iii) and $[K_\varepsilon] \subseteq U$ hold.

We refer to [41] for a deeper study of this type of compact sets in the case $\rho = (\varepsilon)$. Note that any interval $[a,b] \subseteq {}^{\rho}\widetilde{\mathbb{R}}$ with $b-a \in \mathbb{R}_{>0}$, is *not* connected: in fact if $c \in (a,b)$, then both $c + D_\infty$ and $[a,b] \setminus (c + D_\infty)$ are sharply open in $[a,b]$. Once again, this is a general property in several non-Archimedean frameworks (see e.g. [93, 62]). On the other hand, as in the case of functionally compact sets, GSF behave on intervals as if they were connected, in the sense that both the intermediate value theorem Cor. 48 and the extreme value theorem Cor. 51 hold for them (therefore, $f([a,b]) = [f(m), f(M)]$, where we used the notations from the results just mentioned).

We close this section with generalizations of Taylor's theorem in various forms. In the following statement, $\mathrm{d}^k f(x) : {}^{\rho}\widetilde{\mathbb{R}}^{dk} \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}$ is the k-th differential of the GSF f, viewed as an ${}^{\rho}\widetilde{\mathbb{R}}$-multilinear map ${}^{\rho}\widetilde{\mathbb{R}}^d \times \overset{k}{\ldots\ldots} \times {}^{\rho}\widetilde{\mathbb{R}}^d \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}$, and we use the common notation $\mathrm{d}^k f(x) \cdot h^k := \mathrm{d}^k f(x)(h, \ldots, h)$. Clearly, $\mathrm{d}^k f(x) \in {}^{\rho}\mathcal{GC}^\infty({}^{\rho}\widetilde{\mathbb{R}}^{dk}, {}^{\rho}\widetilde{\mathbb{R}})$. For multilinear maps $A : {}^{\rho}\widetilde{\mathbb{R}}^p \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}^q$, we set $|A| := [|A_\varepsilon|] \in {}^{\rho}\widetilde{\mathbb{R}}$, the generalized number defined by the norms of the operators $A_\varepsilon : \mathbb{R}^p \longrightarrow \mathbb{R}^q$.

Theorem 53. *Let $f \in {}^{\rho}\mathcal{GC}^\infty(U, {}^{\rho}\widetilde{\mathbb{R}})$ be a generalized smooth function defined in the sharply open set $U \subseteq {}^{\rho}\widetilde{\mathbb{R}}^d$. Let a, $b \in {}^{\rho}\widetilde{\mathbb{R}}^d$ such that the line segment $[a,b] \subseteq U$, and set $h := b - a$. Then, for all $n \in \mathbb{N}$ we have*

(i) $\exists \xi \in [a,b] : \ f(a+h) = \sum_{j=0}^{n} \frac{\mathrm{d}^j f(a)}{j!} \cdot h^j + \frac{\mathrm{d}^{n+1} f(\xi)}{(n+1)!} \cdot h^{n+1}$.

(ii) $f(a+h) = \sum_{j=0}^{n} \frac{\mathrm{d}^j f(a)}{j!} \cdot h^j + \frac{1}{n!} \cdot \int_0^1 (1-t)^n \, \mathrm{d}^{n+1} f(a+th) \cdot h^{n+1} \, \mathrm{d}t$.

Moreover, there exists some $R \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$ such that

$$\forall k \in B_R(0)\, \exists \xi \in [a, a+k] : \ f(a+k) = \sum_{j=0}^{n} \frac{\mathrm{d}^j f(a)}{j!} \cdot k^j + \frac{\mathrm{d}^{n+1} f(\xi)}{(n+1)!} \cdot k^{n+1} \tag{7.4}$$

$$\frac{\mathrm{d}^{n+1} f(\xi)}{(n+1)!} \cdot k^{n+1} = \frac{1}{n!} \cdot \int_0^1 (1-t)^n \, \mathrm{d}^{n+1} f(a+tk) \cdot k^{n+1} \, \mathrm{d}t \approx 0. \tag{7.5}$$

Formulas (i) and (ii) correspond to a plain generalization of Taylor's theorem for ordinary smooth functions with Lagrange and integral remainder, respectively. Dealing with generalized functions, it is important to note that this direct statement also includes the possibility that the differential $\mathrm{d}^{n+1} f(\xi)$ may be infinite at some point. For this reason, in (7.4) and (7.5), considering a sufficiently small increment k, we get more classical infinitesimal remainders $\mathrm{d}^{n+1} f(\xi) \cdot k^{n+1} \approx 0$.

Proof. Let $f_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R})$ be a net of smooth functions that defines f. We have $a + h = b \in [a,b] \subseteq U$ and U is sharply open, so by the Taylor formula applied to f_ε and by Theorem 33 we have

$$\begin{aligned} f(a+h) &= [f_\varepsilon(a_\varepsilon + h_\varepsilon)] \\ &= \left[\sum_{j=0}^{n} \frac{\mathrm{d}^j f_\varepsilon(a_\varepsilon)}{j!} h_\varepsilon^j + \frac{\mathrm{d}^{n+1} f_\varepsilon(\xi_\varepsilon)}{(n+1)!} h_\varepsilon^{n+1} \right] \\ &= \sum_{j=0}^{n} \frac{\mathrm{d}^j f(a)}{j!} h^j + \frac{\mathrm{d}^{n+1} f(\xi)}{(n+1)!} h^{n+1} \end{aligned}$$

for some $\xi_\varepsilon \in (a_\varepsilon, b_\varepsilon)$, and where $\xi = [\xi_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}$ so that $\xi \in [a, b]$. Analogously, we can prove (ii).

To prove the second part of the theorem, we start by considering a sharp ball $B_r(a) \subseteq U$, where $r = [r_\varepsilon] > 0$. Set $H := \left[\overline{B^{\mathrm{E}}_{r_\varepsilon/2}(a_\varepsilon)}\right]$, and

$$K := \max\left(\left|\mathrm{d}^{n+1}f(M)\right|, \left|\mathrm{d}^{n+1}f(m)\right|\right) \in {}^\rho\widetilde{\mathbb{R}},$$

where $\mathrm{d}^{n+1}f(M)$ and $\mathrm{d}^{n+1}f(m)$ are the maximum and the minimum values of the GSF $\mathrm{d}^{n+1}f : U \times {}^\rho\widetilde{\mathbb{R}}^{d(n+1)} \longrightarrow {}^\rho\widetilde{\mathbb{R}}$ on $H \subseteq U$, see Cor. 51. We hence have $\left|\mathrm{d}^{n+1}f(\xi)\right| \le K$ for all $\xi \in H$. Take any strictly positive number $P \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ such that $P \ge K$ and any strictly positive infinitesimal $p \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ so that $\frac{p}{P} \approx 0$ and hence $\left(\frac{p}{P}\right)^{n+1} \le \frac{p}{P}$. Set $R := \min\left(\frac{r}{2}, \frac{p}{P}\right)$, then $R \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ since both r and $\frac{p}{P}$ are invertible. If $k \in B_R(0)$ then $[a, a+k] \subseteq H \subseteq U$. We can therefore apply (i) to get (7.4). Finally

$$\left|\frac{\mathrm{d}^{n+1}f(\xi)}{(n+1)!}k^{n+1}\right| \le \frac{K}{(n+1)!}R^{n+1} \le \frac{P}{(n+1)!} \cdot \left(\frac{p}{P}\right)^{n+1} \le \frac{P}{(n+1)!} \cdot \frac{p}{P} \approx 0.$$

□

The following definitions allow us to state Taylor formulas in Peano and in infinitesimal form. The latter has no remainder term thanks to the use of an equivalence relation that permits the introduction of a language of nilpotent infinitesimals, see e.g. [35] for a similar formulation. For simplicity, we only present the 1-dimentional case.

Definition 54. (i) Let $U \subseteq {}^\rho\widetilde{\mathbb{R}}$ be a sharp neighborhood of 0 and $P, Q : U \longrightarrow {}^\rho\widetilde{\mathbb{R}}$ be maps defined on U. Then we say that

$$P(u) = o(Q(u)) \quad \text{as } u \to 0$$

if there exists a function $R : U \longrightarrow {}^\rho\widetilde{\mathbb{R}}$ such that

$$\forall u \in U:\ P(u) = R(u) \cdot Q(u) \quad \text{and} \quad \lim_{u\to 0} R(u) = 0,$$

where the limit is taken in the sharp topology.

(ii) Let $x, y \in {}^\rho\widetilde{\mathbb{R}}$ and $k, j \in \mathbb{R}_{>0}$, then we write $x =_j y$ if there exist representatives (x_ε), (y_ε) of x, y, respectively, such that

$$|x_\varepsilon - y_\varepsilon| = O(\rho_\varepsilon^{\frac{1}{j}}). \tag{7.6}$$

We will read $x =_j y$ as *x is equal to y up to j-th order infinitesimals*. Finally, if $k \in \mathbb{N}_{>0}$, we set $D_{kj} := \left\{x \in {}^\rho\widetilde{\mathbb{R}} \mid x^{k+1} =_j 0\right\}$, which is called the *set of k-th order infinitesimals for the equality $=_j$*, and

$$D_{\infty j} := \left\{x \in {}^\rho\widetilde{\mathbb{R}} \mid \exists k \in \mathbb{N}_{>0}:\ x^{k+1} =_j 0\right\}$$

which is called the *set of infinitesimals for the equality $=_j$*.

Of course, the reformulation of Def. 54 (i) for the classical Landau's little-oh is particularly suited to the case of a ring like ${}^\rho\widetilde{\mathbb{R}}$, instead of a field. The intuitive interpretation of $x =_j y$ is that for particular (e.g. physics-related) problems one is not interested in distinguishing quantities whose difference $|x - y|$ is less than an infinitesimal of order j. In fact, if $x =_j y$ we can write $x_\varepsilon = y_\varepsilon + r_\varepsilon$ with $r_\varepsilon \to 0$ of order at most $\rho_\varepsilon^{\frac{1}{j}}$. The idea behind taking $\frac{1}{j}$ in (7.6) is to obtain the property that the greater the order j of the infinitesimal error, the greater the difference $|x-y|$ is allowed to be. This is a typical property in rings with nilpotent infinitesimals (see e.g. [35, 62]). The set D_{ki} represents the neighborhood of infinitesimals of k-th order for the equality $=_j$. Once again, the greater the order k, the bigger is the neighborhood (see Theorem 55 (viii) below). Note that if $x =_j y$, then $x_\varepsilon = y_\varepsilon + o\left(\rho_\varepsilon^{\frac{1}{j}-a}\right)$ for all $a \in (0, 1/j]_{\mathbb{R}}$. In particular, $x_\varepsilon = y_\varepsilon + o\left(\rho_\varepsilon\right)$ implies $x =_1 y$, whereas $x =_1 y$ yields only $x_\varepsilon = y_\varepsilon + o\left(\rho_\varepsilon^{1-a}\right)$ for all $a \in (0,1]_{\mathbb{R}}$. On the other hand, it is not hard to prove the embedding ${}^\bullet\mathbb{R} \subseteq {}^\rho\widetilde{\mathbb{R}}/=_j$ of the ring of Fermat reals ${}^\bullet\mathbb{R}$ of [35] for all $j < 1$.

Theorem 55. *Let $f \in {}^\rho\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}})$ be a generalized smooth function defined in the sharply open set $U \subseteq {}^\rho\widetilde{\mathbb{R}}$. Let x, $\delta \in {}^\rho\widetilde{\mathbb{R}}$, with $\delta > 0$ and $[x-\delta, x+\delta] \subseteq U$. Let k, l, $j \in \mathbb{R}_{>0}$. Then*

(i) $\forall n \in \mathbb{N}: \ f(x+u) = \sum_{r=0}^n \frac{f^{(r)}(x)}{r!}u^r + o(u^n)$ *as* $u \to 0$.

(ii) *The definition of $x =_j y$ does not depend on the representatives of x, y.*

(iii) *$=_j$ is an equivalence relation on ${}^\rho\widetilde{\mathbb{R}}$.*

(iv) *If $x =_j y$ and $l \ge j$, then $x =_l y$. Therefore, $D_{nj} \subseteq D_{nl}$.*

(v) *If $\forall^0 j \in \mathbb{R}_{>0}: \ x =_j y$, then $x = y$.*

(vi) *If $x =_j y$ and $z =_j w$ then $x + z =_j y + w$. If x and z are finite, then $x \cdot z =_j y \cdot w$.*

(vii) $\forall h \in D_{kj}: \ h \approx 0$.

(viii) $D_{mj} \subseteq D_{kj} \subseteq D_{\infty j}$ *if* $m \le k$.

(ix) *D_{kj} is a subring of ${}^\rho\widetilde{\mathbb{R}}$. For all $h \in D_{kj}$ and all finite $x \in {}^\rho\widetilde{\mathbb{R}}$, we have $x \cdot h \in D_{kj}$.*

(x) *Let $n \in \mathbb{N}_{>0}$ and assume that j, k and f satisfy*

$$\forall z \in {}^\rho\widetilde{\mathbb{R}}\,\forall \xi \in [x-\delta, x+\delta]: \ z =_j 0 \ \Rightarrow \ z \cdot f^{(n+1)}(\xi) =_k 0. \tag{7.7}$$

Then, we have

$$\forall u \in D_{nj}: \ f(x+u) =_k \sum_{r=0}^n \frac{f^{(r)}(x)}{r!}u^r.$$

(xi) *For all $n \in \mathbb{N}_{>0}$ there exist $e \in \mathbb{R}_{>0}$ such that $e \le j$, and $\forall u \in D_{ne}$: $f(x+u) =_j \sum_{r=0}^n \frac{f^{(r)}(x)}{r!}u^r$.*

Proof. In order to prove (i) we set $P(u) = f(x+u) - \sum_{r=0}^n \frac{f^{(r)}(x)}{r!}u^r$, $Q(u) = u^n$ and $R(u) = u \cdot \int_0^1 \frac{f^{(n+1)}(x+tu)}{n!}(1-t)^n \,\mathrm{d}t$ for $u \in B_\delta(0)$. The segment $[x-u, x+u] \subseteq B_\delta(x) \subseteq U$, so Thm. 53 (ii) yields $P(u) = Q(u) \cdot R(u)$ for all $u \in U_x$. As in the previous proof, set

$$K := \max\left(\left|f^{(n+1)}(M)\right|, \left|f^{(n+1)}(m)\right|\right)$$

so that $\left|f^{(n+1)}(\xi)\right| \le K$ for all $\xi \in [x-\delta, x+\delta]$, then

$$|R(u)| \le |u| \cdot \left|\int_0^1 \frac{f^{(n+1)}(x+tu)}{n!}(1-t)^n \,\mathrm{d}t\right| \le |u| \cdot \frac{K}{(n+1)!}$$

which goes to 0 as $u \to 0$ in the sharp topology.

The proofs of (ii)-(ix) are simple. We only prove that D_{kj} is closed with respect to sums. Let x, $y \in D_{kj}$ so that

$$\left|\frac{x_\varepsilon^{k+1}}{\rho_\varepsilon^{\frac{1}{j}}}\right| \le M \quad , \quad \left|\frac{y_\varepsilon^{k+1}}{\rho_\varepsilon^{\frac{1}{j}}}\right| \le N \tag{7.8}$$

for ε small and for some M, $N \in \mathbb{R}_{>0}$. Then

$$\begin{aligned}\left|\frac{(x_\varepsilon+y_\varepsilon)^{k+1}}{\rho_\varepsilon^{\frac{1}{j}}}\right| &\le \sum_{r=0}^{k+1}\binom{k+1}{r}\left|\frac{x_\varepsilon^{k+1}}{\rho_\varepsilon^{\frac{1}{j}}}\right|^{\frac{r}{k+1}}\left|\frac{y_\varepsilon^{k+1}}{\rho_\varepsilon^{\frac{1}{j}}}\right|^{\frac{k+1-r}{k+1}}\\ &\le \sum_{r=0}^{k+1}\binom{k+1}{r}M^{\frac{r}{k+1}}N^{\frac{k+1-r}{k+1}},\end{aligned}$$

proving the claim.

In order to show (x), we first note that $x =_j y$ is equivalent to

$$\exists A \in \mathbb{R}_{>0} : \ |x-y| \le A \cdot \mathrm{d}\rho^{\frac{1}{j}}.$$

We again use the notation $K := \max\left(\left|f^{(n+1)}(M)\right|, \left|f^{(n+1)}(m)\right|\right)$ and note that for some $\xi \in [x-\delta, x+\delta]$, $K = \left|f^{(n+1)}(\xi)\right|$. We have

$$\begin{aligned}\left|f(x+u) - \sum_{r=0}^{n}\frac{f^{(r)}(x)}{r!}u^r\right| &\le \frac{K}{(n+1)!}\cdot|u|^{n+1}\\ &= \frac{1}{(n+1)!}\cdot\left|f^{(n+1)}(\xi)\right|\cdot|u|^{n+1}.\end{aligned} \tag{7.9}$$

In particular, if $u \in D_{nj}$ then $u^{n+1} =_j 0$, and assumption (7.7) yields $f^{(n+1)}(\xi)\cdot u^{n+1} =_k 0 =_k \left|f^{(n+1)}(\xi)\right|\cdot|u|^{n+1}$. This and (7.9) yield the conclusion.

To prove (xi), we proceed as above but taking $u \in D_{ne}$ in order to find $0 < e \le j$ such that $\left|f^{(n+1)}(\xi)\right|\cdot|u|^{n+1} =_j 0$. For moderateness $\left|f^{(n+1)}(\xi)\right| \le \mathrm{d}\rho^{-Q}$ and $|u|^{n+1} \le A\cdot\mathrm{d}\rho^{\frac{1}{e}}$ for some Q, $A \in \mathbb{R}_{>0}$ because $u \in D_{ne}$. It suffices to take $e > 0$ sufficiently small so that $\frac{1}{e} - Q \ge \frac{1}{j}$. $\square$

8. Multidimensional integration and hyperlimits

In this section we want to introduce integration of GSF over functionally compact sets with respect to an arbitrary Borel measure μ.

The possibility to achieve results mirroring classical limit theorems for this notion of integral is closely linked to the introduction of the notion of hyperlimit, i.e. of limits of sequences of generalized numbers $a = (a_n)_{n\in\mathbb{N}} : {}^\rho\widetilde{\mathbb{N}} \longrightarrow {}^\sigma\widetilde{\mathbb{R}}$, where σ and ρ are two gauges (see Def. 1) and $n \to +\infty$ along generalized natural numbers, i.e. for

$$n \in {}^\rho\widetilde{\mathbb{N}} := \left\{[n_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}} \mid n_\varepsilon \in \mathbb{N}\ \forall\varepsilon\right\}.$$

Mimicking nonstandard analysis, the numbers $n \in {}^\rho\widetilde{\mathbb{N}}$ are called *hypernatural* numbers. To glimpse the necessity of studying ${}^\rho\widetilde{\mathbb{N}}$, it suffices to note that $\frac{1}{n} < \mathrm{d}\rho^q$ is always false for $n \in \mathbb{N}$ but it can be satisfied for suitable $n \in {}^\rho\widetilde{\mathbb{N}}$. Therefore, if $\lim_{n\to+\infty} a_n = 0$ in the classical sense, i.e. for $n \in \mathbb{N}$ and with respect to the sharp topology, then necessarily a_n is infinitesimal for $n \in \mathbb{N}$ sufficiently large. This represents a severe limitation for this notion of limit. It is also clear from the fact that ${}^\rho\widetilde{\mathbb{R}}$ with the sharp topology is an ultra-pseudometric space, see e.g. [98], and hence a series in ${}^{\rho}\widetilde{\mathbb{R}}$ converges in the sharp topology if and only if its general term $a_n \to 0$ as $n \to +\infty$, $n \in \mathbb{N}$, in the sharp topology, see [61].

