PS Harmonische Analysis

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Note: References refer to the lecture notes.

1. Show that for $n, m \in \mathbb{N}$ we have
$$2\int_0^1 \sin(n\pi x)\sin(m\pi x)dx = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases}$$
Conclude that the Fourier sine coefficients g_n of
$$g(x) = \sum_{n=1}^{\infty} g_n \sin(n\pi x).$$
are given by
$$g_n = 2\int_0^1 \sin(n\pi x) g(x)dx.$$
provided the sum converges uniformly.

2. Use the method of separation of variables as described in the notes to solve the **heat equation**
$$\frac{\partial}{\partial t} u(t,x) = \frac{\partial^2}{\partial x^2} u(t,x)$$
with boundary conditions $u(t,0) = u_0$, $u(t,1) = u_1$ and initial condition $u(0,x) = u(x)$. It models the temperature distribution of a thin wire whose edges are kept at a fixed temperature u_0 and u_1. What can you say about $\lim_{t\to\infty} u(t,x)$? (Hint: If $u(x,t)$ solves the heat equation, so does $u(x,t) + a + bx$. Use this to reduce the boundary conditions to the case $u_0 = u_1 = 0$.)

3. Show that $\ell^2(\mathbb{N})$ is a Hilbert space.

4. Prove the **parallelogram law**
$$\|f+g\|^2 + \|f-g\|^2 = 2\|f\|^2 + 2\|g\|^2$$
for f, g in some Hilbert space $\mathfrak{H}$.

5. Let
$$\ell^p(\mathbb{N}) := \{a := (a_j)_{j\in\mathbb{N}} \mid \|a\|_p < \infty\},$$
where
$$\|a\|_p = \begin{cases} \left(\sum_{j\in\mathbb{N}} |a_j|^p\right)^{1/p}, & 1 \le p < \infty, \\ \sup_{j\in\mathbb{N}} |a_j|, & p = \infty. \end{cases}$$
Show that $\ell^p(\mathbb{N})$, $1 \le p \le \infty$, is a Hilbert space if and only if $p = 2$. (Hint: Previous problem.)

6. Suppose $\mathfrak{Q}$ is a complex vector space. Let $s(f,g)$ be a sesquilinear form on $\mathfrak{Q}$ and $q(f) := s(f,f)$ the associated quadratic form. Prove the **parallelogram law**
$$q(f+g) + q(f-g) = 2q(f) + 2q(g)$$
and the **polarization identity**
$$s(f,g) = \frac{1}{4}\left(q(f+g) - q(f-g) + \mathrm{i}\, q(f - \mathrm{i}g) - \mathrm{i}\, q(f+\mathrm{i}g)\right).$$
Show that $s(f,g)$ is symmetric if and only if $q(f)$ is real-valued.

   Note, that if $\mathfrak{Q}$ is a real vector space, then the parallelogram law is unchanged but the polarization identity in the form $s(f,g) = \frac{1}{4}(q(f+g) - q(f-g))$ will only hold if $s(f,g)$ is symmetric.

7. Let $\{u_j\}_{j=0}^{\infty} \subset \mathfrak{H}$ be a countable orthonormal set and $f \in \mathfrak{H}$. Show that
$$f_n = \sum_{j=0}^{n} \langle u_j, f\rangle u_j$$
is a Cauchy sequence.

8. Let $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Show that $f * g \in L^\infty(\mathbb{R}^n)$ with
$$\|f * g\|_\infty \le \|f\|_p \|g\|_q.$$
Show that for $1 < p < \infty$ or $p = 1$ and $g \in C_0(\mathbb{R}^n)$ we even have $f * g \in C_0(\mathbb{R}^n)$.

9. Let ϕ_ε be a symmetric approximate identity on $\mathbb{R}$, that is, $\phi_\varepsilon(x) = \phi_\varepsilon(-x)$. Show that for every bounded measurable function f we have
$$\lim_{\varepsilon \downarrow 0} (\phi_\varepsilon * f)(x) = \frac{f(x+) + f(x-)}{2},$$
at every point x where both right/left sided limits $f(x\pm) := \lim_{\varepsilon\downarrow 0} f(x \pm \varepsilon)$ exist.

