

PS Topologie und Funktionalanalysis

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Note: References refer to the lecture notes.

1. Suppose $\sum_{n=1}^{\infty} |c_n| < \infty$. Show that

$$u(t, x) := \sum_{n=1}^{\infty} c_n e^{-(\pi n)^2 t} \sin(n\pi x),$$

is continuous for $(t, x) \in [0, \infty) \times [0, 1]$ and solves the heat equation for $(t, x) \in (0, \infty) \times [0, 1]$. (Hint: Weierstrass M-test. When can you interchange the order of summation and differentiation?)

2. Show that for $n, m \in \mathbb{N}$ we have

$$2 \int_0^1 \sin(n\pi x) \sin(m\pi x) dx = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases}$$

Conclude that the Fourier sine coefficients are given by

$$\hat{u}_{0,n} = 2 \int_0^1 \sin(n\pi x) u_0(x) dx$$

provided the sum in (*) converges uniformly. Conclude that in this case the solution can be expressed as

$$u(t, x) = \int_0^1 K(t, x, y) u_0(y) dy, \quad t > 0,$$

where

$$\begin{aligned} K(t, x, y) &:= 2 \sum_{n=1}^{\infty} e^{-(\pi n)^2 t} \sin(n\pi x) \sin(n\pi y) \\ &= \frac{1}{2} \left(\vartheta\left(\frac{x-y}{2}, i\pi t\right) - \vartheta\left(\frac{x+y}{2}, i\pi t\right) \right). \end{aligned}$$

Here

$$\vartheta(z, \tau) := \sum_{n \in \mathbb{Z}} e^{i\pi n^2 \tau + 2\pi i n z} = 1 + 2 \sum_{n \in \mathbb{N}} e^{i\pi n^2 \tau} \cos(2\pi n z), \quad \text{Im}(\tau) > 0,$$

is the Jacobi theta function.

3. Show that $|||f| - |g||| \leq \|f - g\|$.
4. Let X be a Banach space. Show that the norm, vector addition, and multiplication by scalars are continuous. That is, if $f_n \rightarrow f$, $g_n \rightarrow g$, and $\alpha_n \rightarrow \alpha$, then $\|f_n\| \rightarrow \|f\|$, $f_n + g_n \rightarrow f + g$, and $\alpha_n g_n \rightarrow \alpha g$.

5. While $\ell^1(\mathbb{N})$ is separable, it still has room for an uncountable set of linearly independent vectors. Show this by considering vectors of the form

$$a^\alpha = (1, \alpha, \alpha^2, \dots), \quad \alpha \in (0, 1).$$

(Hint: Recall the Vandermonde determinant.)

6. Prove Young's inequality

$$\alpha^{1/p} \beta^{1/q} \leq \frac{1}{p} \alpha + \frac{1}{q} \beta, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad \alpha, \beta \geq 0.$$

Show that equality occurs precisely if $\alpha = \beta$. (Hint: Take logarithms on both sides.)

7. Show that $\ell^\infty(\mathbb{N})$ is a Banach space.
8. Is $\ell^1(\mathbb{N})$ a closed subspace of $\ell^\infty(\mathbb{N})$ (with respect to the $\|\cdot\|_\infty$ norm)? If not, what is its closure?
9. Consider $\ell^1(\mathbb{N})$. Show that $\|a\| := \sup_{k \in \mathbb{N}} |\sum_{j=1}^k a_j|$ is a norm. Is $\ell^1(\mathbb{N})$ complete with this norm?
10. Show that $\ell^\infty(\mathbb{N})$ is not separable. (Hint: Consider sequences which take only the value one and zero. How many are there? What is the distance between two such sequences?)
11. Let X be a normed space. Show that the following conditions are equivalent.
- (i) If $\|x + y\| = \|x\| + \|y\|$ then $y = \alpha x$ for some $\alpha \geq 0$ or $x = 0$.
 - (ii) If $\|x\| = \|y\| = 1$ and $x \neq y$ then $\|\lambda x + (1-\lambda)y\| < 1$ for all $0 < \lambda < 1$.
 - (iii) If $\|x\| = \|y\| = 1$ and $x \neq y$ then $\frac{1}{2}\|x + y\| < 1$.
 - (iv) The function $x \mapsto \|x\|^2$ is strictly convex.

A norm satisfying one of them is called strictly convex.

12. Consider $X = C([-1, 1])$. Which of the following subsets are subspaces of X ? Which of them are closed?
- (i) monotone functions
 - (ii) even functions
 - (iii) polynomials
 - (iv) polynomials of degree at most k for some fixed $k \in \mathbb{N}_0$
 - (v) continuous piecewise linear functions
 - (vi) $C^1([-1, 1])$
 - (vii) $\{f \in C([-1, 1]) \mid f(c) = f_0\}$ for some fixed $c \in [-1, 1]$ and $f_0 \in \mathbb{R}$
13. Let I be a compact interval. Show that the set $Y := \{f \in C(I, \mathbb{R}) \mid f(x) > 0\}$ is open in $X := C(I, \mathbb{R})$. Compute its closure.
14. Compute the closure of the following subsets of $\ell^2(\mathbb{N})$: (i) $B_1 := \{a \in \ell^2(\mathbb{N}) \mid \sum_{j \in \mathbb{N}} |a_j| \leq 1\}$. (ii) $B_\infty := \{a \in \ell^2(\mathbb{N}) \mid \sum_{j \in \mathbb{N}} |a_j| < \infty\}$.

15. Show that, in a Hilbert space,

$$\sum_{1 \leq j < k \leq n} \|f_j - f_k\|^2 + \left\| \sum_{1 \leq j \leq n} f_j \right\|^2 = n \sum_{1 \leq j \leq n} \|f_j\|^2$$

for every $n \in \mathbb{N}$. The case $n = 2$ is the parallelogram law.

16. Show that the maximum norm on $C[0, 1]$ does not satisfy the parallelogram law.