8.1. **Integration over functionally compact sets.** In section 6, we already defined a notion of integral over intervals using the notion of primitive. This notion does not help if we want to define the integral $\int_D f$ of a GSF f over a domain $D \subseteq {}^{\rho}\widetilde{\mathbb{R}}^n$ which is more general than an interval. In this case, it is natural to try an ε-wise definition of the type $\int_D f \, \mathrm{d}\mu := \left[\int_{D_\varepsilon} f_\varepsilon \, \mathrm{d}\mu\right] \in {}^{\rho}\widetilde{\mathbb{R}}$, where the net (f_ε) defines the GSF f and the net (D_ε) determines, in some way, the subset $D \subseteq {}^{\rho}\widetilde{\mathbb{R}}^n$, e.g. $D = [D_\varepsilon]$ in case of internal sets. In pursuing this idea, it is important to recall that the internal set (interval) $[0,1] = [[0,1]_{\mathbb{R}}]$ can also be defined by a net of finite sets. Indeed, if $\mathrm{int}(-)$ is the integer part function, and we set

$$\begin{aligned} N_\varepsilon &:= \mathrm{int}\left(\rho_\varepsilon^{-1/\varepsilon}\right) \\ K_\varepsilon &:= \{\rho_\varepsilon^{1/\varepsilon}, 2\rho_\varepsilon^{1/\varepsilon}, \ldots, N_\varepsilon \rho_\varepsilon^{1/\varepsilon}\} \end{aligned} \tag{8.1}$$

then the Hausdorff distance $d_{\mathrm{H}}([0,1]_{\mathbb{R}}, K_\varepsilon) = \rho_\varepsilon^{1/\varepsilon}$ and hence $[0,1] = [K_\varepsilon]$ (see also [116, 43]). Consequently, if λ is the Lebesgue measure on $\mathbb{R}$, we have that the generalized number $[\lambda([0,1]_{\mathbb{R}})] = 1$, whereas $[\lambda(K_\varepsilon)] = 0$ and, in general, $\left[\int_{[0,1]_{\mathbb{R}}} f_\varepsilon \, \mathrm{d}\lambda\right] \neq \left[\int_{K_\varepsilon} f_\varepsilon \, \mathrm{d}\lambda\right] = 0$. Therefore, even the definition of integral over an interval cannot be easily accomplished by proceeding ε-wise, i.e. on defining nets.

If we try to understand when such an ε-wise definition can be accomplished, it turns out that we have to consider an enlargement $\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(K_\varepsilon)$ and then take $m \to +\infty$. This is indeed quite natural if one keeps in mind that $[K_\varepsilon] = [L_\varepsilon]$ if and only if the Hausdorff distance $d_{\mathrm{H}}(K_\varepsilon, L_\varepsilon)$ defines a negligible nets, (see [116, 43]). In the following, we say that (K_ε) is a *representative of* $K \Subset_{\mathrm{f}} {}^{\rho}\widetilde{\mathbb{R}}^n$ if $K = [K_\varepsilon]$, (K_ε) is sharply bounded, and $K_\varepsilon \Subset \mathbb{R}^n$ for all ε.

Definition 56. Let μ be a Borel measure on $\mathbb{R}^n$ and let K be a functionally compact subset of ${}^{\rho}\widetilde{\mathbb{R}}^n$. Then we call K μ-measurable if the limit

$$\mu(K) := \lim_{\substack{m \to \infty \\ m \in \mathbb{N}}} [\mu(\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(K_\varepsilon))] \tag{8.2}$$

exists for some representative (K_ε) of K. The limit is taken in the sharp topology on ${}^{\rho}\widetilde{\mathbb{R}}$, and $\overline{B^{\mathrm{E}}}_r(A) := \{x \in \mathbb{R}^n : d(x, A) \le r\}$.

In the following result, we will prove that this definition satisfies our requirements. We will occasionally integrate generalized functions more general than GSF:

Definition 57. Let $K \Subset_{\mathrm{f}} {}^{\rho}\widetilde{\mathbb{R}}^n$. Let (Ω_ε) be a net of open subsets of $\mathbb{R}^n$, and (f_ε) be a net of continuous maps $f_\varepsilon\colon \Omega_\varepsilon \longrightarrow \mathbb{R}$. Then we say that

$$(f_\varepsilon) \text{ defines a generalized integrable map} : K \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}$$

if

(i) $K \subseteq \langle \Omega_\varepsilon \rangle$ and $[f_\varepsilon(x_\varepsilon)] \in {}^{\rho}\widetilde{\mathbb{R}}$ for all $[x_\varepsilon] \in K$.

(ii) $\forall (x_\varepsilon), (x'_\varepsilon) \in \mathbb{R}^n_\rho : \ [x_\varepsilon] = [x'_\varepsilon] \in K \ \Rightarrow \ (f_\varepsilon(x_\varepsilon)) \sim_\rho (f_\varepsilon(x'_\varepsilon))$.

If $f \in \mathbf{Set}(K, {}^{\rho}\widetilde{\mathbb{R}})$ is such that

$$\forall [x_\varepsilon] \in K : \ f([x_\varepsilon]) = [f_\varepsilon(x_\varepsilon)] \tag{8.3}$$

we say that $f : K \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}$ is a *generalized integrable function*.
We will again say that f *is defined by the net* (f_ε) or that the net (f_ε) *represents* f. The set of all these generalized integrable functions will be denoted by ${}^{\rho}\mathcal{GI}(K, {}^{\rho}\widetilde{\mathbb{R}})$.

E.g., if $f = [f_\varepsilon(-)]|_K \in {}^{\rho}\mathcal{GC}^\infty(K, {}^{\rho}\widetilde{\mathbb{R}})$, then both f and $|f| = [|f_\varepsilon(-)|]|_K$ are integrable on K.

As in Lemma 18, we may assume without loss of generality that f_ε are continuous maps defined on the whole of $\mathbb{R}^n$.

Theorem 58. *Let $K \subseteq {}^{\rho}\widetilde{\mathbb{R}}^n$ be μ-measurable.*

(i) The definition of $\mu(K)$ is independent of the representative (K_ε).

(ii) There exists a representative (K_ε) of K such that $\mu(K) = [\mu(K_\varepsilon)]$.

(iii) Let (K_ε) be any representative of K and let $f = [f_\varepsilon(-)]|_K \in {}^{\rho}\mathcal{GI}(K, {}^{\rho}\widetilde{\mathbb{R}})$. Then

$$\int_K f \, \mathrm{d}\mu := \lim_{m\to\infty} \left[\int_{\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(K_\varepsilon)} f_\varepsilon \, \mathrm{d}\mu \right]$$

exists and its value is independent of the representative (K_ε).

(iv) There exists a representative (K_ε) of K such that

$$\int_K f \, \mathrm{d}\mu = \left[\int_{K_\varepsilon} f_\varepsilon \, \mathrm{d}\mu \right] \tag{8.4}$$

for each $f = [f_\varepsilon(-)]|_K \in {}^{\rho}\mathcal{GI}(K, {}^{\rho}\widetilde{\mathbb{R}})$. From (8.4), it also follows that $\left|\int_K f \, \mathrm{d}\mu\right| \le \int_K |f| \, \mathrm{d}\mu$.

(v) If (8.4) holds, then the same holds for any representative (L_ε) of K with $L_\varepsilon \supseteq K_\varepsilon$, $\forall^0 \varepsilon$.

Proof. (i) Let (L_ε) be another representative. As $[K_\varepsilon] \subseteq [L_\varepsilon]$, we have that $(\sup_{x\in K_\varepsilon} d(x, L_\varepsilon))_\varepsilon =: (n_\varepsilon)$ is negligible, so $K_\varepsilon \subseteq \overline{B^{\mathrm{E}}}_{n_\varepsilon}(L_\varepsilon)$, and $\mu(\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(K_\varepsilon)) \le \mu(\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^{m-1}}(L_\varepsilon))$. Also using this inequality with the roles of K_ε and L_ε interchanged, we see that $\lim_{m\to\infty}[\mu(\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(L_\varepsilon))]$ exists and that it equals $\lim_{m\to\infty}[\mu(\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(K_\varepsilon))]$.

(ii) Call $[c_\varepsilon] := \mu(K)$ and let $K = [L_\varepsilon]$. By definition of μ-measurable set and by the previous point (i), for any $q \in \mathbb{N}$, there exists $m_q \in \mathbb{N}$ (w.l.o.g. $m_q \ge q$) and $\varepsilon_q > 0$ (w.l.o.g. $\varepsilon_q < \varepsilon_{q-1}$ and $\varepsilon_q < 1/q$) such that

$$|\mu(\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^{m_q}}(L_\varepsilon)) - c_\varepsilon| \le \rho_\varepsilon^q, \qquad \forall \varepsilon \le \varepsilon_q$$

Now let $q_\varepsilon := q$ if $\varepsilon \in (\varepsilon_{q+1}, \varepsilon_q]$. Then $q_\varepsilon \to \infty$ as $\varepsilon \to 0$ and

$$[\mu(\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^{m_{q_\varepsilon}}}(L_\varepsilon))] = [c_\varepsilon] = \mu(K).$$

As also $(\rho_\varepsilon^{m_{q_\varepsilon}})$ is negligible, we have $K = [\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^{m_{q_\varepsilon}}}(L_\varepsilon)]$ and hence the conclusion follows for $K_\varepsilon := \overline{B^{\mathrm{E}}}_{\rho_\varepsilon^{m_{q_\varepsilon}}}(L_\varepsilon)$.

(iii)–(iv). Choose a representative (K_ε) as in part (ii). Then

$$\left| \int_{\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(K_\varepsilon)} f_\varepsilon \, \mathrm{d}\mu - \int_{K_\varepsilon} f_\varepsilon \, \mathrm{d}\mu \right| \le \mu(\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(K_\varepsilon) \setminus K_\varepsilon) \sup_{\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(K_\varepsilon)} |f_\varepsilon|.$$

As $[\mu(\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(K_\varepsilon) \setminus K_\varepsilon)] = [\mu(\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(K_\varepsilon))] - [\mu(K_\varepsilon)] \to 0$ as $m \to \infty$ and since $(\sup_{\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(K_\varepsilon)} |f_\varepsilon|)$ is moderate for some m and decreasing in m, we find that

$$\lim_{m\to\infty} \left[\int_{\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(K_\varepsilon)} f_\varepsilon \, \mathrm{d}\mu \right] = \left[\int_{K_\varepsilon} f_\varepsilon \, \mathrm{d}\mu \right]$$

exists. Independence of the representative of K follows as in part (i), if $f_\varepsilon \geq 0$. The general case follows by considering the positive and negative part of f_ε.

(v) Let $f_\varepsilon \geq 0$. Then by assumption, $[\int_{K_\varepsilon} f_\varepsilon \,\mathrm{d}\mu] \leq [\int_{L_\varepsilon} f_\varepsilon \,\mathrm{d}\mu]$. For the converse inequality, observe that $[\int_{L_\varepsilon} f_\varepsilon \,\mathrm{d}\mu] \leq [\int_{\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(L_\varepsilon)} f_\varepsilon \,\mathrm{d}\mu]$ for each $m \in \mathbb{N}$. Again the general case follows by considering the positive and negative part of f_ε. □

The following Lemma provides an alternative characterization of μ-measurability:

Lemma 59. *A functionally compact set K is μ-measurable if and only if there exists a representative (K_ε) of K such that $[\mu(K_\varepsilon)] = [\mu(L_\varepsilon)]$, for each representative (L_ε) of K with $L_\varepsilon \supseteq K_\varepsilon$, $\forall^0\varepsilon$.*

Proof. $\Rightarrow$: by the previous Thm. 58.

$\Leftarrow$: it suffices to show that $\lim_{m\to\infty}[\mu(\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(K_\varepsilon))] = [\mu(K_\varepsilon)]$. Seeking a contradiction, suppose that there exists $q \in \mathbb{N}$ for which it does not hold that

$$\exists M\,\forall m \geq M\,\forall^0\varepsilon :\ |\mu(\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(K_\varepsilon)) - \mu(K_\varepsilon)| \leq \rho_\varepsilon^q.$$

Then we can construct a strictly increasing sequence $(m_k)_k \to \infty$ and a strictly decreasing sequence $(\varepsilon_k)_k \to 0$ such that $|\mu(\overline{B^{\mathrm{E}}}_{\rho_{\varepsilon_k}^{m_k}}(K_{\varepsilon_k})) - \mu(K_{\varepsilon_k})| > \rho_{\varepsilon_k}^q$, $\forall k$. Let $L_\varepsilon := \overline{B^{\mathrm{E}}}_{\rho_\varepsilon^{m_k}}(K_\varepsilon)$, whenever $\varepsilon \in (\varepsilon_{k+1}, \varepsilon_k]$, $\forall k$. Then $K = [L_\varepsilon]$, but $[\mu(K_\varepsilon)] \neq [\mu(L_\varepsilon)]$, as $|\mu(L_\varepsilon) - \mu(K_\varepsilon)| > \rho_\varepsilon^q$, for each $\varepsilon = \varepsilon_k$ $(k \in \mathbb{N})$. □

Example 60. Let λ denote the Lebesgue-measure.

(i) If $K = \prod_{i=1}^n [a_i, b_i]$, then K is λ-measurable with

$$\int_K f\,\mathrm{d}\lambda = \left[\int_{a_{1,\varepsilon}}^{b_{1,\varepsilon}} dx_1 \ldots \int_{a_{n,\varepsilon}}^{b_{n,\varepsilon}} f_\varepsilon(x_1, \ldots, x_n)\,\mathrm{d}x_n\right]$$

for any representatives $(a_{i,\varepsilon})$, $(b_{i,\varepsilon})$ of a_i and b_i, respectively.

(ii) Let $\rho_\varepsilon = \varepsilon$, and

$$K := \left\{\tfrac{1}{n} \mid n \in \mathbb{N}_{>0}\right\} \cup \{0\}.$$

Then $[K]$ is λ-measurable with $\lambda([K]) = 0$. Indeed, the contribution of $\{1/n \mid n > \varepsilon^{-m/2}\}$ to $\lambda(\overline{B^{\mathrm{E}}}_{\varepsilon^m}(K))$ is at most $\varepsilon^{m/2} + 2\varepsilon^m$, while the contribution of $\{1/n \mid n \leq \varepsilon^{-m/2}\}$ is at most $2\varepsilon^m\varepsilon^{-m/2} = 2\varepsilon^{m/2}$. Thus $\lim_{m\to\infty}[\lambda(\overline{B^{\mathrm{E}}}_{\varepsilon^m}(K))] = 0$.

(iii) Let $\rho_\varepsilon = \varepsilon$, and

$$K := \left\{\frac{1}{\log n} \mid n \in \mathbb{N}_{>1}\right\} \cup \{0\}.$$

Then $[K]$ is not λ-measurable. For, if $n \geq \frac{\varepsilon^{-m}}{(\log \varepsilon^{-m})^2}$, then, by the mean value theorem,

$$\frac{1}{\log n} - \frac{1}{\log(n+1)} \leq \frac{1}{n(\log n)^2} \leq \frac{\varepsilon^m(\log \varepsilon^{-m})^2}{\left(\log\left(\frac{\varepsilon^{-m}}{(\log \varepsilon^{-m})^2}\right)\right)^2} \leq 2\varepsilon^m$$

for small ε. So the contribution of $\left\{\frac{1}{\log n} \mid n \geq \frac{\varepsilon^{-m}}{(\log \varepsilon^{-m})^2}\right\}$ to $\lambda(\overline{B^{\mathrm{E}}}_{\varepsilon^m}(K))$ lies between $\frac{1}{\log(\varepsilon^{-m})}$ and $\frac{2}{\log(\varepsilon^{-m})}$ for small ε. The contribution of $\left\{\frac{1}{\log n} \mid n < \frac{\varepsilon^{-m}}{(\log \varepsilon^{-m})^2}\right\}$ to $\lambda(\overline{B^{\mathrm{E}}}_{\varepsilon^m}(K))$ is at most $\frac{2}{(\log \varepsilon^{-m})^2}$, which is of a lower order. Thus $\lim_{m\to\infty}[\lambda(\overline{B^{\mathrm{E}}}_{\varepsilon^m}(K))]$ does not exist.

8.2. **Hyperfinite limits.** We start by defining the set of hypernatural numbers in ${}^{\rho}\widetilde{\mathbb{R}}$ and the set of ρ-moderate nets of natural numbers. For a deeper study of these notions, see [83].

Definition 61. We set

(i) ${}^{\rho}\widetilde{\mathbb{N}}:=\left\{[n_{\varepsilon}]\in{}^{\rho}\widetilde{\mathbb{R}}\mid n_{\varepsilon}\in\mathbb{N}\quad\forall\varepsilon\right\}$

(ii) $\mathbb{N}_{\rho}:=\left\{(n_{\varepsilon})\in\mathbb{R}_{\rho}\mid n_{\varepsilon}\in\mathbb{N}\quad\forall\varepsilon\right\}$.

Therefore, $n\in{}^{\rho}\widetilde{\mathbb{N}}$ if and only if there exists $(x_{\varepsilon})\in\mathbb{R}_{\rho}$ such that $n=[\text{int}(|x_{\varepsilon}|)]$. Clearly, $\mathbb{N}\subset{}^{\rho}\widetilde{\mathbb{N}}$. Note that the integer part function $\text{int}(-)$ is not well-defined on ${}^{\rho}\widetilde{\mathbb{R}}$. In fact, if $x=1=\left[1-\rho_{\varepsilon}^{1/\varepsilon}\right]=\left[1+\rho_{\varepsilon}^{1/\varepsilon}\right]$, then $\text{int}\left(1-\rho_{\varepsilon}^{1/\varepsilon}\right)=0$, whereas $\text{int}\left(1+\rho_{\varepsilon}^{1/\varepsilon}\right)=1$, for ε sufficiently small. Similar counterexamples can be constructed for floor and ceiling functions.
However, the nearest integer function is well defined on ${}^{\rho}\widetilde{\mathbb{N}}$.