   Show that the same conclusion holds for integrable functions f if ϕ_ε has compact support, $\mathrm{supp}(\phi_\varepsilon) \subseteq [-s,s]$, and for every $r > 0$ we have $\lim_{\varepsilon\downarrow 0} \sup_{r \le |x| \le s} |\phi_\varepsilon(x)| = 0$.

10. Show that the Landau kernel
$$L_n(x) := \begin{cases} \frac{1}{I_n}(1-x^2)^n, & |x| < 1, \\ 0, & |x| \ge 1, \end{cases} \qquad I_n = \int_{-1}^{1} (1-x^2)^n dx$$
is an approximate identity on $\mathbb{R}$ (for $\varepsilon = \frac{1}{n}$ with $n \in \mathbb{N}$).

    Use this to prove the Weierstraß approximation theorem, that every continuous function on $[-\frac{1}{2}, \frac{1}{2}]$ can be uniformly approximated by polynomials.

11. Show that the Poisson kernel
$$P_\varepsilon(x) := \frac{1}{\pi}\frac{\varepsilon}{x^2+\varepsilon^2}$$
is an approximate identity on $\mathbb{R}$.
Show that the Cauchy transform (also Borel transform)
$$F(z) := \frac{1}{\pi}\int_{\mathbb{R}} \frac{f(\lambda)}{\lambda - z} d\lambda$$
of a real-valued function $f \in L^p(\mathbb{R})$, $1 \le p < \infty$ is analytic in the upper half-plane with imaginary part given by
$$\mathrm{Im}(F(x+\mathrm{i}y)) = (P_y * f)(x).$$
In particular, by Young's inequality $\|\mathrm{Im}(F(.+\mathrm{i}y))\|_p \le \|f\|_p$ and thus also $\sup_{y>0}\|\mathrm{Im}(F(.+\mathrm{i}y))\|_p = \|f\|_p$. Such harmonic functions are said to be in the Hardy space $h^p(\mathbb{C}_+)$.
(Hint: To see analyticity of F use Problem A.1 from the notes plus the estimate
$$\left|\frac{1}{\lambda - z}\right| \le \frac{1}{1+|\lambda|}\frac{1+|z|}{|\mathrm{Im}(z)|}.)$$

12. Let $f \in L^1(-\pi,\pi)$ be periodic and $a \in \mathbb{R}$, $n \in \mathbb{Z}$. Show

$g(x)$	$\hat{g}_k$
$f(-x)$	$\hat{f}_{-k}$
$f(x)^*$	$\hat{f}^*_{-k}$
$f(x+a)$	$\mathrm{e}^{\mathrm{i}ak}\hat{f}_k$
$\mathrm{e}^{\mathrm{i}nx}f(x)$	$\hat{f}_{k-n}$

13. Show the following estimates for the Dirichlet and Fejér kernels: $|D_n(x)| \le \min(2n+1, \frac{\pi}{|x|})$ and $F_n(x) \le \min(n, \frac{\pi^2}{nx^2})$ for $|x| \le \pi$.

14. Show that the Dirichlet kernel satisfies
$$\frac{8}{\pi}\sum_{k=1}^{n}\frac{1}{k} \le \|D_n\|_1 \le 2\pi(1+\log(2n+1))$$
and note that the harmonic series diverges.

15. Compute the Fourier series of the Dirichlet kernel D_n and the Fejér kernel F_n.

16. Compute the Fourier series of $f(x) := |x|$ on $[-\pi,\pi]$. For which $x \in [-\pi,\pi]$ does $S_n(f)(x)$ converge to $f(x)$?

17. Compute the Fourier series of $f(x) := x^2$ on $[-\pi,\pi]$ and use this to show (Basel problem)
$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$
(Hint: Evaluate the series at $x = \pi$.)

