PS Topologie und Funktionalanalysis

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Note: References refer to the lecture notes.

1. Suppose $\sum_{n=1}^{\infty} |c_n| < \infty$. Show that
$$u(t,x) := \sum_{n=1}^{\infty} c_n \mathrm{e}^{-(\pi n)^2 t} \sin(n\pi x),$$
is continuous for $(t,x) \in [0,\infty) \times [0,1]$ and solves the heat equation for $(t,x) \in (0,\infty) \times [0,1]$. (Hint: Weierstrass M-test. When can you interchange the order of summation and differentiation?)

2. Show that for $n, m \in \mathbb{N}$ we have
$$2\int_0^1 \sin(n\pi x)\sin(m\pi x)dx = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases}$$
Conclude that the Fourier sine coefficients are given by
$$\hat{u}_{0,n} = 2\int_0^1 \sin(n\pi x) u_0(x) dx$$
provided the sum in $(*)$ converges uniformly. Conclude that in this case the solution can be expressed as
$$u(t,x) = \int_0^1 K(t,x,y) u_0(y) dy, \qquad t > 0,$$
where
$$\begin{aligned} K(t,x,y) &:= 2\sum_{n=1}^{\infty} \mathrm{e}^{-(\pi n)^2 t} \sin(n\pi x)\sin(n\pi y) \\ &= \frac{1}{2}\big(\vartheta(\frac{x-y}{2}, \mathrm{i}\pi t) - \vartheta(\frac{x+y}{2}, \mathrm{i}\pi t)\big). \end{aligned}$$
Here
$$\vartheta(z,\tau) := \sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\pi n^2\tau + 2\pi \mathrm{i} n z} = 1 + 2\sum_{n\in\mathbb{N}} \mathrm{e}^{\mathrm{i}\pi n^2 \tau}\cos(2\pi n z), \quad \mathrm{Im}(\tau) > 0,$$
is the Jacobi theta function.

3. Show that $|\|f\| - \|g\|| \leq \|f - g\|$.

4. Let X be a Banach space. Show that the norm, vector addition, and multiplication by scalars are continuous. That is, if $f_n \to f$, $g_n \to g$, and $\alpha_n \to \alpha$, then $\|f_n\| \to \|f\|$, $f_n + g_n \to f + g$, and $\alpha_n g_n \to \alpha g$.

5. While $\ell^1(\mathbb{N})$ is separable, it still has room for an uncountable set of linearly independent vectors. Show this by considering vectors of the form
$$a^\alpha = (1, \alpha, \alpha^2, \dots), \qquad \alpha \in (0,1).$$
(Hint: Recall the Vandermonde determinant.)

6. Prove Young's inequality
$$\alpha^{1/p}\beta^{1/q} \le \frac{1}{p}\alpha + \frac{1}{q}\beta, \qquad \frac{1}{p} + \frac{1}{q} = 1, \quad \alpha, \beta \ge 0.$$
Show that equality occurs precisely if $\alpha = \beta$. (Hint: Take logarithms on both sides.)

7. Show that $\ell^\infty(\mathbb{N})$ is a Banach space.

8. Is $\ell^1(\mathbb{N})$ a closed subspace of $\ell^\infty(\mathbb{N})$ (with respect to the $\|.\|_\infty$ norm)? If not, what is its closure?

9. Consider $\ell^1(\mathbb{N})$. Show that $\|a\| := \sup_{k\in\mathbb{N}} |\sum_{j=1}^k a_j|$ is a norm. Is $\ell^1(\mathbb{N})$ complete with this norm?

10. Show that $\ell^\infty(\mathbb{N})$ is not separable. (Hint: Consider sequences which take only the value one and zero. How many are there? What is the distance between two such sequences?)