Lemma 62. *Let $(n_{\varepsilon})\in\mathbb{N}_{\rho}$ and $(x_{\varepsilon})\in\mathbb{R}_{\rho}$ be such that $[n_{\varepsilon}]=[x_{\varepsilon}]$. Let $\text{rpi}:\mathbb{R}\longrightarrow\mathbb{N}$ be the function rounding to the nearest integer with tie breaking towards positive infinity. Then $\text{rpi}(x_{\varepsilon})=n_{\varepsilon}$ for ε small. The same result holds using $\text{rni}:\mathbb{R}\longrightarrow\mathbb{N}$, the function rounding half towards $-\infty$.*

Proof. We have $\text{rpi}(x)=\lfloor x+\frac{1}{2}\rfloor$, where $\lfloor-\rfloor$ is the floor function. For ε small, $\rho_{\varepsilon}<\frac{1}{2}$ and, since $[n_{\varepsilon}]=[x_{\varepsilon}]$, for such ε we can also have $n_{\varepsilon}-\rho_{\varepsilon}+\frac{1}{2}<x_{\varepsilon}+\frac{1}{2}<n_{\varepsilon}+\rho_{\varepsilon}+\frac{1}{2}$. But $n_{\varepsilon}\le n_{\varepsilon}-\rho_{\varepsilon}+\frac{1}{2}$ and $n_{\varepsilon}+\rho_{\varepsilon}+\frac{1}{2}<n_{\varepsilon}+1$. Therefore $\lfloor x_{\varepsilon}+\frac{1}{2}\rfloor=n_{\varepsilon}$. An analogous argument can be applied to $\text{rni}(-)$. □

Actually, this lemma does not allow us to define a *nearest integer* function $\text{ni}:{}^{\rho}\widetilde{\mathbb{N}}\longrightarrow\mathbb{N}_{\rho}$ as $\text{ni}([x_{\varepsilon}]):=\text{rpi}(x_{\varepsilon})$ because if $[x_{\varepsilon}]=[n_{\varepsilon}]$, the equality $n_{\varepsilon}=\text{rpi}(x_{\varepsilon})$ holds only for ε small. We should therefore consider the function ni as valued in the germs for $\varepsilon\to0^{+}$ generated by nets in $\mathbb{N}_{\rho}$. A simpler approach is to choose a representative $(n_{\varepsilon})\in\mathbb{N}_{\rho}$ for each $x\in{}^{\rho}\widetilde{\mathbb{N}}$ and to define $\text{ni}(x):=(n_{\varepsilon})$. Clearly, we must consider the net $\left(\text{ni}(x)_{\varepsilon}\right)$ only for ε small, such as in equalities of the form $x=\left[\text{ni}(x)_{\varepsilon}\right]$. This is what we do in the following

Definition 63. The nearest integer function $\text{ni}(-)$ is defined by:

(i) $\text{ni}:{}^{\rho}\widetilde{\mathbb{N}}:\longrightarrow\mathbb{N}_{\rho}$

(ii) If $[x_{\varepsilon}]\in{}^{\rho}\widetilde{\mathbb{N}}$ and $\text{ni}\left([x_{\varepsilon}]\right)=(n_{\varepsilon})$ then $\forall^{0}\varepsilon:\ n_{\varepsilon}=\text{rpi}(x_{\varepsilon})$.

In other words, if $x\in{}^{\rho}\widetilde{\mathbb{N}}$, then $x=[\text{ni}(x)_{\varepsilon}]$ and $\text{ni}(x)_{\varepsilon}\in\mathbb{N}$ for all ε.

We first consider the notion of hyperlimit. As we will see clearly in Example 66(i), a key point in the definition of hyperlimit is to consider *two* gauges. This is a natural way of proceeding because different gauges define different topologies. On the other hand, the notion of hyperlimit corresponds exactly to that of limit in the sharp topology on ${}^{\rho}\widetilde{\mathbb{R}}$ of a generalized sequence (hypersequence), i.e. defined on the directed set ${}^{\sigma}\widetilde{\mathbb{N}}$.

Definition 64. Let ρ, σ be two gauges (see Def. 1). Let $(a_{n})_{n}:{}^{\sigma}\widetilde{\mathbb{N}}\longrightarrow{}^{\rho}\widetilde{\mathbb{R}}$ be a σ-hypersequence of ρ-generalized numbers. Finally let $l\in{}^{\rho}\widetilde{\mathbb{R}}$. Then we say that

$$l\text{ is the hyperlimit of }(a_{n})_{n}$$

if

$$\forall q\in\mathbb{N}\,\exists M\in{}^{\sigma}\widetilde{\mathbb{N}}\,\forall n\in{}^{\sigma}\widetilde{\mathbb{N}}:\ n\ge M\ \Rightarrow\ |a_{n}-l|<\text{d}\rho^{q}. \tag{8.5}$$

Remark 65.

(i) In a hyperlimit, we are considering ${}^{\sigma}\widetilde{\mathbb{N}}$ as an ordered set directed by $\le$:

$$n, m \in {}^{\sigma}\widetilde{\mathbb{N}} \;\Rightarrow\; n \vee m = [\max(\mathrm{ni}(n)_\varepsilon, \mathrm{ni}(m)_\varepsilon)] \in {}^{\sigma}\widetilde{\mathbb{N}}.$$

On the other hand, on ${}^{\rho}\widetilde{\mathbb{R}}$ we are considering the sharp topology (which is Hausdorff). In fact, if l, λ are hyperlimits of $a : {}^{\sigma}\widetilde{\mathbb{N}} \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}$, then

$$|l - \lambda| \le |l - a_M| + |a_M - \lambda| \le 2\mathrm{d}\rho^{q+1} < \mathrm{d}\rho^q$$

for all q. So $l = \lambda \in {}^{\rho}\widetilde{\mathbb{R}}$. We will therefore use the notations

$$l = {}^{\rho}\lim_{n \in {}^{\sigma}\widetilde{\mathbb{N}}} a_n$$

or simply $l = \lim_{n \in {}^{\rho}\widetilde{\mathbb{N}}} a_n$ if $\sigma = \rho$.

(ii) A sufficient condition to extend an ordinary sequence $a : \mathbb{N} \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}$ of ρ-generalized numbers to the whole of ${}^{\sigma}\widetilde{\mathbb{N}}$ is

$$\forall n \in {}^{\sigma}\widetilde{\mathbb{N}} : \; \left(a_{\mathrm{ni}(n)_\varepsilon}\right) \in \mathbb{R}_\rho. \tag{8.6}$$

In fact, in this way a_n is well-defined because of Lem. 62; on the other hand, using (8.6), we have defined an extension of the old sequence a because if $n \in \mathbb{N}$, then $\mathrm{ni}(n)_\varepsilon = n$ for ε small and hence we get $a_n = [a_n]$. For example, the sequence of infinities $a_n = \frac{1}{n} + \mathrm{d}\rho^{-1}$ for all $n \in \mathbb{N}$ can be extended to any ${}^{\sigma}\widetilde{\mathbb{N}}$, whereas $a_n = \mathrm{d}\sigma^{-n}$ can be extended as $a : {}^{\sigma}\widetilde{\mathbb{N}} \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}$ only for certain gauges ρ, e.g. if the gauges satisfy

$$\exists N \in \mathbb{N}\, \forall n \in \mathbb{N}\, \forall^0 \varepsilon : \; \sigma_\varepsilon^n \ge \rho_\varepsilon^N,$$

e.g. $\sigma_\varepsilon \ge -\log(\rho_\varepsilon)^{-1}$.

Example 66.

(i) The following example strongly motivates the use of two gauges. Let ρ be a gauge and set $\sigma_\varepsilon := \exp\left(-\rho_\varepsilon^{-\frac{1}{\rho_\varepsilon}}\right)$, so that also σ is a gauge. We have

$${}^{\rho}\lim_{n \in {}^{\sigma}\widetilde{\mathbb{N}}} \frac{1}{\log n} = 0 \in {}^{\rho}\widetilde{\mathbb{R}} \quad \text{whereas} \quad \nexists\, {}^{\rho}\lim_{n \in {}^{\rho}\widetilde{\mathbb{N}}} \frac{1}{\log n}.$$

In fact, if $n > 1$, we have $0 < \frac{1}{\log n} < \mathrm{d}\rho^q$ if and only if $\log n > \mathrm{d}\rho^{-q}$, i.e. $n > e^{\mathrm{d}\rho^{-q}}$ (in ${}^{\sigma}\widetilde{\mathbb{R}}$). We can thus take $M := \left[\mathrm{int}\left(e^{\rho_\varepsilon^{-q}}\right) + 1\right] \in {}^{\sigma}\widetilde{\mathbb{N}}$ because $e^{\rho_\varepsilon^{-q}} < \exp\left(\rho_\varepsilon^{-\frac{1}{\rho_\varepsilon}}\right) = \sigma_\varepsilon^{-1}$ for ε small.

Vice versa, by contradiction, if $\exists\, {}^{\rho}\lim_{n \in {}^{\rho}\widetilde{\mathbb{N}}} \frac{1}{\log n} =: l \in {}^{\rho}\widetilde{\mathbb{R}}$, then by the definition of hyperlimit from ${}^{\rho}\widetilde{\mathbb{N}}$ to ${}^{\rho}\widetilde{\mathbb{R}}$ we would get the existence of $M \in {}^{\rho}\widetilde{\mathbb{N}}$ such that

$$\forall n \in {}^{\rho}\widetilde{\mathbb{N}} : \; n \ge M \;\Rightarrow\; \frac{1}{\log n} - \mathrm{d}\rho < l < \frac{1}{\log n} + \mathrm{d}\rho. \tag{8.7}$$

Since M is ρ-moderate, we always have $0 < \frac{1}{\log M} - \mathrm{d}\rho$, so $l > 0$. Thus $\mathrm{d}\rho^p < |l|$ for some $p \in \mathbb{N}$. Setting

$$q := \min\{p \in \mathbb{N} \mid \mathrm{d}\rho^p < |l|\} + 1,$$

we get that $|l_{\bar{\varepsilon}_k}| < \rho^q_{\bar{\varepsilon}_k}$ for some sequence $(\bar{\varepsilon}_k)_k \downarrow 0$. Therefore

$$\frac{1}{\log M_{\bar{\varepsilon}_k}} < l_{\bar{\varepsilon}_k} + \rho_{\bar{\varepsilon}_k} \le |l_{\bar{\varepsilon}_k}| + \rho_{\bar{\varepsilon}_k} < \rho^q_{\bar{\varepsilon}_k} + \rho_{\bar{\varepsilon}_k}$$

and hence $M_{\bar{\varepsilon}_k} > \exp\left(\frac{1}{\rho^q_{\bar{\varepsilon}_k} + \rho_{\bar{\varepsilon}_k}}\right)$ for all $k \in \mathbb{N}$, which is in contradiction with $M \in {}^{\rho}\widetilde{\mathbb{R}}$ because $q \ge 1$.

(ii) For all $k \in \mathbb{N}_{>0}$, we have $\lim_{n \in {}^{\rho}\widetilde{\mathbb{N}}} \frac{1}{n^k} = 0$. In fact, for all $n \in {}^{\rho}\widetilde{\mathbb{N}}_{>0}$, we have $0 < \frac{1}{n^k} < \mathrm{d}\rho^q$ if and only if $n^k > \mathrm{d}\rho^{-q}$, i.e. $n > \mathrm{d}\rho^{-\frac{q}{k}}$. Thus, it suffices to take $M_\varepsilon := \mathrm{int}\left(\rho_\varepsilon^{-\frac{q}{k}}\right) + 1$ in the definition of hyperlimit. Analogously, we can treat rational functions having degree of denominator greater or equal to that of the numerator.

8.3. **Properties of multidimensional integral.** We start by proving the change of variable formula.

Lemma 67. *Let $K = [K_\varepsilon]$ be functionally compact and $\varphi = [\varphi_\varepsilon] \in {}^{\rho}\mathcal{GC}^\infty(K, {}^{\rho}\widetilde{\mathbb{R}}^d)$ with $\det(d\varphi)(x)$ invertible for each $x \in K$. If φ is injective on K, then the φ_ε are injective on K_ε, $\forall^0 \varepsilon$.*

Proof. By contradiction, suppose that for each $\eta > 0$, there exists $\varepsilon < \eta$ such that φ_ε is not injective on K_ε. Then we find for such ε some x_ε, $y_\varepsilon \in K_\varepsilon$ with $x_\varepsilon \ne y_\varepsilon$ and $\varphi_\varepsilon(x_\varepsilon) = \varphi_\varepsilon(y_\varepsilon)$. For all other ε, define $x_\varepsilon = y_\varepsilon \in K_\varepsilon$ arbitrary. Then $x := [x_\varepsilon]$, $y := [y_\varepsilon] \in K$ and $\varphi(x) = \varphi(y)$. As φ is injective, $x = y$. But then this contradicts the local injectivity of φ_ε on $B^{\mathrm{E}}_{r_\varepsilon}(x_\varepsilon)$ for some $[r_\varepsilon] > 0$, see also [42, Thm. 6] and [26]. $\square$

Theorem 68. *Let $K \subseteq {}^{\rho}\widetilde{\mathbb{R}}^n$ be λ-measurable, where λ is the Lebesgue measure, and let $\varphi \in {}^{\rho}\mathcal{GC}^\infty(K, {}^{\rho}\widetilde{\mathbb{R}}^d)$ be such that $\varphi^{-1} \in {}^{\rho}\mathcal{GC}^\infty(\varphi(K), {}^{\rho}\widetilde{\mathbb{R}}^n)$. Then $\varphi(K)$ is λ-measurable and*

$$\int_{\varphi(K)} f \,\mathrm{d}\lambda = \int_K (f \circ \varphi) \left|\det(\mathrm{d}\varphi)\right| \mathrm{d}\lambda$$

for each $f \in {}^{\rho}\mathcal{GC}^\infty(\varphi(K), {}^{\rho}\widetilde{\mathbb{R}})$.

Proof. Let $x \in \overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(K_\varepsilon)$. Then there exists $y \in K_\varepsilon$ such that $|x - y| \le \rho_\varepsilon^m$. As $\varphi \in {}^{\rho}\mathcal{GC}^\infty(K, {}^{\rho}\widetilde{\mathbb{R}}^d)$,

$$|\varphi_\varepsilon(x) - \varphi_\varepsilon(y)| \le |x - y| \sup_{\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(K_\varepsilon)} \|\mathrm{d}\varphi_\varepsilon\| \le \rho_\varepsilon^{m-M}$$

for some $M \in \mathbb{N}$ (not depending on m). Thus $\varphi_\varepsilon(\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(K_\varepsilon)) \subseteq \overline{B^{\mathrm{E}}}_{\rho_\varepsilon^{m-M}}(\varphi_\varepsilon(K_\varepsilon))$. Applying this to φ^{-1}, we find that also $\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(\varphi_\varepsilon(K_\varepsilon)) \subseteq \varphi_\varepsilon(\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^{m-M}}(K_\varepsilon))$ for some $M \in \mathbb{N}$. Now let $f_\varepsilon \ge 0$. As $(\det(\mathrm{d}\varphi_\varepsilon^{-1}))$ is moderate, $|\det(\mathrm{d}\varphi_\varepsilon)(x)| > 0$ for each $x \in K$. Thus by Lem. 67, w.l.o.g φ_ε are injective. Then

$$\int_{\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(\varphi_\varepsilon(K_\varepsilon))} f_\varepsilon \,\mathrm{d}\lambda \le \int_{\varphi_\varepsilon(\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^{m-M}}(K_\varepsilon))} f_\varepsilon \,\mathrm{d}\lambda = \int_{\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^{m-M}}(K_\varepsilon)} (f_\varepsilon \circ \varphi_\varepsilon) |\det \mathrm{d}\varphi_\varepsilon| \,\mathrm{d}\lambda$$

and

$$\int_{\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^{m+M}}(K_\varepsilon)} (f_\varepsilon \circ \varphi_\varepsilon) |\det \mathrm{d}\varphi_\varepsilon| \,\mathrm{d}\lambda = \int_{\varphi_\varepsilon(\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^{m+M}}(K_\varepsilon))} f_\varepsilon \,\mathrm{d}\lambda \le \int_{\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(\varphi_\varepsilon(K_\varepsilon))} f_\varepsilon \,\mathrm{d}\lambda$$

Since $\lim_{m\to\infty}\left[\int_{\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^{m-M}(K_\varepsilon)}}(f_\varepsilon\circ\varphi_\varepsilon)|\det \mathrm{d}\varphi_\varepsilon|\,\mathrm{d}\lambda\right]$ exists, it follows from the previous inequalities that also $\lim_{m\to\infty}\left[\int_{\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^{m}(\varphi_\varepsilon(K_\varepsilon))}} f_\varepsilon\,\mathrm{d}\lambda\right]$ exists, with the same value. The general case follows by considering the positive and negative part of f_ε. $\square$

We now consider the problem of additivity of the integral.

Definition 69. Let K, L be functionally compact. Then we call K and L *strongly disjoint* if the following equivalent conditions hold:

(i) for each representative (K_ε) of K and (L_ε) of L, $K_\varepsilon\cap L_\varepsilon=\emptyset$, $\forall^0\varepsilon$

(ii) for some (and thus each) representative (K_ε) of K and (L_ε) of L, there exists $m\in\mathbb{N}$ such that $\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(K_\varepsilon)\cap\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(L_\varepsilon)=\emptyset$, $\forall^0\varepsilon$

(iii) $\forall e\in{}^{\rho}\widetilde{\mathbb{R}}:\ e^2=e, e\neq 0\ \Rightarrow\ Ke\cap Le=\emptyset$

(iv) $\forall H\subseteq_0 I:\ K|_H\cap L|_H=\emptyset$

(v) $x\neq y$ for each subpoint x of K and y of L.