18. Show that if a sequence of complex numbers S_n converges to S, then the sequence of mean values
$$\bar{S}_n := \frac{1}{n}\sum_{k=0}^{n-1} S_k$$
also converges to S.

19. Suppose f and g are periodic (with period 2π) integrable functions. Show
$$(\widehat{f * g})_k = 2\pi \hat{f}_k \hat{g}_k,$$
where
$$(f * g)(x) = \int_{-\pi}^{\pi} f(x-y)g(y)dy.$$
(Hint: Fubini.)

20. Compute the Fourier series of $f(x) := x$ on $[-\pi,\pi]$ and use this to solve again the Basel problem. (Hint: Parseval's relation.)

21. Show that the Hilbert transform H satisfies
$$(\widehat{Hf}_k) = \frac{\operatorname{sign}(k)}{\mathrm{i}} \hat{f}_k$$
for f a trigonometric polynomial. (Hint: To evaluate the integral consider the closed form of the Dirichlet kernel and split off the $+1/2$.)

22. Compute the Fourier series of $f(x) := \operatorname{sign}(x)$. For which $x \in [-\pi,\pi]$ does $S_n(f)(x)$ converge to $f(x)$?

23. Compute the Fourier series of $f(x) := \frac{\pi}{\sin(\alpha\pi)}\mathrm{e}^{\mathrm{i}\alpha x}$ on $[-\pi,\pi]$ for $\alpha \in \mathbb{C}\setminus\mathbb{Z}$. For which $x \in [-\pi,\pi]$ does $S_n(f)(x)$ converge to $f(x)$?

    Establish the partial fraction decomposition
$$\pi z \cot(\pi z) = 1 + 2z^2 \sum_{k=1}^{\infty} \frac{1}{z^2 - k^2}, \qquad z \in \mathbb{C}\setminus\mathbb{Z}.$$

24. Show that for $\delta \in \mathbb{R}$ and $f \in L^1(0,\pi)$ we have
$$\lim_{n\to\infty} \int_0^{\pi} f(x)\sin((n+\delta)x)dx = 0.$$
Use this to compute the Dirichlet integral
$$\lim_{R\to\infty} \int_0^R \frac{\sin(x)}{x} dx = \frac{\pi}{2}.$$
(Hint: To evaluate the Dirichlet integral start from $\int_0^{\pi} D_n(x)dx = \pi$ and observe that $\frac{2}{x} - \frac{1}{\sin(x/2)}$ is continuous on $[-\pi,\pi]$.)

25. Show the summation by parts formula

$$\sum_{j=m}^{n} g_j (\partial f)_j = g_n f_{n+1} - g_{m-1} f_m + \sum_{j=m}^{n} (\partial^* g)_j f_j$$

where,

$$(\partial f)_n = f_{n+1} - f_n, \qquad (\partial^* f)_n = f_{n-1} - f_n,$$

are the forward/backward difference operators.

Suppose the real-valued sequence a_n converges to zero and satisfies $b_n := -(\partial\partial^* a)_n = a_{n+1} + a_{n-1} - 2a_n \geq 0$ for $n \in \mathbb{N}$. Show that

- $\alpha_n := a_{n-1} - a_n \geq 0$ is decreasing and converges to zero,
- $n\alpha_n \to 0$,
- $a_m = \sum_{j=m+1}^{\infty} (j-m) b_j$.

(Hint: To show the second claim use $a_m - a_n \geq (n-m)\alpha_n$ for $n \geq m$. For the third claim use summation by parts.)

26. Let $(a_k)_{k\in\mathbb{N}}$ be a nonnegative monotone decreasing sequence which converges to zero and $(b_k(x))_{k\in\mathbb{N}}$ a sequence of complex-valued functions for which $B_n(x) = \sum_{k=1}^{n-1} b_k(x)$ is uniformly bounded, $|B_n(x)| \leq C$. Show that (Dirichlet criterion) the series

$$\sum_{k=0}^{\infty} a_k b_k(x)$$

converges uniformly.