11. Let X be a normed space. Show that the following conditions are equivalent.
    (i) If $\|x + y\| = \|x\| + \|y\|$ then $y = \alpha x$ for some $\alpha \ge 0$ or $x = 0$.
    (ii) If $\|x\| = \|y\| = 1$ and $x \ne y$ then $\|\lambda x + (1-\lambda)y\| < 1$ for all $0 < \lambda < 1$.
    (iii) If $\|x\| = \|y\| = 1$ and $x \ne y$ then $\frac{1}{2}\|x + y\| < 1$.
    (iv) The function $x \mapsto \|x\|^2$ is strictly convex.

    A norm satisfying one of them is called strictly convex.

12. Consider $X = C([-1,1])$. Which of the following subsets are subspaces of X? Which of them are closed?
    (i) monotone functions
    (ii) even functions
    (iii) polynomials
    (iv) polynomials of degree at most k for some fixed $k \in \mathbb{N}_0$
    (v) continuous piecewise linear functions
    (vi) $C^1([-1,1])$
    (vii) $\{f \in C([-1,1]) \mid f(c) = f_0\}$ for some fixed $c \in [-1,1]$ and $f_0 \in \mathbb{R}$

13. Let I be a compact interval. Show that the set $Y := \{f \in C(I,\mathbb{R}) \mid f(x) > 0\}$ is open in $X := C(I,\mathbb{R})$. Compute its closure.

14. Compute the closure of the following subsets of $\ell^2(\mathbb{N})$: (i) $B_1 := \{a \in \ell^2(\mathbb{N}) \mid \sum_{j\in\mathbb{N}} |a_j| \le 1\}$. (ii) $B_\infty := \{a \in \ell^2(\mathbb{N}) \mid \sum_{j\in\mathbb{N}} |a_j| < \infty\}$.

15. Show that, in a Hilbert space,
$$\sum_{1\le j<k\le n} \|f_j - f_k\|^2 + \|\sum_{1\le j\le n} f_j\|^2 = n \sum_{1\le j\le n} \|f_j\|^2$$
for every $n \in \mathbb{N}$. The case $n = 2$ is the parallelogram law.

16. Show that the maximum norm on $C[0,1]$ does not satisfy the parallelogram law.

17. Let X and Y be real normed spaces. Show that an additive (i.e. $T(x+y) = T(x) + T(y)$) continuous map $T : X \to Y$ is linear. What about complex spaces?

18. Suppose $\mathfrak{Q}$ is a complex vector space. Let $s(f,g)$ be a sesquilinear form on $\mathfrak{Q}$ and $q(f) := s(f,f)$ the associated quadratic form. Prove the **parallelogram law**
$$q(f+g) + q(f-g) = 2q(f) + 2q(g)$$
and the **polarization identity**
$$s(f,g) = \frac{1}{4}\left(q(f+g) - q(f-g) + \mathrm{i}\, q(f-\mathrm{i}g) - \mathrm{i}\, q(f+\mathrm{i}g)\right).$$
Show that $s(f,g)$ is symmetric if and only if $q(f)$ is real-valued.
Note, that if $\mathfrak{Q}$ is a real vector space, then the parallelogram law is unchanged but the polarization identity in the form $s(f,g) = \frac{1}{4}(q(f+g) - q(f-g))$ will only hold if $s(f,g)$ is symmetric.

19. Show that the integral defined in Example 1.15 satisfies
$$\int_c^e f(x)dx = \int_c^d f(x)dx + \int_d^e f(x)dx, \qquad \left|\int_c^d f(x)dx\right| \le \int_c^d |f(x)|dx.$$
How should $|f|$ be defined here?

20. Is it possible to define $\sin(f)$ for $f \in L^1(I)$? If yes, how should this be done? What about $\exp(f)$?

21. Show that in a metric space X a totally bounded set U is bounded.

22. Find a compact subset of $\ell^\infty(\mathbb{N})$ which does not satisfy (ii) from Theorem 1.12.

23. Which of the following families are relatively compact in $C[0,1]$?
    (i) $F := \{f \in C^1[0,1] \mid \|f\|_\infty \le 1\}$
    (ii) $F := \{f \in C^1[0,1] \mid \|f'\|_\infty \le 1\}$
    (iii) $F := \{f \in C^1[0,1] \mid \|f\|_\infty \le 1, \|f'\|_2 \le 1\}$