Definition 70. Let K, L be functionally compact. Then we call K and L *almost strongly disjoint* if the following equivalent conditions hold:

(i) for each representative (K_ε) of K and (L_ε) of L, $[\mu(K_\varepsilon\cap L_\varepsilon)]=0$

(ii) for some (and thus each) representative (K_ε) of K and (L_ε) of L

$$\lim_{m\to\infty}[\mu(\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(K_\varepsilon)\cap\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(L_\varepsilon))]=0.$$

The equivalence of the conditions follows by a similar argument as in Lem. 59, e.g.:

Lemma 71. *The conditions in Def. 70 are equivalent.*

Proof. $(i)\Rightarrow(ii)$: let $K=[K_\varepsilon]$ and $L=[L_\varepsilon]$. Seeking a contradiction, suppose that there exists $q\in\mathbb{N}$ for which it does not hold that

$$\exists M\,\forall m\geq M\,\forall^0\varepsilon:\ \mu(\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(K_\varepsilon)\cap\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(L_\varepsilon))\leq\rho_\varepsilon^q.$$

Then we can construct a strictly increasing sequence $(m_k)_k\to\infty$ and a strictly decreasing sequence $(\varepsilon_k)_k\to 0$ s.t. $\mu(\overline{B^{\mathrm{E}}}_{\rho_{\varepsilon_k}^{m_k}}(K_{\varepsilon_k})\cap\overline{B^{\mathrm{E}}}_{\rho_{\varepsilon_k}^{m_k}}(L_{\varepsilon_k}))>\rho_{\varepsilon_k}^q$, $\forall k$.
Let $K'_\varepsilon:=\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^{m_k}}(K_\varepsilon)$ and $L'_\varepsilon:=\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^{m_k}}(L_\varepsilon)$, whenever $\varepsilon\in(\varepsilon_{k+1},\varepsilon_k]$, $\forall k$. Then $K=[K'_\varepsilon]$ and $L=[L'_\varepsilon]$, but $[\mu(K'_\varepsilon\cap L'_\varepsilon)]\neq 0$, as $\mu(K'_\varepsilon\cap L'_\varepsilon)>\rho_\varepsilon^q$, for each $\varepsilon=\varepsilon_k$ $(k\in\mathbb{N})$.
$(ii)\Rightarrow(i)$: let (K_ε), (L_ε) as in (ii), and $K=[K'_\varepsilon]$ and $L=[L'_\varepsilon]$. For each $q\in\mathbb{N}$, we have that

$$[\mu(\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(K_\varepsilon)\cap\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(L_\varepsilon))]\leq\mathrm{d}\rho^q$$

for sufficiently large $m\in\mathbb{N}$. As $[K'_\varepsilon]\subseteq[K_\varepsilon]$, $K'_\varepsilon\subseteq\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(K_\varepsilon)$, $\forall^0\varepsilon$, and similarly for L, and thus also $[\mu(K'_\varepsilon\cap L'_\varepsilon)]\leq\mathrm{d}\rho^q$. $\square$

E.g., if $a\leq b\leq c$, then $[a,b]$ and $[b,c]$ are almost strongly disjoint. Obviously, strongly disjoint sets are almost strongly disjoint. Recall that the union of two internal sets is usually not internal, but

$$K\vee L:=[K_\varepsilon\cup L_\varepsilon]=\{e_Sx+e_{S^c}y : x\in K, y\in L, S\subseteq\,]0,1]\}$$

is the smallest internal set containing K and L [87]. E.g., $[a,b]\vee[b,c]=[a,c]$.

Theorem 72. *If K, L are almost strongly disjoint μ-measurable subsets of ${}^{\rho}\widetilde{\mathbb{R}}^{n}$, then $K \vee L$ is also μ-measurable and for each $f \in {}^{\rho}\mathcal{GI}(K \vee L, {}^{\rho}\widetilde{\mathbb{R}})$*

$$\int_{K\vee L} f \,\mathrm{d}\mu = \int_K f \,\mathrm{d}\mu + \int_L f \,\mathrm{d}\mu.$$

Proof. As $\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(K_\varepsilon \cup L_\varepsilon) = \overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(K_\varepsilon) \cup \overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(L_\varepsilon)$,

$$\left[\int_{\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(K_\varepsilon\cup L_\varepsilon)} f \,\mathrm{d}\mu\right] = \left[\int_{\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(K_\varepsilon)} f \,\mathrm{d}\mu\right] + \left[\int_{\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(L_\varepsilon)} f \,\mathrm{d}\mu\right] - \\ - \left[\int_{\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(K_\varepsilon)\cap\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(L_\varepsilon)} f \,\mathrm{d}\mu\right].$$

Since

$$\left|\left[\int_{\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(K_\varepsilon)\cap\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(L_\varepsilon)} f \,\mathrm{d}\mu\right]\right| \le \left[\mu(\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(K_\varepsilon) \cap \overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(L_\varepsilon))\right] \cdot \\ \cdot \left[\sup_{\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(K_\varepsilon)\cap\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(L_\varepsilon)} |f|\right] \overset{m\to\infty}{\to} 0$$

we see that $K \vee L$ is μ-measurable with $\int_{K\vee L} f \,\mathrm{d}\mu = \int_K f \,\mathrm{d}\mu + \int_L f \,\mathrm{d}\mu$. □

Example 73. Let $S \subseteq [0,1[$ with $0 \in \overline{S}$ and $0 \in \overline{S^c}$. Let $K_\varepsilon = \begin{cases} [0,1], & \varepsilon \in S \\ [0,3], & \varepsilon \in S^c \end{cases}$ and $L_\varepsilon = \begin{cases} [2,3], & \varepsilon \in S \\ [0,3], & \varepsilon \in S^c. \end{cases}$ Then $K \cap L = \emptyset$ and $\mu(K \vee L) = 2e_S + 3e_{S^c} \ne 2e_S + 6e_{S^c} = \mu(K) + \mu(L)$. Thus the condition that K and L are almost strongly disjoint cannot be replaced by the condition that $K \cap L = \emptyset$.

Theorem 74. *Let $K \Subset_f {}^{\rho}\widetilde{\mathbb{R}}^{n}$. Let $f_n \in {}^{\rho}\mathcal{GI}(K, {}^{\rho}\widetilde{\mathbb{R}}^{d})$, $\forall n \in {}^{\sigma}\widetilde{\mathbb{N}}$. If ${}^{\rho}\lim_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} f_n(x)$ exists for each $x \in K$, then the convergence is uniform over K and the limit function is integrable on K.*

Proof. We first show that the sequence is uniformly Cauchy, i.e. that for each $m \in \mathbb{N}$

$$\exists N \in \mathbb{N}\, \forall k, l \in {}^{\sigma}\widetilde{\mathbb{N}}\, \forall x \in K: \ k, l \ge \mathrm{d}\sigma^{-N} \ \Rightarrow\ |f_k(x) - f_l(x)| \le \mathrm{d}\rho^m. \tag{8.8}$$

Seeking a contradiction, suppose that for some $m \in \mathbb{N}$, we have

$$\forall N \in \mathbb{N}\, \exists k, l \in {}^{\sigma}\widetilde{\mathbb{N}}, k, l \ge \mathrm{d}\sigma^{-N}\, \exists x \in K: \ |f_k(x) - f_l(x)| \not\le \mathrm{d}\rho^m.$$

We thus construct sequences $(k_N)_N$ and $(l_N)_N$ in ${}^{\sigma}\widetilde{\mathbb{N}}$, with $k_N, l_N \ge \mathrm{d}\sigma^{-N}$ for which there exist $x_N \in K$ s.t. $|f_{k_N}(x_N) - f_{l_N}(x_N)| \not\le \mathrm{d}\rho^m$, $\forall N \in \mathbb{N}$. Let $K = [K_\varepsilon]$ and $f_k = [f_{k,\varepsilon}]$. We thus find $S_N \subseteq (0,1]$ with $0 \in \overline{S_N}$ such that $|f_{k_N,\varepsilon}(x_{N,\varepsilon}) - f_{l_N,\varepsilon}(x_{N,\varepsilon})| \ge \rho_\varepsilon^m$ for each $\varepsilon \in S_N$. Then choose a decreasing sequence $(\varepsilon_n)_n \to 0$ such that

$$\begin{aligned} &\varepsilon_1 \in S_1 \\ &\varepsilon_2 \in S_2; \varepsilon_3 \in S_1; \\ &\varepsilon_4 \in S_3; \varepsilon_5 \in S_2; \varepsilon_6 \in S_1; \\ &\dots \end{aligned}$$

Let $x_{\varepsilon_n} := x_{N,\varepsilon_n}$ if $\varepsilon_n \in S_N$, for each $n \in \mathbb{N}$. Extend to a net $(x_\varepsilon)_\varepsilon$ with $x_\varepsilon \in K_\varepsilon$, $\forall\varepsilon$. Then $xe_{U_N} = x_N e_{U_N}$ for some $U_N \subseteq S_N$ with $0 \in \overline{U_N}$, and therefore $|f_{k_N}(x) - f_{l_N}(x)|e_{U_N} \ge \mathrm{d}\rho^m e_{U_N}$, $\forall N$. We thus contradict the fact that $(f_n(x))_{n\in{}^\sigma\widetilde{\mathbb{N}}}$ is a convergent hypersequence.
By taking the limit for $l \to \infty$ in (8.8), we conclude that $(f_n)_{n\in{}^\sigma\widetilde{\mathbb{N}}}$ is uniformly convergent on K.
For each $n \in \mathbb{N}$, fix a representative $(f_{[\mathrm{d}\sigma^{-n}],\varepsilon})$, where $[\mathrm{d}\sigma^{-n}] := [\mathrm{int}\,(\sigma_\varepsilon^{-n})] \in {}^\sigma\widetilde{\mathbb{N}}$, of $f_{[\mathrm{d}\sigma^{-n}]}$ with $\sup_{x\in K_\varepsilon} |f_{[\mathrm{d}\sigma^{-n}],\varepsilon}(x)| \le \rho_\varepsilon^{-M}$, $\forall\varepsilon \in (0,1]$, $\forall n \in \mathbb{N}$ (some $M \in \mathbb{N}$, independent of n). For each $m \in \mathbb{N}$, there exists $N_m \in \mathbb{N}$ such that

$$\forall n, n' \ge N_m\, \forall^0\varepsilon : \sup_{x\in K_\varepsilon} |f_{[\mathrm{d}\sigma^{-n}],\varepsilon}(x) - f_{[\mathrm{d}\sigma^{-n'}],\varepsilon}(x)| \le \rho_\varepsilon^m.$$

Then for each $k \in \mathbb{N}$, there exists some $\varepsilon_k > 0$ such that

$$\forall\varepsilon \le \varepsilon_k\, \forall m \le k\, \forall n, n' \in [N_m, k] : \sup_{x\in K_\varepsilon} |f_{[\mathrm{d}\sigma^{-n}],\varepsilon}(x) - f_{[\mathrm{d}\sigma^{-n'}],\varepsilon}(x)| \le \rho_\varepsilon^m.$$

W.l.o.g., $(\varepsilon_k)_k \downarrow 0$. Let $n_\varepsilon := k$, for $\varepsilon \in (\varepsilon_{k+1}, \varepsilon_k]$. Then

$$\forall m \in \mathbb{N}\, \forall n \in \mathbb{N}, n \ge N_m\, \forall^0\varepsilon : \sup_{x\in K_\varepsilon} |f_{[\mathrm{d}\sigma^{-n}],\varepsilon}(x) - f_{[\mathrm{d}\sigma^{-n_\varepsilon}],\varepsilon}(x)| \le \rho_\varepsilon^m.$$

Thus the limit function $f = [f_{[\mathrm{d}\sigma^{-n_\varepsilon}],\varepsilon}(-)] \in {}^\rho\mathcal{GI}(K, {}^\rho\widetilde{\mathbb{R}})$. □

Theorem 75. *Let $K \Subset_f {}^\rho\widetilde{\mathbb{R}}^n$ be μ-measurable. Let $f_n \in {}^\rho\mathcal{GI}(K, {}^\rho\widetilde{\mathbb{R}}^d)$, $\forall n \in {}^\sigma\widetilde{\mathbb{N}}$. If ${}^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} f_n(x)$ exists for each $x \in K$, then ${}^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} f_n$ is integrable on K and*

$$ {}^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} \int_K f_n \,\mathrm{d}\mu = \int_K {}^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} f_n \,\mathrm{d}\mu.$$

Proof. By Thm. 74, $(f_n)_n$ uniformly converges to some $f = [f_\varepsilon(-)]$. Then for all $q \in \mathbb{N}$ we have

$$\left| \int_K f_n - \int_K f \right| \le \int_K |f_n - f| \le \mathrm{d}\rho^q \mu(K)$$

as soon as $n \in {}^\sigma\widetilde{\mathbb{N}}$ is large enough. □

9. Sheaf properties

The aim of this section is to establish appropriate sheaf properties for GSF. That this task is not entirely straightforward can be seen from the following example, which can be easily reformulated in other non-Archimedean settings:

Example 76. Let $i : {}^\rho\widetilde{\mathbb{R}} \to {}^\rho\widetilde{\mathbb{R}}$ be as in Rem. 40, i.e., $i(x) := 1$ if $x \approx 0$ and $i(x) := 0$ otherwise. The domain ${}^\rho\widetilde{\mathbb{R}}$ of this function is the disjoint union of the sharply open sets $D_\infty = \{x \in {}^\rho\widetilde{\mathbb{R}} \mid x \approx 0\}$ and its complement D_∞^{c}. Moreover, $i|_{D_\infty} \equiv 1$ and $i|_{D_\infty^{\mathrm{c}}} \equiv 0$ are both GSF. However, as we have seen in the remark following Cor. 48, i itself is *not* a GSF. This shows that ${}^\rho\mathcal{GC}^\infty$ is not a sheaf with respect to the sharp topology.

Trivially, if we introduce the space of (sharply) locally defined GSF by means of $f \in {}^\rho\mathcal{GC}^\infty_{\mathrm{loc}}(X, Y)$ if $f : X \to Y$, and $\forall x \in X\, \exists r \in {}^\rho\widetilde{\mathbb{R}}_{>0} : f|_{B_r(x)\cap X} \in {}^\rho\mathcal{GC}^\infty(B_r(x) \cap X, Y)$, then ${}^\rho\mathcal{GC}^\infty_{\mathrm{loc}}(-, Y)$ is naturally a sheaf with respect to the sharp topology. By Example 76, however, ${}^\rho\mathcal{GC}^\infty_{\mathrm{loc}}(X, Y)$ is strictly larger than ${}^\rho\mathcal{GC}^\infty(X, Y)$. This fact can be viewed as a necessary trade-off between the classical statement of locality for generalized functions, on the one hand, and the requirement to preserve classical theorems from smooth analysis on the other. In the above example, it is the validity of an intermediate value theorem in our setting that precludes the function i from qualifying as a GSF. Conversely, it follows that this result does not hold in ${}^{\rho}\mathcal{GC}^{\infty}_{\text{loc}}(X,Y)$. Any theory of generalized functions that is based on set-theoretical functions and includes actual infinitesimals has to face these dichotomies related to the total disconnectedness of its non-Archimedean ring of scalars.

The general scheme of this section is:

(a) We are searching for a new compatibility/coherence condition for an arbitrarily indexed family $(f_j)_{j\in J}$ of GSF (throughout this section, J will be an arbitrary set), which allows us to prove a corresponding sheaf property. We will call this property *dynamic compatibility condition* (DCC).
(b) The DCC must imply the classical one. Note that for particular types of covers the classic coherence condition may still work, e.g. covers made of near-standard points and large open sets (see Cor. 87 below) or those made of increasing sequences of internal sets (see Thm. 77 below).
(c) The DCC must be a necessary condition if we assume that the sections $(f_j)_{j\in J}$ glue together into a GSF.
(d) The sheaf property based on the DCC should be a particular case of the general abstract notion of sheaf (see Sec. 10).

We start from the following sheaf property (originally proved in [116]):

Theorem 77. *Let $X\subseteq{}^{\rho}\widetilde{\mathbb{R}}^{n}$, $Y\subseteq{}^{\rho}\widetilde{\mathbb{R}}^{d}$ and $K_q\Subset_f{}^{\rho}\widetilde{\mathbb{R}}^{n}$, $f_q\in{}^{\rho}\mathcal{GC}^{\infty}(K_q,Y)$ for all $q\in\mathbb{N}$, where $X=\bigcup_{q\in\mathbb{N}}K_q$, $K_q\subseteq\text{int}\left(K_{q+1}\right)$ and $f_{q+1}|_{K_q}=f_q$ for each $q\in\mathbb{N}$. Then there exists a unique $f\in{}^{\rho}\mathcal{GC}^{\infty}(X,Y)$ such that $f|_{K_q}=f_q$ for all $q\in\mathbb{N}$.*

Proof. Let $f_q=[f_{q,\varepsilon}(-)]$ and $K_q=[K_{q,\varepsilon}]$, for each $q\in\mathbb{N}$. By Lem. 11(v), there exist $k_q\in\mathbb{N}$ (k_q recursively chosen so that $(k_q)_q$ is increasing) such that $B^{\text{E}}_{\rho_\varepsilon^{k_q}}(K_{q,\varepsilon})\subseteq K_{q+1,\varepsilon}$, for each q, ε. We may assume $Y\subseteq{}^{\rho}\widetilde{\mathbb{R}}$ (in general, one can apply the one-dimensional case componentwise). Let $\theta\in\mathcal{C}^{\infty}(\mathbb{R}^n)$ with $\theta(x)=0$, if $|x|\ge1$ and $\theta(x)\ge0$, for each $x\in\mathbb{R}^n$ with $\int_{\mathbb{R}^n}\theta=1$ and let $r\odot\theta_r(x):=r^{-n}\theta(r^{-1}x)$, for $r\in\mathbb{R}_{>0}$. Let 1_A denote the characteristic function of a set $A\subseteq\mathbb{R}^n$, and set

$$\varphi_{q,\varepsilon}:=1_{K_{q+3,\varepsilon}\setminus K_{q,\varepsilon}}*\rho_\varepsilon^{k_{q+3}}\odot\theta,\quad\forall q,\varepsilon.$$