Conclude that the Fourier cosine/sine series

$$\sum_{k=0}^{\infty} a_k \cos(kx), \qquad \sum_{k=0}^{\infty} a_k \sin(kx)$$

converge uniformly on every compact subinterval $0 < |x| < 2\pi$. (Hint: To show the Dirichlet criterion use summation by parts from Problem 25.)

27. Show that the operator

$$I : L^1(-\pi, \pi) \to C_{per}[-\pi, \pi], \quad f(x) \mapsto F(x) - \hat{f}_0 x - \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x)dx,$$

where $F(x) := \int_0^x f(y)dy$, satisfies

$$\widehat{I(f)}_k = \begin{cases} 0, & k = 0, \\ \dfrac{\hat{f}_k}{\mathrm{i}k}, & k \in \mathbb{Z}\setminus\{0\}. \end{cases}$$

Show also $\|I(f)\|_\infty \leq \frac{5}{2}\|f\|_1$.

28. Show that $f \in C^\infty_{per}[-\pi, \pi]$ if and only if $|k^m \hat{f}_k|$ is bounded for all $m \in \mathbb{N}$.

29. Define the Sobolev space

$$H^s_{per}(-\pi,\pi) := \{f \in L^2(-\pi,\pi) \mid |k|^s \hat{f}_k \in \ell^2(\mathbb{Z})\}.$$

Show that $H^1_{per}(-\pi,\pi) = \{f \in AC_{per}[-\pi,\pi] \mid f' \in L^2(-\pi,\pi)\}$. Show that the Fourier coefficients of $f \in H^s_{per}(-\pi,\pi)$ are summable for $s > \frac{1}{2}$. Show that this fails for $s = \frac{1}{2}$. (Hint: Problem 27.)

30. Show that if $f \in C^{0,\gamma}_{per}[-\pi,\pi]$ is Hölder continuous, then

$$|\hat{f}_k| \le \frac{[f]_\gamma}{2}\left(\frac{\pi}{|k|}\right)^\gamma, \qquad k \ne 0.$$

Conclude that if $f \in C^{l,\gamma}_{per}[-\pi,\pi]$ (the set of periodic functions which are C^l and for which the highest derivative is Hölder continuous of exponent γ) we have

$$|\hat{f}_k| \le \frac{[f^{(l)}]_\gamma}{2}\left(\frac{\pi}{|k|}\right)^{l+\gamma}, \qquad k \ne 0.$$

(Hint: What changes if you replace $\mathrm{e}^{-\mathrm{i}ky}$ by $\mathrm{e}^{-\mathrm{i}k(y+\pi/k)}$ in a Fourier series? Now make a change of variables $y \to y - \pi/k$ in the integral.)

31. Consider the function

$$f(x) := \sum_{j=0}^\infty 2^{-\gamma j}\mathrm{e}^{\mathrm{i}2^j x}, \qquad \gamma > 0.$$

Show:

(i) f is in the Wiener algebra (and in particular continuous).
(ii) f is not of bounded variation for $0 < \gamma < 1$.
(iii) f is not absolutely continuous for $0 < \gamma \le 1$.
(iv) f is Hölder continuous of exponent γ for $0 < \gamma < 1$ but not Lipschitz continuous for $\gamma = 1$.

(Hint for (iv): To estimate $|f(x+\delta) - f(x)|$ split the sum at an index n and use $|\mathrm{e}^{\mathrm{i}x} - 1| \le |x|$, $x \in \mathbb{R}$, for the first part and $|\mathrm{e}^{\mathrm{i}x} - 1| \le 2$, $x \in \mathbb{R}$, for the second. Now find a suitable value for n.)