24. Consider $X = \mathbb{C}^n$ and let $A \in \mathfrak{L}(X)$ be a matrix. Equip X with the norm (show that this is a norm)
$$\|x\|_\infty := \max_{1\le j\le n} |x_j|$$
and compute the operator norm $\|A\|$ with respect to this norm in terms of the matrix entries. Do the same with respect to the norm
$$\|x\|_1 := \sum_{1\le j\le n} |x_j|.$$

25. Let $X := C[0,1]$. Investigate the operator $A : X \to X$, $f(x) \mapsto x\,f(x)$. Show that this is a bounded linear operator and compute its norm. What is the closure of $\mathrm{Ran}(A)$?

26. Show that the **Fredholm integral operator**
$$(Kf)(x) := \int_0^1 K(x,y) f(y) dy,$$
where $K(x,y) \in C([0,1]\times[0,1])$, defined on $\mathfrak{D}(K) := C[0,1]$, is a bounded operator both in $X := C[0,1]$ (max norm) and $X := L^2(0,1)$. Show that the norm in the $X = C[0,1]$ case is given by
$$\|K\| = \max_{x\in[0,1]} \int_0^1 |K(x,y)| dy.$$

27. Let $A \in \mathfrak{L}(X)$ be a bijection. Show
$$\|A^{-1}\|^{-1} = \inf_{x\in X, \|x\|=1} \|Af\|.$$

28. Prove Lemma 1.19. (Hint: Hölder's inequality in $\mathbb{C}^n$ and note that equality is attained.)

29. Let X, Y be normed spaces and suppose $A \in \mathfrak{L}(X,Y)$. Show that $\mathrm{Ker}(A)$ is closed. Suppose $M \subseteq \mathrm{Ker}(A)$ is a closed subspace. Show that the induced map $\tilde{A} : X/M \to Y$, $[x] \mapsto Ax$ is a well-defined operator satisfying $\|\tilde{A}\| = \|A\|$ and $\mathrm{Ker}(\tilde{A}) = \mathrm{Ker}(A)/M$, $\mathrm{Ran}(\tilde{A}) = \mathrm{Ran}(A)$. In particular, $\tilde{A}$ is injective for $M = \mathrm{Ker}(A)$.

30. Given some vectors $f_1, \dots, f_n$ we define their **Gram determinant** as
$$\Gamma(f_1, \dots, f_n) := \det\left(\langle f_j, f_k\rangle\right)_{1\le j,k\le n}.$$
Show that the Gram determinant is nonzero if and only if the vectors are linearly independent. Moreover, show that in this case
$$\mathrm{dist}(g, \mathrm{span}\{f_1, \dots, f_n\})^2 = \frac{\Gamma(f_1, \dots, f_n, g)}{\Gamma(f_1, \dots, f_n)}$$
and
$$\Gamma(f_1, \dots, f_n) \le \prod_{j=1}^n \|f_j\|^2.$$
with equality if the vectors are orthogonal. (Hint: First establish $\Gamma(f_1, \dots, f_j + \alpha f_k, \dots, f_n) = \Gamma(f_1, \dots, f_n)$ for $j \ne k$ and use it to investigate how Γ changes when you apply the Gram–Schmidt procedure?)

31. Let $\{u_j\}$ be some orthonormal basis. Show that a bounded linear operator A is uniquely determined by its matrix elements $A_{jk} := \langle u_j, Au_k\rangle$ with respect to this basis.