If $y\in B^{\text{E}}_{\rho_\varepsilon^{k_{q+3}}}(x)\cap K_{q,\varepsilon}$, then $x\in B^{\text{E}}_{\rho_\varepsilon^{k_q}}(K_{q,\varepsilon})\subseteq K_{q+1,\varepsilon}$, and hence $B^{\text{E}}_{\rho_\varepsilon^{k_{q+3}}}(x)\cap K_{q,\varepsilon}=\emptyset$ if $x\notin K_{q+1,\varepsilon}$. If $x\in K_{q+2,\varepsilon}$, then $B^{\text{E}}_{\rho_\varepsilon^{k_{q+3}}}(x)\subseteq B^{\text{E}}_{\rho_\varepsilon^{k_{q+2}}}(K_{q+2,\varepsilon})\subseteq K_{q+3,\varepsilon}$. Thereby, $\varphi_{q,\varepsilon}(x)=1$, for each $x\in K_{q+2,\varepsilon}\setminus K_{q+1,\varepsilon}$. Moreover, $\text{stsupp}\varphi_{q,\varepsilon}\subseteq K_{q+3,\varepsilon}+B^{\text{E}}_{\rho_\varepsilon^{k_{q+3}}}(0)=B^{\text{E}}_{\rho_\varepsilon^{k_{q+3}}}(K_{q+3,\varepsilon})\subseteq K_{q+4,\varepsilon}$. Further, $\sup_{x\in\mathbb{R}^n}|\partial^\alpha\varphi_{q,\varepsilon}(x)|\le\rho_\varepsilon^{-k_{q+3}|\alpha|}\int_{\mathbb{R}^n}|\partial^\alpha\theta|$ by the properties of the convolution. Let $\varphi_\varepsilon:=\sum_{q\in\mathbb{N}}\varphi_{q,\varepsilon}$. Then $\varphi_\varepsilon\in\mathcal{C}^\infty(\bigcup_{q\in\mathbb{N}}K_{q,\varepsilon})$ and for each q, $(\sup_{x\in K_{q,\varepsilon}}|\partial^\alpha\varphi_\varepsilon(x)|)\in\mathbb{R}_\rho$. Also $\varphi_\varepsilon(x)\ge1$, for each $x\in\bigcup_{q\in\mathbb{N}}K_{q,\varepsilon}$. Let $\psi_{q,\varepsilon}:=\varphi_{q,\varepsilon}/\varphi_\varepsilon\in\mathcal{C}^\infty(\mathbb{R}^n)$. Then $\sum_{q\in\mathbb{N}}\psi_{q,\varepsilon}(x)=1$, for each $x\in\bigcup_{q\in\mathbb{N}}K_{q,\varepsilon}$. Since $\sup_{x\in K_{q,\varepsilon}}|1/\varphi_\varepsilon(x)|\le1$, we find that $(\sup_{x\in\mathbb{R}^n}|\partial^\alpha\psi_{q,\varepsilon}(x)|)\in\mathbb{R}_\rho$, for each q. Let $f_\varepsilon:=\sum_{q\in\mathbb{N}}\psi_{q,\varepsilon}\cdot f_{q+3,\varepsilon}\in\mathcal{C}^\infty(\bigcup_{q\in\mathbb{N}}K_{q,\varepsilon})$, for each ε (recall that $\text{stsupp}\psi_{q,\varepsilon}\subseteq K_{q+3,\varepsilon}$). Then for each $N\in\mathbb{N}$, $\alpha\in\mathbb{N}^d$ and $x=[x_\varepsilon]\in K_N$ (without loss of generality, $x_\varepsilon\in K_{N,\varepsilon}$, for each ε),

$$|\partial^\alpha f_\varepsilon(x_\varepsilon)|\le\sum_{q\le N+3}|\partial^\alpha(\psi_{q,\varepsilon}\cdot f_{q+3,\varepsilon})(x_\varepsilon)|\in\mathbb{R}_\rho$$

and

$$|f_\varepsilon(x_\varepsilon) - f_{N,\varepsilon}(x_\varepsilon)| \le \sum_{q\le N+3} |\psi_{q,\varepsilon}(x_\varepsilon)|\,|f_{q+3,\varepsilon}(x_\varepsilon) - f_{N,\varepsilon}(x_\varepsilon)| \sim_\rho 0$$

since $\text{stsupp}\psi_{q,\varepsilon} \subseteq K_{q+3,\varepsilon}$. Hence $f \in {}^\rho\mathcal{GC}^\infty(X,Y)$ with $f|_{K_N} = f_N$. □

This property allows us to firstly prove a sheaf property for $K = [K_\varepsilon] \Subset_{\text{f}} {}^\rho\widetilde{\mathbb{R}}^n$ and secondly to use Thm. 77 to extend it to domains X that satisfy the previous assumptions, i.e. the following

Definition 78. Let $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$, then we say that X *admits a functionally compact exhaustion* if there exists a sequence $(K_q)_{q\in\mathbb{N}}$ such that

(i) $K_q \Subset_{\text{f}} {}^\rho\widetilde{\mathbb{R}}^n$;
(ii) $K_q \subseteq \text{int}\,(K_{q+1})$;
(iii) $X = \bigcup_{q\in\mathbb{N}} K_q$.

For example, every strongly internal set $X = \langle A_\varepsilon\rangle$ admits a functionally compact exhaustion since we can consider

$$\begin{aligned} K_{q\varepsilon} &:= \overline{B^{\text{E}}}_{\rho_\varepsilon^{-q}}(0) \cap \overline{B^{\text{E}}}_{-\rho_\varepsilon^q}(A_\varepsilon) \qquad \forall q\in\mathbb{N} \\ K_q &:= [K_{q\varepsilon}] \Subset_{\text{f}} X \\ X &= \bigcup_{q\in\mathbb{N}} K_q \end{aligned} \tag{9.1}$$

where $\overline{B^{\text{E}}}_{-r}(A) := \{x \in A \mid d(x, A^c) \ge r\}$. Other simple examples are e.g. the intervals $(0, a]$ or $(-\infty, a]$.

9.1. **The Lebesgue generalized number.** We first introduce a notation for a specified Lebesgue number:

Lemma 79. *Let $K \Subset \mathbb{R}^n$ and $(V_j)_{j\in J}$ be an open cover of K. For $x \in K$, set*

$$\sigma(x) := \sup\{r \in \mathbb{R}_{>0} \mid \exists j \in J :\ B^{\text{E}}_r(x) \subseteq V_j\} \tag{9.2}$$

$$\sigma := \frac{1}{2}\min\{\sigma(x) \mid x \in K\};\ \textit{if } K = \emptyset,\ \textit{set } \sigma := 1.$$

Then $\sigma(-) : K \longrightarrow \mathbb{R}_{>0}$ is a continuous function and σ is a Lebesgue number of $(V_j)_{j\in J}$ for K, i.e.

$$\forall x \in K\, \exists j \in J :\ B^{\text{E}}_\sigma(x) \subseteq V_j. \tag{9.3}$$

We use the notation $\text{Lebnum}\,((V_j)_{j\in J}, K) =: \sigma$.

Proof. See the proof of [12, Thm. 1.6.11]. □

Lemma 80. *Let $K = [K_\varepsilon] \Subset_f {}^\rho\widetilde{\mathbb{R}}^n$ with $K_\varepsilon \Subset \mathbb{R}^n$ for all ε. Assume that*

$$K \subseteq \bigcup_{j\in J} U_j, \tag{9.4}$$

where $U_j = \langle U_{j\varepsilon}\rangle$ are strongly internal sets, and set

$$s_\varepsilon := \text{Lebnum}\left((U_{j\varepsilon})_{j\in J}, K_\varepsilon\right)$$

$$s := [s_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}_{\ge 0}.$$

Then $s > 0$.

Proof. By contradiction, assume that $s \not> 0$, so that $s_{\varepsilon_k} < \rho^k_{\varepsilon_k}$ for all $k \in \mathbb{N}$ and for some sequence $(\varepsilon_k)_{k\in\mathbb{N}} \downarrow 0$. By definition of Lebnum we can write $s_{\varepsilon_k} = \frac{1}{2}\sigma(x_{\varepsilon_k}) < \rho^k_{\varepsilon_k}$ for some $x_{\varepsilon_k} \in K_{\varepsilon_k} \Subset \mathbb{R}^n$. By (9.2) we hence have

$$\forall k \in \mathbb{N}\, \forall j \in J: \ B^{\mathrm{E}}_{2\rho^k_{\varepsilon_k}}(c_{\varepsilon_k}) \nsubseteq U_{j\varepsilon_k}. \tag{9.5}$$

Note that the conclusion is trivial if $K_\varepsilon = \emptyset$ for ε small. We can therefore assume that there exists some $h_\varepsilon \in K_\varepsilon$ for all ε. Let $x_\varepsilon := x_{\varepsilon_k}$ if $\varepsilon = \varepsilon_k$ and $x_\varepsilon := h_\varepsilon$ otherwise, so that $x_\varepsilon \in K_\varepsilon \subseteq \bigcup_{j\in J} U_{j\varepsilon}$ for small ε (see Lem. 11(i) and recall that $K \subseteq \bigcup_{j\in J} U_j \subseteq \langle \bigcup_{j\in J} U_{j\varepsilon}\rangle$). Thereby, $x := [x_\varepsilon] \in K \subseteq \bigcup_{j\in J} U_j$. So, $x \in U_j$ for some $j \in J$, and hence $\overline{B_R(x)} \subseteq U_j = \langle U_{j\varepsilon}\rangle$ for some $R \in {}^\rho\widetilde{\mathbb{R}}_{>0}$, so that $B^{\mathrm{E}}_{R_{\varepsilon_k}}(x_{\varepsilon_k}) \subseteq U_{j\varepsilon_k}$ for $k \in \mathbb{N}$ sufficiently large by Lem. 11(i) and (2.6). Since also $2\rho^k_{\varepsilon_k} < R_{\varepsilon_k}$ for $k \in \mathbb{N}$ sufficiently large, we can finally say that $B^{\mathrm{E}}_{2\rho^k_{\varepsilon_k}}(x_{\varepsilon_k}) \subseteq B^{\mathrm{E}}_{R_{\varepsilon_k}}(x_{\varepsilon_k}) \subseteq U_{j\varepsilon_k}$, which contradicts (9.5). □

On the basis of this result, we can set

$$\mathrm{Lebnum}\left(\left(U_j\right)_{j\in J}, K\right) =: s \in {}^\rho\widetilde{\mathbb{R}}_{>0}.$$

Assumption (9.4) cannot be replaced by the weaker $K \subseteq \langle \bigcup_{j\in J} U_{j\varepsilon}\rangle$: let $K_\varepsilon := [-1,1]_{\mathbb{R}}$, $J := \{1,2\}$, $c_{1\varepsilon} := -1$, $c_{2\varepsilon} := 1$, $r_{1\varepsilon} := 1 + e^{-1/\varepsilon} =: r_{2\varepsilon}$. Then

$$[K_\varepsilon] \subseteq \langle B^{\mathrm{E}}_{r_{1\varepsilon}}(c_{1\varepsilon}) \cup B^{\mathrm{E}}_{r_{2\varepsilon}}(c_{2\varepsilon})\rangle = \langle (-2 - e^{-1/\varepsilon}, 2 + e^{-1/\varepsilon})_{\mathbb{R}}\rangle$$

but $s_\varepsilon = \mathrm{Lebnum}((U_{j\varepsilon})_{j\in J}, K_\varepsilon) \le e^{-1/\varepsilon}$ because for $x = 0$ the largest ball contained in a set of the covering is $B^{\mathrm{E}}_{e^{-1/\varepsilon}}(0)$.

9.2. **The dynamic compatibility condition.**

Definition 81.

(i) Let $[J] := \{(j_\varepsilon) \mid j_\varepsilon \in J, \forall \varepsilon\} = J^I$.

(ii) Let $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$, $Y \subseteq {}^\rho\widetilde{\mathbb{R}}^d$ and $f \in \mathbf{Set}(X,Y)$. Let $K \Subset_{\mathrm{f}} X \subseteq \bigcup_{j\in J} U_j$, and assume that for all $j \in J$ we have $f_j := f|_{U_j\cap X} \in {}^\rho\mathcal{GC}^\infty(U_j \cap X, Y)$. Then we say that $\left(f_j\right)_{j\in J}$ *satisfies the dynamic compatibility condition (DCC) on the cover* $\left(K \cap U_j\right)_{j\in J}$ if for all $j \in J$ there exist nets $(f_{j\varepsilon})$ defining f_j, for each $j \in J$, such that setting $U_{\bar{\jmath}} := \langle U_{j_\varepsilon,\varepsilon}\rangle$, we have:
  (a) $\forall \bar{\jmath} = (j_\varepsilon) \in [J]\, \forall [x_\varepsilon] \in U_{\bar{\jmath}} \cap K\, \forall \alpha \in \mathbb{N}^n: \ (\partial^\alpha f_{j_\varepsilon,\varepsilon}(x_\varepsilon)) \in \mathbb{R}^d_\rho$.
  (b) $\forall \bar{\jmath} = (j_\varepsilon), \bar{h} = (h_\varepsilon) \in [J]\, \forall [x_\varepsilon] \in K \cap U_{\bar{\jmath}} \cap U_{\bar{h}}: \ [f_{j_\varepsilon,\varepsilon}(x_\varepsilon)] = [f_{h_\varepsilon,\varepsilon}(x_\varepsilon)]$.

Finally, we say that $\left(f_j\right)_{j\in J}$ *satisfies the DCC on the cover* $\left(U_j\right)_{j\in J}$ if it satisfies the DCC on each functionally compact set contained in X. The adjective *dynamic* underscores that we are considering ε-depending indices $\bar{\jmath} = (j_\varepsilon) \in [J]$.

Remark 82.

(i) Taking constant $\bar{\jmath}$ and $\bar{h}$ in Def. 81(b), we have that DCC is stronger than the classical compatibility condition for $\left(f_j\right)_{j\in J}$ on $K \Subset_{\mathrm{f}} X$.

(ii) DCC is a necessary condition if the sections $\left(f_j\right)_{j\in J}$ glue into a GSF $f = [f_\varepsilon(-)]$ because in this case we can take $f_{j\varepsilon} = f_\varepsilon$ for all $j \in J$.

(iii) The notation $U_{\bar{\jmath}} := \langle U_{j_\varepsilon,\varepsilon}\rangle$ used to state the DCC was introduced merely for simplicity of notations. In fact, in general it is not possible to prove the independence from the representative net $(U_{j\varepsilon})$: if $U_j = \langle V_{j\varepsilon}\rangle$, we can have

$d_{\mathrm{H}}(U^c_{j\varepsilon}, V^c_{j\varepsilon}) = \rho_\varepsilon^{j/\varepsilon}$, but taking $\bar{\jmath} = (\varepsilon)$ we would have $d_{\mathrm{H}}(U^c_{\varepsilon,\varepsilon}, V^c_{\varepsilon,\varepsilon}) = \rho_\varepsilon$ and hence $U_{\bar{\jmath}} = \langle U_{\varepsilon,\varepsilon}\rangle \neq V_{\bar{\jmath}} = \langle V_{\varepsilon,\varepsilon}\rangle$.

In the next and final subsection we prove that the DCC implies the sheaf property.

9.3. **Proof of the sheaf property.**

Lemma 83. *Let K, K^+ be functionally compact sets with $K \subseteq int(K^+) \subseteq {}^\rho\widetilde{\mathbb{R}}^n$. Let $Y \subseteq {}^\rho\widetilde{\mathbb{R}}^d$ and $f \in \mathbf{Set}(K^+, Y)$. Let $K^+ \subseteq \bigcup_{j\in J} U_j$, where $U_j = \langle U_{j\varepsilon}\rangle$ are strongly internal sets and for all $j \in J$ we have $f_j := f|_{U_j\cap K^+} \in {}^\rho\mathcal{GC}^\infty(U_j \cap K^+, Y)$. Let $K^+ = [K^+_\varepsilon]$. Assume that, for some representatives $(f_{j\varepsilon})_{\substack{j\in J\\ \varepsilon\in I}}$ of $(f_j)_{j\in J}$ we have*

$$\forall m \in \mathbb{N}\, \forall^0\varepsilon\, \forall j, k \in J : \sup_{K^+_\varepsilon\cap U_{j\varepsilon}\cap U_{k\varepsilon}} |f_{k\varepsilon} - f_{j\varepsilon}| \le \rho_\varepsilon^m. \tag{9.6}$$

Then $f \in {}^\rho\mathcal{GC}^\infty(K, Y)$.

Proof. Let $K = [K_\varepsilon]$. By changing the representative $(K^+_\varepsilon)_\varepsilon$ of K^+, we may assume that there exists $S \in \mathbb{N}$ such that $B_{\rho_\varepsilon^S}(K_\varepsilon) \subseteq K^+_\varepsilon$, $\forall\varepsilon$ [116, Lemma 3.12]. We have that $K^+_\varepsilon \subseteq \bigcup_{j\in J} U_{j,\varepsilon}$, $\forall\varepsilon \le \varepsilon_0$, for some $\varepsilon_0 > 0$ (proof by contradiction). Let $\varepsilon \in (0, \varepsilon_0]$. By compactness of K^+_ε, $K^+_\varepsilon \subseteq U_{j_1,\varepsilon} \cup \cdots \cup U_{j_{l_\varepsilon},\varepsilon}$ for some $l_\varepsilon \in \mathbb{N}$. Call $V_k := U_{j_k,\varepsilon} \setminus (U_{j_1,\varepsilon} \cup \cdots \cup U_{j_{k-1},\varepsilon})$, and let

$$f_\varepsilon := \sum_{k=1}^{l_\varepsilon} f_{j_k,\varepsilon} 1_{V_k}.$$

Then f_ε is locally integrable. Let $m \in \mathbb{N}$. By (9.6), we find $\varepsilon_m > 0$ such that

$$\forall j, k \in J\, \forall\varepsilon \le \varepsilon_m : \sup_{K^+_\varepsilon\cap U_{j,\varepsilon}\cap U_{k,\varepsilon}} |f_{k,\varepsilon} - f_{j,\varepsilon}| \le \rho_\varepsilon^m.$$

Further, by the definition of f_ε, we then also have

$$\forall j \in J\, \forall\varepsilon \le \varepsilon_m : \sup_{K^+_\varepsilon\cap U_{j,\varepsilon}} |f_\varepsilon - f_{j,\varepsilon}| \le \rho_\varepsilon^m.$$

W.l.o.g., $\varepsilon_m \downarrow 0$. Let $q_\varepsilon := q$, for each $\varepsilon \in (\varepsilon_{(q+1)^2}, \varepsilon_{q^2}]$ ($q \in \mathbb{N}$). Then

$$\forall j \in J\, \forall\varepsilon \le \varepsilon_0 : \sup_{K^+_\varepsilon\cap U_{j,\varepsilon}} |f_\varepsilon - f_{j,\varepsilon}| \le \rho_\varepsilon^{q_\varepsilon^2}.$$

Let b be a smooth map $\mathbb{R}^n \to \mathbb{R}$ with $\int b = 1$ and $\mathrm{stsupp}(b) \subseteq B_1(0)$, and let

$$\delta_{q,\varepsilon}(x) := \rho_\varepsilon^{-nq} b\left(\frac{x}{\rho_\varepsilon^q}\right).$$

We show that $(f_\varepsilon * \delta_{q_\varepsilon,\varepsilon})_\varepsilon$ is a representative of f, thereby proving that $f \in {}^\rho\mathcal{GC}^\infty(K, {}^\rho\widetilde{\mathbb{R}}^d)$. We therefore take $x = [x_\varepsilon] \in K$, and we prove that

(i) $(\partial^\alpha(f_\varepsilon * \delta_{q_\varepsilon,\varepsilon})(x_\varepsilon)) \in \mathbb{R}^d_\rho$, $\forall\alpha \in \mathbb{N}^n$
(ii) $[(f_\varepsilon * \delta_{q_\varepsilon,\varepsilon})(x_\varepsilon)] = f(x)$.

Let $j \in J$ such that $x \in U_j$. Then there exists $S \in \mathbb{N}$ such that $B_{\rho_\varepsilon^S}(x_\varepsilon) \subseteq K^+_\varepsilon \cap U_{j,\varepsilon}$, $\forall^0\varepsilon$. To prove (i) and (ii), it suffices to see that $(\partial^\alpha(f_\varepsilon * \delta_{q_\varepsilon,\varepsilon})(x_\varepsilon) - \partial^\alpha f_{j,\varepsilon}(x_\varepsilon))_\varepsilon$ is negligible for each $\alpha \in \mathbb{N}^n$.