32. Let $S_n = \sum_{k=0}^n a_k$ and $\bar{S}_n := \frac{1}{n}\sum_{k=0}^{n-1} S_k$. Show that for $m < n$

$$S_n - \bar{S}_{n+1} = \frac{m+1}{n-m}\big(\bar{S}_{n+1} - \bar{S}_{m+1}\big) + \frac{1}{n-m}\sum_{k=m+1}^{n}(S_n - S_k).$$

33. Show that $F^1(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ with

$$\|f\|_p \le (2\pi)^{\frac{n}{2}(1-\frac{1}{p})}\|f\|_1^{\frac{1}{p}}\|\hat{f}\|_1^{1-\frac{1}{p}}.$$

Moreover, show that $\mathcal{S}(\mathbb{R}^n) \subset F^1(\mathbb{R}^n)$ and conclude that $F^1(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for $p \in [1,\infty)$. (Hint: Use $x^p \le x$ for $0 \le x \le 1$ to show $\|f\|_p \le \|f\|_\infty^{1-1/p}\|f\|_1^{1/p}$.)

34. Suppose $f_j \in L^1(\mathbb{R})$, $j = 1, \dots, n$ and set $f(x) = \prod_{j=1}^n f_j(x_j)$. Show that $f \in L^1(\mathbb{R}^n)$ with $\|f\|_1 = \prod_{j=1}^n \|f_j\|_1$ and $\hat{f}(p) = \prod_{j=1}^n \hat{f}_j(p_j)$.

35. Compute the Fourier transform of the following functions $f : \mathbb{R} \to \mathbb{C}$:
(i) $f(x) = \chi_{(-1,1)}(x)$. (ii) $f(x) = \frac{e^{-k|x|}}{k}$, $\mathrm{Re}(k) > 0$.

36. Show that
$$\psi_n(x) := H_n(x) e^{-\frac{x^2}{2}} \in \mathcal{S}(\mathbb{R}),$$
where $H_n(x)$ is the Hermite polynomial DLMF (12.7.2) of degree n given by
$$H_n(x) := e^{\frac{x^2}{2}} \left(x - \frac{d}{dx} \right)^n e^{-\frac{x^2}{2}},$$
are eigenfunctions of the Fourier transform: $\hat{\psi}_n(p) = (-\mathrm{i})^n \psi_n(p)$.

37. Prove the Poisson summation formula
$$\sum_{n\in\mathbb{Z}} f(n) e^{-\mathrm{i}nx} = \sqrt{2\pi} \sum_{m\in\mathbb{Z}} \hat{f}(x + 2\pi m),$$
where $f \in L^1(\mathbb{R})$ satisfies $|f(x)| + |\hat{f}(x)| \le \frac{C}{(1+|x|)^\alpha}$ for some $\alpha > 1$. (Hint: Compute the Fourier coefficients of the right-hand side. To this end observe that the integrals over $[-\pi, \pi]$ give a tiling of $\mathbb{R}$ when m runs through all values in $\mathbb{Z}$.)

38. Suppose $f \in L^1(\mathbb{R})$ satisfies $|f(x)| + |\hat{f}(x)| \le \frac{C}{(1+|x|)^\alpha}$ for some $\alpha > 1$. Prove the Whittaker–Shannon interpolation formula:
$$f(x) = \sum_{n\in\mathbb{Z}} f(n) \operatorname{sinc}(\pi(x - n))$$
provided $\mathrm{supp}(\hat{f}) \subseteq [-\pi, \pi]$. Here $\operatorname{sinc}(x) := \frac{\sin(x)}{x}$. (Hint: Use the Poisson summation formula to express $\hat{f}$ and take the inverse Fourier transform.)

39. Show
$$\int_0^\infty \frac{\sin(x)^2}{x^2} dx = \frac{\pi}{2}.$$
(Hint: Problem 35 (i).)

40. Suppose $f(x) e^{k|x|} \in L^1(\mathbb{R})$ for some $k > 0$. Then $\hat{f}(p)$ has an analytic extension to the strip $|\mathrm{Im}(p)| < k$.