Let $\alpha \in \mathbb{N}^n$. Then

$$\begin{aligned}
&|\partial^\alpha(f_\varepsilon * \delta_{q_\varepsilon,\varepsilon})(x_\varepsilon) - \partial^\alpha(f_{j,\varepsilon} * \delta_{q_\varepsilon,\varepsilon})(x_\varepsilon)| = |((f_\varepsilon - f_{j,\varepsilon}) * \partial^\alpha \delta_{q_\varepsilon,\varepsilon})(x_\varepsilon)| \\
&= \rho_\varepsilon^{-|\alpha| q_\varepsilon} \left| \int (f_\varepsilon - f_{j,\varepsilon})(x_\varepsilon - \rho_\varepsilon^{q_\varepsilon} u) \partial^\alpha b(u)\, du \right| \le C_\alpha \rho_\varepsilon^{-|\alpha| q_\varepsilon} \sup_{B_{\rho_\varepsilon^{q_\varepsilon}}(x_\varepsilon)} |f_\varepsilon - f_{j,\varepsilon}| \\
&\le C_\alpha \rho_\varepsilon^{(q_\varepsilon - |\alpha|) q_\varepsilon}
\end{aligned}$$

and

$$\begin{aligned}
&|\partial^\alpha(f_{j,\varepsilon} * \delta_{q_\varepsilon,\varepsilon})(x_\varepsilon) - \partial^\alpha f_{j,\varepsilon}(x_\varepsilon)| = \left| \int (\partial^\alpha f_{j,\varepsilon}(x_\varepsilon - y) - \partial^\alpha f_{j,\varepsilon}(x_\varepsilon)) \delta_{q_\varepsilon,\varepsilon}(y)\, dy \right| \\
&\le C \sup_{u \in B_{\rho_\varepsilon^{q_\varepsilon}}(x_\varepsilon)} |\partial^\alpha f_{j,\varepsilon}(u) - \partial^\alpha f_{j,\varepsilon}(x_\varepsilon)| \le C \rho_\varepsilon^{q_\varepsilon} \max_{|\beta| = |\alpha|+1} \sup_{u \in B_{\rho_\varepsilon^{q_\varepsilon}}(x_\varepsilon)} |\partial^\beta f_{j,\varepsilon}(u)| \\
&\le C \rho_\varepsilon^{q_\varepsilon - N_{j,\alpha}}
\end{aligned}$$

for sufficiently small ε, by Thm. 17 (i). Combining both inequalities, we conclude that $(\partial^\alpha(f_\varepsilon * \delta_{q_\varepsilon,\varepsilon})(x_\varepsilon) - \partial^\alpha f_{j,\varepsilon}(x_\varepsilon))_\varepsilon$ is ρ-negligible. ☐

Lemma 84. *Let K, K^+ be functionally compact sets with $K \subseteq int(K^+) \subseteq {}^\rho\widetilde{\mathbb{R}}^n$. Let $Y \subseteq {}^\rho\widetilde{\mathbb{R}}^d$ and $f \in \mathbf{Set}(K^+, Y)$. Let $K^+ \subseteq \bigcup_{j \in J} U_j$, where for all $j \in J$ we have $f_j := f|_{U_j \cap K^+} \in {}^\rho\mathcal{GC}^\infty(U_j \cap K^+, Y)$ and U_j is a strongly internal set. Assume that $(f_j)_{j \in J}$ satisfies the DCC on* the cover $(K^+ \cap U_j)_{j \in J}$. *Then $f \in {}^\rho\mathcal{GC}^\infty(K, Y)$.*

Proof. Let $s_\varepsilon := \frac{1}{2}\text{Lebnum}\left((U_{j\varepsilon})_{j \in J}, K_\varepsilon^+\right)$. By Lemma 80, $s := [s_\varepsilon] > 0$.
We define a cover $(V_x)_{x \in K^+}$ of K^+ as follows. Let $x = [x_\varepsilon] \in K^+$, where $x_\varepsilon \in K_\varepsilon^+$, $\forall \varepsilon$. Then there exist $j_\varepsilon \in J$ such that $B^E_{2 s_\varepsilon}(x_\varepsilon) \subseteq U_{j_\varepsilon,\varepsilon}$, $\forall \varepsilon$. We then define $V_x := \langle B^{\mathrm{E}}_{s_\varepsilon}(x_\varepsilon) \rangle$, and we denote $\bar{\jmath}(x) := [j_\varepsilon] \in [J]$. By condition (a), $(f_{j_\varepsilon,\varepsilon})$ defines a generalized smooth map $f_{\bar{\jmath}(x)} \in {}^\rho\mathcal{GC}^\infty(V_x \cap K^+, Y)$.
In order to conclude that $f \in {}^\rho\mathcal{GC}^\infty(K, Y)$ by Lemma 83, it suffices to show that:

(i) $f_{\bar{\jmath}(x)} = f|_{V_x \cap K^+}$, $\forall x \in K^+$.
(ii) $\forall m \in \mathbb{N}\, \forall^0 \varepsilon\, \forall x, y \in K^+ :\ \sup_{K_\varepsilon^+ \cap V_{x,\varepsilon} \cap V_{y,\varepsilon}} |f_{\bar{\jmath}(y),\varepsilon} - f_{\bar{\jmath}(x),\varepsilon}| \le \rho_\varepsilon^m$.

Proof of (i): Let $x \in K^+$. Let $y \in V_x \cap K^+$. We want to show that $f_{\bar{\jmath}(x)}(y) = f(y)$. As $y \in K^+$, we have $y \in U_j$ for some $j \in J$. Thus $y \in K^+ \cap U_j \cap V_x \subseteq K^+ \cap U_j \cap U_{\bar{\jmath}(x)}$, and condition (b) yields $f(y) = f_j(y) = f_{\bar{\jmath}(x)}(y)$.
Proof of (ii): By contradiction, suppose that there exists $m \in \mathbb{N}$ and, for each $n \in \mathbb{N}$, there exist $\varepsilon_n > 0$ with $(\varepsilon_n)_n$ decreasingly tending to 0 and $x_n, y_n \in K^+$ and $z_{\varepsilon_n} \in K^+_{\varepsilon_n} \cap V_{x_n,\varepsilon_n} \cap V_{y_n,\varepsilon_n}$ such that

$$|f_{j_{\varepsilon_n},\varepsilon_n}(z_{\varepsilon_n}) - f_{h_{\varepsilon_n},\varepsilon_n}(z_{\varepsilon_n})| > \rho_{\varepsilon_n}^m \tag{9.7}$$

where we denote $j_{\varepsilon_n} := \bar{\jmath}(x_n)_{\varepsilon_n}$ and $h_{\varepsilon_n} := \bar{\jmath}(y_n)_{\varepsilon_n} \in J$. Then there exist $x_{\varepsilon_n}, y_{\varepsilon_n} \in K^+_{\varepsilon_n}$ such that $V_{x_n,\varepsilon_n} = B^E_{s_{\varepsilon_n}}(x_{\varepsilon_n})$ and $V_{y_n,\varepsilon_n} = B^E_{s_{\varepsilon_n}}(y_{\varepsilon_n})$. For $\varepsilon \notin \{\varepsilon_n : n \in \mathbb{N}\}$, let $z_\varepsilon \in K^+_\varepsilon$ be arbitrary. Then $z \in K^+$. Let $j_\varepsilon := h_\varepsilon := \bar{\jmath}(z)_\varepsilon$, for $\varepsilon \notin \{\varepsilon_n : n \in \mathbb{N}\}$. Let (z'_ε) be any representative of z. Then

$$\begin{cases} z'_\varepsilon \in B^E_{s_\varepsilon}(z_\varepsilon) \subseteq U_{j_\varepsilon,\varepsilon} = U_{h_\varepsilon,\varepsilon} & \forall^0 \varepsilon \notin \{\varepsilon_n : n \in \mathbb{N}\} \\ z'_{\varepsilon_n} \in B^E_{s_{\varepsilon_n}}(z_{\varepsilon_n}) \subseteq B^E_{2 s_{\varepsilon_n}}(x_{\varepsilon_n}) \subseteq U_{j_{\varepsilon_n},\varepsilon_n} & \forall n \in \mathbb{N} \text{ large enough} \end{cases}$$

and similarly $z'_{\varepsilon_n} \in U_{h_{\varepsilon_n},\varepsilon_n}$ for large enough $n \in \mathbb{N}$. Thus $z \in K^+ \cap U_{\bar{\jmath}} \cap U_{\bar{h}}$, and condition (b) contradicts (9.7). ☐

Using Thm. 77 and a functionally compact exhaustion, see Def. 78, we get

Theorem 85. *Let $X \subseteq {}^{\rho}\widetilde{\mathbb{R}}^n$ be a set that admits a functionally compact exhaustion, $Y \subseteq {}^{\rho}\widetilde{\mathbb{R}}^d$ and $f \in \mathbf{Set}(X,Y)$. Let $X \subseteq \bigcup_{j\in J} U_j$, where for all $j \in J$ we have $f_j := f|_{U_j \cap X} \in {}^{\rho}\mathcal{GC}^{\infty}(U_j \cap X, Y)$ and U_j is a strongly internal set. Assume that $(f_j)_{j\in J}$ satisfies the DCC on* the cover $(U_j)_{j\in J}$. *Then $f \in {}^{\rho}\mathcal{GC}^{\infty}(X,Y)$.*

The usual sheaf properties both for Schwartz distributions and for Colombeau generalized functions do not need any stronger compatibility condition, which ultimately stems from the possibility to use, for these generalized functions, only (near-)standard points (recall Thm. 20 and Thm. 21). This is proved in the following result, which generalizes the aforementioned sheaf properties (see e.g. [50]).

Theorem 86. *Let $X \subseteq \left({}^{\rho}\widetilde{\mathbb{R}}^n\right)^{\bullet}$, $Y \subseteq {}^{\rho}\widetilde{\mathbb{R}}^d$ and let $f : X \longrightarrow Y$ be a set-theoretical map. Suppose that $X \subseteq \bigcup_{x\in X} B_{r_x}(x)$, where $r_x \in \mathbb{R}_{>0}$ for all x, and that*

$$f|_{B_{r_x}(x)\cap X} \in {}^{\rho}\mathcal{GC}^{\infty}(B_{r_x}(x) \cap X, Y)$$

for all $x \in X$. Then $f \in {}^{\rho}\mathcal{GC}^{\infty}(X,Y)$.

Proof. For every $x \in X$, let $f|_{B_{r_x}(x)} =: v^x \in {}^{\rho}\mathcal{GC}^{\infty}(B_{r_x}(x) \cap X)$ and let v^x be defined by the net (v^x_ε) with $v^x_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^d)$. Recall that by $x^\circ \in \mathbb{R}^n$ we denote the standard part of any $x \in ({}^{\rho}\widetilde{\mathbb{R}}^n)^{\bullet}$. Pick a countable, locally finite open (in $\mathbb{R}^n$) refinement $(U_i)_{i\in\mathbb{N}}$ of $({B^{\mathrm{E}}}_{r_x/2}(x^\circ))_{x\in X}$ and let $(\chi_i)_{i\in\mathbb{N}}$ be a partition of unity with $\mathrm{stsupp}\chi_i \Subset U_i$ for all $i \in \mathbb{N}$. For any $i \in \mathbb{N}$ pick $x_i \in X$ such that $U_i \subseteq B^{\mathrm{E}}_{r_{x_i}/2}(x_i^\circ)$ and set

$$f_\varepsilon := \sum_{i\in\mathbb{N}} \chi_i v^{x_i}_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^d).$$

Then the net (f_ε) defines a GSF of the type $X \longrightarrow Y$: indeed, we will show that $f(z) = v^z(z) = [f_\varepsilon(z_\varepsilon)]$ for all $z = [z_\varepsilon] \in X$:

$$\begin{aligned} f_\varepsilon(z_\varepsilon) - v^z_\varepsilon(z_\varepsilon) &= \sum_{i\in\mathbb{N}} \chi_i(z_\varepsilon)(v^{x_i}_\varepsilon(z_\varepsilon) - v^z_\varepsilon(z_\varepsilon)) = \\ &= \sum_{\{i \mid z^\circ \in B_{3r_{x_i}/4}(x_i)\}} \chi_i(z_\varepsilon)(v^{x_i}_\varepsilon(z_\varepsilon) - v^z_\varepsilon(z_\varepsilon)) + \\ &\quad + \sum_{\{i \mid z^\circ \notin B_{3r_{x_i}/4}(x_i)\}} \chi_i(z_\varepsilon)(v^{x_i}_\varepsilon(z_\varepsilon) - v^z_\varepsilon(z_\varepsilon)) = \\ &=: A_\varepsilon + B_\varepsilon. \end{aligned}$$

Since $z_\varepsilon \to z^\circ$, for small ε all z_ε remain in a compact set and since the supports of the χ_i form a locally finite family it follows that both A_ε and B_ε are in fact finite sums for small ε. To estimate the summands in A_ε, note that $z^\circ \in B_{3r_{x_i}/4}(x_i)$ implies that $z \in B_{r_{x_i}}(x_i)$, so $v^{x_i}(z) = f|_{B_{r_{x_i}}(x_i)}(z) = f(z) = v^z(z)$. Hence $[A_\varepsilon] = 0$. Concerning B_ε, $z^\circ \notin B_{3r_{x_i}/4}(x_i)$ implies that $|z^\circ - x_i^\circ| > r_{x_i}/2$. On the other hand, if $\chi_i(z_\varepsilon) \neq 0$ then $z_\varepsilon \in U_i \subseteq B^{\mathrm{E}}_{r_{x_i}/2}(x_i^\circ)$, implying $|z^\circ - x_i^\circ| \le r_{x_i}/2$. Hence $B_\varepsilon = 0$. Consequently, $[f_\varepsilon(z_\varepsilon)] = [v^z_\varepsilon(z_\varepsilon)]$, as claimed. ☐

The following is the sheaf property for Fermat covers.

Corollary 87. *Let $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$, $Y \subseteq {}^\rho\widetilde{\mathbb{R}}^d$ and let $f : X \longrightarrow Y$ be a set-theoretical map. Suppose that $X \subseteq \bigcup_{j\in J} U_j$, where each U_j is a large open set, and that*

$$f|_{U_j\cap X} \in {}^\rho\mathcal{GC}^\infty(U_j \cap X, Y)$$

for all $j \in J$. Then

(i) *If $X \subseteq \left({}^\rho\widetilde{\mathbb{R}}^n\right)^\bullet$, then $f \in {}^\rho\mathcal{GC}^\infty(X, Y)$.*

(ii) *If X contains its converging subpoints and all points of X are finite, then $f \in {}^\rho\mathcal{GC}^\infty(X, Y)$.*

Proof. To prove property (i), let $s = x^\circ$, $x \in X$. Then $x \in U_{j_x}$ for some $j_x \in J$, and hence $B_{r_{j_x}}(x) \subseteq U_{j_x}$ for some $r_{j_x} \in \mathbb{R}_{>0}$. Therefore $s = x^\circ \in B^{\mathrm{E}}_{r_{j_x}}(x^\circ)$ and so $X^\circ \subseteq \bigcup_{x\in X} B^{\mathrm{E}}_{r_{j_x}}(x^\circ)$. Claim (i) now follows directly from Cor. 86.
Property (ii) follows by (i) and Thm. 21 applied to $f|_{X'}$, where $X' := \{x \in X \mid x \text{ is near-standard}\}$. □

It is now natural to ask whether the sheaf property Thm. 85 could be inscribed into the general notion of sheaf on a site. This is one of the aims of the next Sec. 10.

10. The Grothendieck topos of generalized smooth functions

As we argued in the introduction, function spaces and Cartesian closedness are considered by many authors as important features for mathematics and mathematical physics. Even if Colombeau's theory of generalized functions can be extended to any locally convex space E, on the other hand, in [72] (p. 2) it is stated that: *"locally convex topology is not appropriate for non-linear questions in infinite dimensions"*, and indeed a different approach to infinite dimensional spaces is to embed smooth manifolds into a Cartesian closed category $\mathcal{C}$ (see [37] for a review of this type of approaches). Similar lines of thought can be found in [63, 64], but where generalized functions are seen as functionals, hence not following Cauchy-Dirac's original conception but Schwartz' conception instead. We first motivate and introduce the few notions of category theory that we need in the present section. Indeed, only basic preliminaries of category theory are needed to understand this section: definition of category and basic examples, functors and natural transformation. Our basic references for this section are [78, 59, 7]. As it is customary, we write $D \in \mathbb{D}$ to denote that D is an object of the category $\mathbb{D}$, we write $A \xrightarrow{f} B$ *in* $\mathbb{D}$ to say that $f \in \mathbb{D}(A, B)$ and $\mathbb{D}^{\mathrm{op}}$ for the opposite of $\mathbb{D}$ (see e.g. [79]). Only in this section, we use both the notations $f \cdot g := g \circ f$ for arrows $X \xrightarrow{f} Y \xrightarrow{g} Z$ in some category, and the notation $\bar{\jmath} = (\bar{\jmath}_\varepsilon) \in [J]$.

10.1. Coverages, sheaves and sites.

The notion of coverage on a category allows one to define more abstractly the concept of sheaf without being forced to consider a topological space. Nevertheless, the classical example to keep in mind to have a first understanding of the following definitions is a sheaf (e.g. of continuous functions) defined on the poset of open sets $\mathbb{D} = \mathbb{D}(X)$ in some topological space X.

We first define families with common codomain D:

Definition 88. Let $\mathbb{D}$ be a category and let $D \in \mathbb{D}$. Then we say that $\mathcal{F} \in \mathrm{Fam}(D)$ is a *family with common codomain* D if there exist a set $J \in$ **Set** and families $(D_j)_{j\in J}$, $(i_j)_{j\in J}$ such that:

(i) $D_j \xrightarrow{i_j} D$ in $\mathbb{D}$ for all $j \in J$.
(ii) $\mathcal{F} = \left(D_j \xrightarrow{i_j} D\right)_{j\in J}$.

A coverage is a class of families with a common codomain that is closed with respect to pullback, in the precise sense stated in the following

Definition 89. Let $\mathbb{D}$ be a category, then we say that Γ is a *coverage on* $\mathbb{D}$ if:

(i) $\Gamma : \mathrm{Obj}(\mathbb{D}) \longrightarrow \mathbf{Set}$, where $\mathrm{Obj}(\mathbb{D})$ is the class of objects of the category $\mathbb{D}$.
(ii) $\forall D \in \mathbb{D} : \ \Gamma(D) \subseteq \mathrm{Fam}(D)$. Families in $\Gamma(D)$ are called *covering families of* D.
(iii) If $D \in \mathbb{D}$, $\left(D_j \xrightarrow{i_j} D\right)_{j\in J} \in \Gamma(D)$ is a covering family of D, and $C \xrightarrow{g} D$ is an arbitrary arrow of $\mathbb{D}$, then there exists a covering family of C, $\left(C_k \xrightarrow{h_k} C\right)_{k\in K} \in \Gamma(C)$ such that

$$\forall k \in K\, \exists j \in J\, \exists \bar{g} : \quad \begin{array}{ccc} C_k & \xrightarrow{h_k} & C \\ \downarrow{\scriptstyle \bar{g}} & & \downarrow{\scriptstyle g} \\ D_j & \xrightarrow{i_j} & D \end{array} \tag{10.1}$$

(iv) A pair $(\mathbb{D}, \Gamma)$, of a category and a coverage on it, is called a *site*.

For example, let ${}^{\rho}\mathcal{OGC}^{\infty}$ be the category of sharply open sets $U \subseteq {}^{\rho}\widetilde{\mathbb{R}}^{u}$ (all possible dimensions $u \in \mathbb{N}$ are included) and GSF. Let $\Gamma(U)$ contains open coverings and inclusions: $\left(U_j \xrightarrow{i_j} U\right)_{j\in J} \in \Gamma(U)$ if and only if $U_j \in {}^{\rho}\mathcal{OGC}^{\infty}$, $i_j : U_j \hookrightarrow U$ and $\bigcup_{j\in J} U_j = U$. Then $({}^{\rho}\mathcal{OGC}^{\infty}, \Gamma)$ is a site and property (10.1) holds simply by taking $K = J$ and $C_j := g^{-1}(U_j) \in {}^{\rho}\mathcal{OGC}^{\infty}$ as covering family of C, and $\bar{g} := g|_{C_j}$. Note that these simple steps do not work in the category ${}^{\rho}\mathcal{SGC}^{\infty}$ of strongly internal sets and GSF because in general $g^{-1}(U_j)$ is not strongly internal (only the inclusion $g^{-1}(\langle A_\varepsilon\rangle) \subseteq \langle g_\varepsilon^{-1}(A_\varepsilon)\rangle$ holds). In this case, a general method is to express the open set $g^{-1}(U_j)$ as a union of strongly internal sets. This implies that we have to take a different index set K for the covering family $C_k \hookrightarrow C$.

Using the notion of coverage, we can define the notion of compatible family:

Definition 90. Let $(\mathbb{D}, \Gamma)$ be a site and $F : \mathbb{D}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ be a presheaf. Let $\mathcal{F} = \left(D_j \xrightarrow{i_j} D\right)_{j\in J} \in \Gamma(D)$ be a covering family of $D \in \mathbb{D}$. Then, we say that $(f_j)_{j\in J}$ *are compatible on* $\mathcal{F}$ *(rel. F)* if the following conditions hold:

(i) $f_j \in F(D_j)$ for all $j \in J$. In this case f_j is called a *section*.
(ii) For all g, c and j, $h \in J$, we have

$$\begin{array}{ccc} C & \xrightarrow{c} & D_h \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle i_h} \\ D_j & \xrightarrow{i_j} & D \end{array} \quad \Rightarrow \quad F(g)(f_j) = F(c)(f_h). \tag{10.2}$$

A typical way to apply (10.2) is to construct a sort of intersection object $C = D_h \cap D_j$ and to take as c, g the inclusions. Then, if $F = \mathbb{D}(-, Y)$, the equality in (10.2) reduces to the usual compatibility condition $f_j|_{D_h\cap D_j} = f_h|_{D_h\cap D_j}$.

We can finally define the notion of sheaf on a site:

Definition 91. Let $(\mathbb{D},\Gamma)$ be a site. Then we say that F is a *sheaf on* $(\mathbb{D},\Gamma)$, and we write $F\in\mathrm{Sh}(\mathbb{D},\Gamma)$ if

(i) $F:\mathbb{D}^{\mathrm{op}}\longrightarrow\mathbf{Set}$ (i.e. F is a presheaf).

(ii) If $\mathcal{F}=\left(D_j\xrightarrow{i_j}D\right)_{j\in J}\in\Gamma(D)$ is a covering family of $D\in\mathbb{D}$ and $(f_j)_{j\in J}$ are compatible on $\mathcal{F}$ (rel. F), then

$$\exists!f\in F(D)\,\forall j\in J:\ F(i_j)(f)=f_j. \tag{10.3}$$

In the classical example of continuous functions on a topological space, $F=\mathcal{C}^0(-,Y)$ and the equality in (10.3) becomes $f|_{D_j}=f_j$. Note that, even if the category of open sets and GSF ${}^\rho\mathcal{OGC}^\infty$ is a site, example 76 shows that ${}^\rho\mathcal{OGC}^\infty(-,Y)$ is not a sheaf.

10.2. **The category of glueable functions.** As we mentioned in the introduction to Sec. 9, our main aim here is to show that the DCC is strictly related to the aforementioned definition of sheaf on a site. The strategy we will follow is:

(a) Define a category ${}^\rho\mathcal{G}\ell^\infty\supseteq{}^\rho\mathcal{SGC}^\infty$.
(b) Define a coverage on ${}^\rho\mathcal{G}\ell^\infty$.
(c) Show that ${}^\rho\mathcal{G}\ell^\infty(-,\mathcal{Y})$ is a sheaf using Thm. 85 and hence the DCC.

Intuitively, we already think that a family of strongly internal sets $(U_j)_{j\in J}$ is a coverage of the strongly internal set U if, simply, $U\subseteq\bigcup_{j\in J}U_j$; we also intuitively think that $(f_j)_{j\in J}$ are compatible sections if DCC holds. The following definition reflects this intuition:

Definition 92. Let ${}^\rho\mathcal{G}\ell^\infty$ be the category of *glueable families*, whose objects are non empty families $(U_j)_{j\in J}\in{}^\rho\mathcal{G}\ell^\infty$ of strongly internal sets in some space ${}^\rho\widetilde{\mathbb{R}}^u$:

$$J\neq\emptyset,\ \exists u\in\mathbb{N}\,\forall j\in J:\ {}^\rho\widetilde{\mathbb{R}}^u\supseteq U_j\in{}^\rho\mathcal{SGC}^\infty.$$

We say that

$$\mathcal{X}\xrightarrow{\varphi}\mathcal{Y}\quad\text{in}\quad{}^\rho\mathcal{G}\ell^\infty$$

if $\mathcal{X}=(U_j)_{j\in J}$, $\mathcal{Y}=(V_h)_{h\in H}\in{}^\rho\mathcal{G}\ell^\infty$ and $\varphi=\left((f_j)_{j\in J},\alpha\right)$, where:

(i) The map $\alpha\in\mathbf{Set}(J,H)$ is called a *reparametrization*.

(ii) The family of GSF $f_j\in{}^\rho\mathcal{SGC}^\infty(U_j,V_{\alpha(j)})$, $j\in J$, satisfies the DCC on $U:=\bigcup_{j\in J}U_j$.

To state condition (ii) more explicitly, let u, $v\in\mathbb{N}$ be the dimensions of $(U_j)_{j\in J}$ and $(V_h)_{h\in H}$ resp. (i.e. ${}^\rho\widetilde{\mathbb{R}}^u\supseteq U_j$ and ${}^\rho\widetilde{\mathbb{R}}^v\supseteq V_h$ for all j, h), and set $V:=\bigcup_{h\in H}V_h$. Then (ii) asks that there exists $(U_{j\varepsilon})_{\substack{j\in J\\ \varepsilon\in I}}$ such that for all $K\Subset_{\mathrm{f}}U$ there exists $(f_{j\varepsilon})_{\substack{j\in J\\ \varepsilon\in I}}\in\mathcal{C}^\infty(\mathbb{R}^u,\mathbb{R}^v)$ such that:

(ii.a) $f_j=[f_{j\varepsilon}(-)]\,|_{U_j}$ for each $j\in J$.

(ii.b) $[f_{\bar{\jmath}_\varepsilon,\varepsilon}(-)]\in{}^\rho\mathcal{GC}^\infty(U_{\bar{\jmath}}\cap K,V_{\bar{\jmath}\cdot\alpha}\cap V)$ for all $\bar{\jmath}=(\bar{\jmath}_\varepsilon)\in[J]$, where $U_{\bar{\jmath}}:=\langle U_{\bar{\jmath}_\varepsilon,\varepsilon}\rangle$. Note that $I\xrightarrow{\bar{\jmath}}J\xrightarrow{\alpha}H$ and hence $\bar{\jmath}\cdot\alpha=\alpha\circ\bar{\jmath}\in[H]$.

(ii.c) $[f_{\bar{\jmath}_\varepsilon,\varepsilon}(-)]=\left[f_{\bar{h}_\varepsilon,\varepsilon}(-)\right]$ on $U_{\bar{\jmath}}\cap U_{\bar{h}}\cap K$ for all $\bar{\jmath}$, $\bar{h}\in[J]$.

Composition and identities in ${}^\rho\mathcal{G}\ell^\infty$ are defined as follows: Let

$$\mathcal{X}\xrightarrow{\varphi}\mathcal{Y}\xrightarrow{\psi}\mathcal{Z}\quad\text{in}\quad{}^\rho\mathcal{G}\ell^\infty \tag{10.4}$$

and set $\mathcal{X} = (U_j)_{j\in J}$, $\mathcal{Y} = (V_h)_{h\in H}$, $\mathcal{Z} = (W_l)_{l\in L}$, $\varphi = \left((f_j)_{j\in J}, \alpha\right)$, $\psi = \left((g_h)_{h\in H}, \beta\right)$. Then

$$\begin{aligned} & U_j \xrightarrow{f_j} V_{\alpha(j)} \xrightarrow{g_{\alpha(j)}} W_{\beta(\alpha(j))} \quad \forall j \in J \\ & J \xrightarrow{\alpha} H \xrightarrow{\beta} L, \end{aligned}$$

and we hence set

$$\begin{aligned} \varphi \cdot \psi &:= \left(\left(f_j \cdot g_{\alpha(j)} \right)_{j\in J}, \alpha \cdot \beta \right) \\ 1_{\mathcal{X}} &:= \left(\left(1_{U_j} \right)_{j\in J}, 1_J \right). \end{aligned}$$

The following lemma confirms the correctness of this definition.

Lemma 93. *${}^{\rho}\mathcal{G}\ell^{\infty}$ is a category.*

Proof. We essentially have to prove the closure with respect to composition, i.e. that (10.4) implies $\varphi \cdot \psi : \mathcal{X} \longrightarrow \mathcal{Z}$ in ${}^{\rho}\mathcal{G}\ell^{\infty}$. We implicitly use the notations of the previous definition. For all $K = [K_\varepsilon] \Subset_{\text{f}} U$, we have $f_{\bar{\jmath}} := [f_{\bar{\jmath}_\varepsilon,\varepsilon}(-)] \in {}^{\rho}\mathcal{GC}^{\infty}(U_{\bar{\jmath}} \cap K, V_{\bar{\jmath}\cdot\alpha} \cap V)$, and hence for $\bar{\jmath} \in [J]$ we get $f_{\bar{\jmath}}([U_{\bar{\jmath}_\varepsilon,\varepsilon} \cap K_\varepsilon]) =: \hat{K} \Subset_{\text{f}} V_{\bar{\jmath}\cdot\alpha} \cap V$. Using $\hat{K}$ and $\bar{\jmath} \cdot \alpha =: \bar{h} \in [H]$ with the arrow ψ, we obtain $g_{\bar{h}} := \left[g_{\bar{h}_\varepsilon,\varepsilon}(-)\right] \in {}^{\rho}\mathcal{GC}^{\infty}(V_{\bar{h}} \cap \hat{K}, W_{\bar{h}\cdot\beta} \cap W)$ for some nets $(g_{h\varepsilon})_{h,\varepsilon}$. Therefore

$$(f_{\bar{\jmath}} \cdot g_{\bar{\jmath}\cdot\alpha})\,|_{U_{\bar{\jmath}}\cap K} = \left[g_{\alpha(\bar{\jmath}_\varepsilon),\varepsilon}\left(f_{\bar{\jmath}_\varepsilon,\varepsilon}(-)\right)\right] \in {}^{\rho}\mathcal{GC}^{\infty}(U_{\bar{\jmath}} \cap K, W_{\bar{\jmath}\cdot\alpha\cdot\beta} \cap W).$$

This shows that condition (ii.b) of Def. 92 holds for $\varphi \cdot \psi = \left(\left(f_j \cdot g_{\alpha(j)} \right)_{j\in J}, \alpha \cdot \beta \right)$. To prove condition (ii.c) take $x = [x_\varepsilon] \in U_{\bar{\jmath}} \cap U_{\bar{l}} \cap K$, then $f_{\bar{\jmath}}(x) = f_{\bar{l}}(x) \in V_{\bar{\jmath}\cdot\alpha} \cap V_{\bar{l}\cdot\alpha} \cap \hat{K}$ and hence $g_{\bar{\jmath}\cdot\alpha}\left(f_{\bar{\jmath}}(x)\right) = g_{\bar{l}\cdot\alpha}\left(f_{\bar{l}}(x)\right)$, which is our conclusion. Properties of identities trivially hold. □

Note that ${}^{\rho}\mathcal{SGC}^{\infty} \subseteq {}^{\rho}\mathcal{G}\ell^{\infty}$ through the embedding:

$$\begin{aligned} & U \in {}^{\rho}\mathcal{SGC}^{\infty} \mapsto (U)_{\bar{\mathbb{1}}} \in {}^{\rho}\mathcal{G}\ell^{\infty} \\ & f \in {}^{\rho}\mathcal{SGC}^{\infty}(U,V) \mapsto ((f)_{\bar{\mathbb{1}}}, \bar{\mathbb{1}} \longrightarrow \bar{\mathbb{1}}) \in {}^{\rho}\mathcal{G}\ell^{\infty}\left((U)_{\bar{\mathbb{1}}}, (V)_{\bar{\mathbb{1}}}\right), \end{aligned}$$

where $\bar{\mathbb{1}} := \{*\}$ is any singleton set. The converse is also possible using the sheaf Thm. 85: In fact, if $\left((f_j)_{j\in J}, \alpha\right) \in {}^{\rho}\mathcal{G}\ell^{\infty}\left((U_j)_{j\in J}, (V_h)_{h\in H}\right)$, then the DCC holds and hence there exists a unique $f \in {}^{\rho}\mathcal{SGC}^{\infty}(U,V)$ such that $f|_{U_j} = f_j$ for all $j \in J$. If we set $\text{gl}\left((f_j)_{j\in J}, \alpha\right) := f$ then

$$\text{gl}\,(\varphi \cdot \psi) = \text{gl}\left(\left(f_j \cdot g_{\alpha(j)} \right)_{j\in J}, \alpha \cdot \beta \right) = \text{gl}\,(\varphi) \cdot \text{gl}\,(\psi)$$

because setting $\text{gl}(\varphi) =: f$ and $\text{gl}(\psi) =: g$, we have $(f \cdot g)\,|_{U_j} = f|_{U_j} \cdot g|_{V_{\alpha(j)}} = f_j \cdot g_{\alpha(j)}$, i.e. the unique GSF obtained by gluing $\left(f_j \cdot g_{\alpha(j)}\right)_{j\in J}$ is $f \cdot g$. Finally $\text{gl}\,(1_{\mathcal{X}}) = 1_U = 1_{\text{gl}(\mathcal{X})}$ and hence $\text{gl} : {}^{\rho}\mathcal{G}\ell^{\infty} \longrightarrow {}^{\rho}\mathcal{SGC}^{\infty}$ is a (clearly non injective) functor.

10.3. **Coverage of glueable functions.** We now introduce a coverage on the category ${}^{\rho}\mathcal{G}\ell^{\infty}$ of glueable families:

Definition 94. Let $\mathcal{E} = (W_e)_{e\in E} \in {}^{\rho}\mathcal{G}\ell^{\infty}$. Then we say that $\gamma \in \Gamma(\mathcal{E})$ if there exists a non empty $J \in \mathbf{Set}$ such that:

(i) $\gamma = (\gamma_j)_{j\in J}$ and $\gamma_j = \left((i_h)_{h\in J}, \delta\right)$ for all $j \in J$.

(ii) $J \xrightarrow{\delta} E$ is a surjective map.

(iii) $i_j : D_j \hookrightarrow W_{\delta(j)}$ for all $j \in J$, where $(D_j)_{j\in J} \in {}^{\rho}\mathcal{G}\ell^{\infty}$. Because of this property, δ is called a *refinement map*.

(iv) $W_e = \bigcup \{D_j \mid \delta(j) = e,\ j \in J\}$ for all $e \in E$.

Remark 95.

(i) Note that the index set J of $\gamma = (\gamma_j)_{j\in J}$ is the same used in $(i_h)_{h\in J}$. Moreover, the two components of γ_j do not depend on $j \in J$. This may appear to be a strange property for a coverage, but note that our intuition here is guided by viewing the inclusions $(D_j \xhookrightarrow{i_j} W)_{j\in J}$ as a coverage of $W := \bigcup_{e\in E} W_e$. On the other hand, note that in Def. 92 of glueable families, in the DCC (ii) we need the whole family $(f_j)_{j\in J}$ and not only the single GSF f_j. Similarly, we need to consider the entire family of inclusions $(i_h)_{h\in J}$ and not the single i_h. For this reason, we cannot directly consider the family $(D_j \xhookrightarrow{i_j} W)_{j\in J}$, made of single inclusions, as a coverage.

(ii) Condition (iv) implies $W = \bigcup_{j\in J} D_j = \bigcup_{j\in J} W_{\delta(j)}$. We will see more clearly later that this condition allows us to prove the uniqueness part of (10.3).

(iii) If $W_e \neq \emptyset$ for all $e \in E$, then (iv) directly implies that δ has to be a surjective map.

Theorem 96. Γ *is a coverage on* ${}^{\rho}\mathcal{G}\ell^{\infty}$.

Proof. From Def. 94(iii) it follows that $\gamma_j \in {}^{\rho}\mathcal{G}\ell^{\infty}\left((D_j)_{j\in J}, \mathcal{E}\right)$, i.e. property Def. 88(i). To prove the closure with respect to pullbacks, take $\eta \in {}^{\rho}\mathcal{G}\ell^{\infty}(\mathcal{C}, \mathcal{E})$, where $\mathcal{C} =: (V_c)_{c\in C}$ and $\eta =: \left((g_c)_{c\in C}, \beta\right)$. Since $(g_c)_{c\in C}$ satisfies the DCC (see Def. 92(ii), i.e. Def. 92(ii.a), (ii.b)), we can use Thm. 85 to get

$$\exists! g \in {}^{\rho}\mathcal{GC}^{\infty}(V, W)\, \forall c \in C :\ g|_{V_c} = g_c,$$

where $V := \bigcup_{c\in C} V_c$. We can hence consider $g^{-1}(D_j)$ and cover it with strongly internal sets:

$$\forall j \in J\, \exists H_j \neq \emptyset\, \exists\, (B_{jh})_{h\in H_j}\, \forall h \in H_j :\ B_{jh} \in {}^{\rho}\mathcal{SGC}^{\infty},\ g^{-1}(D_j) = \bigcup_{h\in H_j} B_{jh}.$$

Set $B_{jhc} := B_{jh} \cap V_c \in {}^{\rho}\mathcal{SGC}^{\infty}$, $K := \{(j, h, c) \mid j \in J,\ h \in H_j,\ c \in C\}$, $\nu : (j, h, c) \in K \mapsto c \in C$, $a_k : B_k \hookrightarrow V_{\nu(k)}$, and $\alpha_k := \left((a_k)_{k\in K}, \nu\right)$. Then K is non empty because C, H_j, $J \neq \emptyset$, and we have

$$B = \bigcup_{k\in K} B_k = \bigcup_{c\in C}\bigcup_{j\in J}\bigcup_{h\in H_j} B_{jhc} = \bigcup_{c\in C} V_c \cap \bigcup_{j\in J}\bigcup_{h\in H_j} B_{jh} = V \cap \bigcup_{j\in J} g^{-1}(D_j) =$$
$$= V \cap g^{-1}\left(\bigcup_{j\in J} D_j\right) = V \cap g^{-1}(D) = V \cap g^{-1}(W) = V.$$

To prove property (iv) of Def. 94 for the new covering family $(\alpha_k)_{k\in K}$, take $x \in V_c \subseteq V$, so that $g(x) \in W$. Thereby, $x \in g^{-1}(D_j)$ for some $j \in J$, and hence $x \in B_{jh} \subseteq B_{jhc}$ for some $h \in H_j$. Setting $k := (j, h, c) \in K$, we have $\nu(k) = c$ and $x \in B_k$. This shows that $(\alpha_k)_{k\in K} \in \Gamma(\mathcal{C})$. Finally, let $(\delta)_1^{-1}$ be any left inverse of δ, i.e. $(\delta)_1^{-1} \cdot \delta = \delta \circ (\delta)_1^{-1} = 1_E$ (recall Def. 94(ii)), and setting $\mathcal{B}_k := (B_k)_{k\in K}$, $\mu := \left((g_k|_{B_k})_{k\in K}, \nu \cdot \beta \cdot (\delta)_1^{-1}\right)$, we have

$$\forall k \in K\, \exists j \in J\, \exists \mu : \quad \begin{array}{ccc} \mathcal{B}_k & \xrightarrow{\alpha_k} & \mathcal{C} \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \eta} \\ \mathcal{D}_j & \xrightarrow{\gamma_j} & \mathcal{E} \end{array}$$

i.e. the claim Def. 89(iii). □

Definition 97. The category of sheaves ${}^\rho\mathrm{T}\mathcal{GC}^\infty := \mathrm{Sh}\left({}^\rho\mathcal{G}\ell^\infty, \Gamma\right)$ (and natural transformations as arrows) is called the *Grothendieck topos of generalized smooth functions* (see e.g. [78, 59, 7] and references therein).

10.4. **The sheaf of glueable functions.** We are now able to show that the DCC is the key property to prove the following

Theorem 98. *For each $\mathcal{Y} \in {}^\rho\mathcal{G}\ell^\infty$, the functor ${}^\rho\mathcal{G}\ell^\infty(-, \mathcal{Y})$ is a sheaf on the site $({}^\rho\mathcal{G}\ell^\infty, \Gamma)$, i.e. it satisfies Def. 91: ${}^\rho\mathcal{G}\ell^\infty(-, \mathcal{Y}) \in {}^\rho\,\mathrm{T}\mathcal{GC}^\infty\left({}^\rho\mathcal{G}\ell^\infty, \Gamma\right)$.*

Proof. We use the notations of Def. 94. Let $(\gamma_j)_{j\in J} = \left((i_h)_{h\in J}, \delta\right)_{j\in J} \in \Gamma(\mathcal{E})$ be a covering family and let $\varphi_j = \left(\left(f_h^j\right)_{h\in J}, \alpha^j\right) \in {}^\rho\mathcal{G}\ell^\infty(\mathcal{D}_j, \mathcal{Y})$, $j \in J$, be a compatible family of sections, where $\mathcal{D}_j := (D_h)_{h\in J} =: \mathcal{D}_0$ (recall Rem. 95(i) about the independence from $j \in J$). Note explicitly that by Def. 94(i) the covering family $(\gamma_j)_{j\in J}$ is indexed by the same set J as its inclusions $(i_h)_{h\in J}$; moreover, the glueable family $\left(f_h^j\right)_{h\in J}$ is also indexed by J because by Def. 92, any arrow in the category ${}^\rho\mathcal{G}\ell^\infty$ is indexed by the same set of its domain which, in this case, is $\mathcal{D}_0 = (D_h)_{h\in J}$. Set $\mathcal{Y} =: (V_l)_{l\in L}$. We first want to prove that the compatibility of sections $(\varphi_j)_{j\in J}$ allows us to show that both $\left(f_h^j\right)_{h\in J}$ and α^j do not actually depend on j. We therefore take $i \in J$ and define $\mathcal{D}_0 \cap \mathcal{D}_0 := (D_h \cap D_k)_{(h,k)\in J^2} \in {}^\rho\mathcal{G}\ell^\infty$,

$$i_{hk} : D_h \cap D_k \hookrightarrow D_h, \qquad i_{kh} : D_h \cap D_k \hookrightarrow D_k,\ \ \nu_1 : (h,k) \in J^2 \mapsto h \in J,$$

$\nu_2 : (h,k) \in J^2 \mapsto k \in J$, $\iota_1 := \left((i_{hk})_{(h,k)\in J^2}, \nu_1\right)$, $\iota_2 := \left((i_{kh})_{(h,k)\in J^2}, \nu_2\right)$. The compatibility condition (10.2) for the functor ${}^{\rho}\mathcal{G}\ell^{\infty}(-,\mathcal{Y})$ yields

$$
\begin{gathered}
{}^{\rho}\mathcal{G}\ell^{\infty}(-,\mathcal{Y})(\iota_1)(\varphi_j) = {}^{\rho}\mathcal{G}\ell^{\infty}(-,\mathcal{Y})(\iota_2)(\varphi_i)\\
\left(\left(i_{hk}\cdot f^j_{\nu_1(h,k)}\right)_{(h,k)\in J^2},\nu_1\cdot\alpha^j\right) = \left(\left(i_{kh}\cdot f^i_{\nu_2(h,k)}\right)_{(h,k)\in J^2},\nu_2\cdot\alpha^i\right)\\
\left(\left(f^j_h|_{D_h\cap D_k}\right)_{hk},\nu_1\cdot\alpha^j\right) = \left(\left(f^i_k|_{D_h\cap D_k}\right)_{hk},\nu_2\cdot\alpha^i\right).
\end{gathered}
$$

Thereby, for $h=k$ we get $f^j_h = f^i_h$ and, for arbitrary h, $k\in J$, we also have $(\nu_1\cdot\alpha^j)(h,k) = (\nu_2\cdot\alpha^i)(h,k)$, i.e. $\alpha^j(h)=\alpha^i(k)$. This equality, since $J\ne\emptyset$, implies that $\alpha^i=\alpha^j =: \alpha_{\text{c}}\in L$ is constant. Therefore, as a consequence of the compatibility condition, we have that both components of φ_j do not depend on $j\in J$: our sections can hence be simply written as $\varphi_j =: \left((f_h)_{h\in J},\alpha_{\text{c}}\right)$. Note also that all the GSF $f_h : D_h \longrightarrow V := V_{\alpha_{\text{c}}}$ have the same codomain. The glueable family $(f_h)_{h\in J}$ satisfies the DCC because of Def. 92(ii) and we can hence apply Thm. 85 to obtain a unique $f\in{}^{\rho}\mathcal{GC}^{\infty}(D,V)$ such that $f|_{D_j}=f_j$ for each $j\in J$, where $D=\bigcup_{j\in J}D_j$. We can finally set $\bar\alpha : e\in E\mapsto\alpha_{\text{c}}\in L$ and $\varphi := \left((f|_{W_e})_{e\in E},\bar\alpha\right)$ to obtain the existence part of the conclusion:

$$
{}^{\rho}\mathcal{G}\ell^{\infty}(-,\mathcal{Y})(\gamma_j)(\varphi) = \gamma_j\cdot\varphi = \left(\left(f|_{W_{\delta(h)}\cap D_h}\right)_{h\in J},\delta\cdot\bar\alpha\right) = \left((f|_{D_h})_{h\in J},\alpha\right) = \varphi_j.
$$

To prove the uniqueness of the glued section, assume that $\hat\varphi = \left(\left(\hat f_e\right)_{e\in E},\hat\alpha\right)\in{}^{\rho}\mathcal{G}\ell^{\infty}(\mathcal{E},\mathcal{Y})$ is another section such that ${}^{\rho}\mathcal{G}\ell^{\infty}(-,\mathcal{Y})(\gamma_j)(\hat\varphi)=\varphi_j$ for all $j\in J$. This equality gives

$$
\left(\left(\hat f_{\delta(h)}|_{D_h}\right)_{h\in J},\delta\cdot\hat\alpha\right) = \left((f_h)_{h\in J},\alpha\right) \tag{10.5}
$$

Take $e\in E$ and $x\in W_e$, then condition Def. 94(iv) yields the existence of $h\in J$ such that $\delta(h)=e$ and $x\in D_h$. Therefore, using (10.5) we obtain

$$
\hat f_e(x) = \hat f_{\delta(h)}|_{D_h\cap W_e}(x) = f_h|_{W_e}(x) = f|_{W_e}(x),
$$

i.e. $\hat f_e = f|_{W_e}$ for all $e\in E$. Finally, (10.5) also gives $\hat\alpha = (\delta)_1^{-1}\cdot\alpha$. □

10.5. **Concrete sites and generalized diffeological spaces.** In this final section, we want to sketch one of the many possibilities that we can start to explore using the Grothendieck topos ${}^{\rho}\mathrm{T}\mathcal{GC}^{\infty}$. The idea is to show that the site $({}^{\rho}\mathcal{G}\ell^{\infty},\Gamma)$ is a concrete site. In this way, considering the space of concrete sheaves over this site, we get a category of spaces that extends usual smooth manifolds but it is closed with respect to operations such as: arbitrary subspaces, products, sums, function spaces, etc. (see [58, 62, 7, 34, 37] and references therein for similar approaches). As above, all the necessary categorical notions will be introduced).

Definition 99. Let $\mathbb{D}$ be a category and $F:\mathbb{D}^{\text{op}}\longrightarrow\mathbf{Set}$ be a functor. Then, we say that F *is representable* if $F\simeq\mathbb{D}(-,D)$ for some $D\in\mathbb{D}$. Moreover, if $(\mathbb{D},\Gamma)$ is a site, then we say that $(\mathbb{D},\Gamma)$ *is a subcanonical site* if every representable functor F is a sheaf $F\in\mathrm{Sh}(\mathbb{D},\Gamma)$.
Since in Thm. 98 we proved that ${}^{\rho}\mathcal{G}\ell^{\infty}(-,\mathcal{Y})$ is a sheaf, directly from Def. 99 it follows that $({}^{\rho}\mathcal{G}\ell^{\infty},\Gamma)$ is a subcanonical site.

A concrete site is a site whose objects can be thought of as an underlying set with a structure. The idea is that if $\mathbb{1}\in\mathbb{D}$ is a terminal object, then $|D| := \mathbb{D}(\mathbb{1},D)\in\mathbf{Set}$

is the underlying set of $D \in \mathbb{D}$ and if $f : C \longrightarrow D$ in $\mathbb{D}$, then $|f| := \mathbb{D}(\mathbb{1}, f) : x \in |C| \mapsto x \cdot f = f \circ x \in |D|$ is the set-theoretical map corresponding to the arrow f. These maps have a natural relation with covering families of Γ, as stated in the following

Definition 100. We say that $(\mathbb{D}, \Gamma, \mathbb{1})$ *is a concrete site* if:

(i) $(\mathbb{D}, \Gamma)$ is a subcanonical site.

(ii) $\mathbb{1} \in \mathbb{D}$ is a terminal object, i.e. $\mathbb{1} \in \mathbb{D}$ and $\forall D \in \mathbb{D}\, \exists! t \in \mathbb{D}(D, \mathbb{1})$.

$\mathbb{D}(\mathbb{1}, D) =: |D|$ is called the *underlying set of* $D \in \mathbb{D}$. For $f \in \mathbb{D}(C, D)$, the map $|f| := \mathbb{D}(\mathbb{1}, f) : |C| \longrightarrow |D|$ is called the *function associated to the morphism* f.

(iii) The functor $\mathbb{D}(\mathbb{1}, -) : \mathbb{D} \longrightarrow \mathbf{Set}$ is faithful, i.e. for all $f, g \in \mathbb{D}(C, D)$, if $|f| = |g|$, then $f = g$.

(iv) If $\left(D_j \xrightarrow{i_j} D\right)_{j \in J} \in \Gamma(D)$, then the associated maps trivially cover $|D|$, i.e.:
$$\bigcup_{j \in J} |i_j| \left(|D_i|\right) = |D|. \tag{10.6}$$

For example, let us define a terminal object in the category ${}^{\rho}\mathcal{G}\ell^{\infty}$ of glueable spaces as:
$$\mathbb{1} := (\{0\})_{\bar{\mathbb{1}}} \in {}^{\rho}\mathcal{G}\ell^{\infty}$$
where $\bar{\mathbb{1}} = \{*\}$. Note that, if we view $\mathbb{R}^n = \mathbf{Set}\left(\{1, \ldots, n\}, \mathbb{R}\right)$, then $\mathrm{Card}\left(\mathbb{R}^0\right) = 1$ and hence $\mathrm{Card}\left({}^{\rho}\widetilde{\mathbb{R}}^0\right) = \mathrm{Card}\left(\left(\mathbb{R}^0\right)^I / \sim_\rho\right) = 1$. Therefore, ${}^{\rho}\widetilde{\mathbb{R}}^0 = \{0\}$ is the trivial ring. It is also a strongly internal set because $B_1(0) = \left\{x \in {}^{\rho}\widetilde{\mathbb{R}}^0 \mid |x - 0| < 1\right\} = \{0\}$.

What is $\varphi \in {}^{\rho}\mathcal{G}\ell^{\infty}(\mathbb{1}, \mathcal{X}) = |\mathcal{X}|$ in this case? Set $\varphi = ((f)_{\bar{\mathbb{1}}}, \alpha)$ and $\mathcal{X} = (U_j)_{j \in J}$, then $\alpha : \bar{\mathbb{1}} \longrightarrow J$ and $f : \{0\} \longrightarrow U_{\alpha(*)}$, which can be identified with the pair $(f(0), \alpha(*)) \in U_{\alpha(*)} \times \{\alpha(*)\}$. Therefore,
$$|\mathcal{X}| = {}^{\rho}\mathcal{G}\ell^{\infty}(\mathbb{1}, \mathcal{X}) \simeq \sum_{j \in J} U_j = \bigcup_{j \in J} U_j \times \{j\}.$$

Similarly, $\psi = \left((f_j)_{j \in J}, \alpha\right) \in {}^{\rho}\mathcal{G}\ell^{\infty}\left((U_j)_{j \in J}, (V_l)_{l \in L}\right)$ can be identified with the map:
$$|\psi| : (x, j) \in \sum_{j \in J} U_j \mapsto (f_j(x), \alpha(j)) \in \sum_{l \in L} V_l,$$
i.e. with the map $(x, j) \mapsto (f(x), \alpha(j))$, where $f \in {}^{\rho}\mathcal{GC}^{\infty}\left(\bigcup_{j \in J} U_j, \bigcup_{l \in L} V_l\right)$ is obtained by gluing $(f_j)_{j \in J}$. This implies condition Def. 100(iii), whereas Def. 100(iv) follows from Rem. 95(ii):

Theorem 101. $({}^{\rho}\mathcal{G}\ell^{\infty}, \Gamma)$ *is a concrete site.*

It is well known that a sheaf $F \in \mathrm{Sh}(\mathbb{D}, \Gamma)$ can be thought of as a generalized space defined by the information $F(D) \in \mathbf{Set}$ associated to each test space $D \in \mathbb{D}$. The idea of *concrete sheaf* is that it is this kind of generalized space defined by an underlying set of points $F(\mathbb{1})$. For example, any $y \in {}^{\rho}\mathcal{G}\ell^{\infty}(\mathbb{1}, \mathcal{Y})$ can be identified with the map $\sum_0 \{0\} \times \{0\} = \{(0, 0)\} \longrightarrow \sum_{l \in L} V_l$, and hence ${}^{\rho}\mathcal{G}\ell^{\infty}(\mathbb{1}, \mathcal{Y}) \simeq \sum_{l \in L} V_l$.

Definition 102. Let $(\mathbb{D}, \Gamma, \mathbb{1})$ be a concrete site. Then we say that F *is a concrete sheaf* (and we write $F \in \mathrm{CSh}(\mathbb{D}, \Gamma, \mathbb{1})$) if:

(i) $F \in \mathrm{Sh}(\mathbb{D}, \Gamma)$.

(ii) For all $s \in F(D)$, let $\underline{s} : p \in |D| \mapsto F(p)(s) \in F(\mathbb{1})$, then we have
$$\forall D \in \mathbb{D}\, \forall s, t \in F(D) : \ \underline{s} = \underline{t} \ \Rightarrow \ s = t.$$

Similarly to what we did above, we can prove that ${}^{\rho}\mathcal{G}\ell^{\infty}(-, \mathcal{Y})$ is a concrete sheaf.

11. Conclusions and future perspectives

Sobolev and Schwartz solved the problem "how to derive continuous functions?". Also Sebastiao e Silva (see [100]) solved the same problem without relying on functional analysis at all, but instead using only a formal approach and arriving at an isomorphic solution. We solved the problem: "how to derive continuous functions obtaining set-theoretical functions, unrestrictedly composable, extending the usual classical theorems of calculus and allowing for infinitesimal and infinite values?". This second problem doesn't appear to have a trivial formal solution.

We have shown that GSF theory has features that closely resemble classical smooth functions. In contrast, some differences have to be carefully considered, such as the fact that the new ring of scalars ${}^{\rho}\widetilde{\mathbb{R}}$ is not a field, it is not totally ordered, it is not order complete, so that its theory of supremum and infimum is more involved (see [83]), and its intervals are not connected in the sharp topology because the set of all the infinitesimals is a clopen set. Almost all these properties are necessarily shared by other non-Archimedean rings because their opposites are incompatible with the existence of infinitesimal numbers.

Conversely, the ring of Robinson-Colombeau generalized numbers ${}^{\rho}\widetilde{\mathbb{R}}$ is a framework where the use of infinitesimal and infinite quantities is available, it is defined using elementary mathematics, and with a strong connection with infinitesimal and infinite functions of classical analysis. As proved in [28], this leads to a better understanding and opens the possibility to define new models of physical systems. We can hence state that GSF theory is potentially a good framework for mathematical physics.

As we started to see in Sec. 10.5, the category of concrete sheaves over the concrete site of gluable families contains the category of strongly open sets and GSF and hence, also the category of ordinary smooth functions on open sets. In future works, we will build on this and show that it also contains the category of smooth manifolds (more generally all diffeological spaces). This opens the possibility to study singular differential geometry using non-Archimedean methods and, as is typical of topos theory, interesting connections with logic.

Finally, as we will see in the next two papers of this series ([77, 44]), GSF theory is also an interesting non-Archimedian framework for the mathematical analysis of singular non-linear ordinary and partial differential equations.

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