32. Let $I = (a,b)$ be some interval and consider the scalar product

$$\langle f, g\rangle := \int_a^b f(x)^* g(x)\, w(x) dx$$

associated with some positive weight function $w(x)$. Let $P_j(x) = x^j + \beta_j x^{j-1} + \gamma_j x^{j-2} + \dots$ be the corresponding monic orthogonal polynomials obtained by applying the Gram–Schmidt procedure (without normalization) to the monomials:

$$\int_a^b P_i(x) P_j(x) w(x) dx = \begin{cases} \alpha_j^2, & i = j, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\bar{P}_j(x) := \alpha_j^{-1} P_j(x)$ be the corresponding orthonormal polynomials and show that they satisfy the three term recurrence relation

$$a_j \bar{P}_{j+1}(x) + b_j \bar{P}_j(x) + a_{j-1}\bar{P}_{j-1}(x) = x\bar{P}_j(x),$$

or equivalently

$$P_{j+1}(x) = (x - b_j) P_j(x) - a_{j-1}^2 P_{j-1}(x),$$

where

$$a_j := \int_a^b x\bar{P}_{j+1}(x)\bar{P}_j(x) w(x) dx, \qquad b_j := \int_a^b x\bar{P}_j(x)^2 w(x) dx.$$

Here we set $P_{-1}(x) = \bar{P}_{-1}(x) \equiv 0$ for notational convenience (in particular we also have $\beta_0 = \gamma_0 = \gamma_1 = 0$).

33. Consider $M := \{f \in L^2(0,1) | \int_0^1 f(x)dx = 0\} \subset L^2(0,1)$. Compute $M^\perp$.

34. Consider the subspace $M_e := \{f \in C[-1,1] | f(x) = f(-x)\}$ of all even continuous functions in $L^2(-1,1)$. Is M_e closed? What is $P_{\overline{M_e}}$ and $M_e^\perp$? Compute $\operatorname{dist}(\exp(x), M_e)$.

35. Show that $\ell(a) = \sum_{j=1}^\infty \frac{a_j + a_{j+2}}{2^j}$ defines a bounded linear functional on $X := \ell^2(\mathbb{N})$. Compute its norm.

36. Let $\mathfrak{H}_1$, $\mathfrak{H}_2$ be Hilbert spaces and let $u \in \mathfrak{H}_1$, $v \in \mathfrak{H}_2$. Show that the operator

$$Af := \langle u, f\rangle v$$

is bounded and compute its norm. Compute the adjoint of A.

37. Let $\mathfrak{H}$ be a Hilbert space and $\{f_j\}_{j=1}^n \subset \mathfrak{H}$ some vectors. Show that $A : \mathbb{C}^n \to \mathfrak{H}$, $\alpha \mapsto \sum_{j=1}^n \alpha_j f_j$ is bounded and compute A^*.

38. Let $\mathfrak{H}_1$, $\mathfrak{H}_2$ be Hilbert spaces and suppose $A \in \mathfrak{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ has a bounded inverse $A^{-1} \in \mathfrak{L}(\mathfrak{H}_2, \mathfrak{H}_1)$. Show $(A^{-1})^* = (A^*)^{-1}$.

39. Let $\mathfrak{H}$ be a Hilbert space and $s : \mathfrak{H}^2 \to \mathbb{C}$ a bounded coercive sesquilinear form. Show that the solution of

$$s(h,f) = \langle h, g\rangle, \qquad \forall h \in \mathfrak{H}. \tag{*}$$

is also the unique minimizer of

$$h \mapsto \mathrm{Re}\Big(\frac{1}{2}s(h,h) - \langle h, g\rangle\Big)$$

if s is nonnegative with $s(w,w) \ge \varepsilon\|w\|^2$ for all $w \in \mathfrak{H}$.

40. Consider $A \in \mathfrak{L}(\mathfrak{H})$ and denote the set of fixed points by $\mathrm{Fix}(A) := \mathrm{Ker}(A-\mathbb{I}) = \{f \in \mathfrak{H} | Af = f\}$. Show $\mathrm{Fix}(A) = \mathrm{Fix}(A^*)$ provided $\|A\| \le 1$.

    Use this to establish the mean ergodic theorem for $A \in \mathfrak{L}(\mathfrak{H})$ with $\|A\| \le 1$:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} A^j f = P_{\mathrm{Fix}} f,$$

where P_{Fix} denotes the orthogonal projection onto $\mathrm{Fix}(A)$. (Hint: To see $\mathrm{Fix}(A) = \mathrm{Fix}(A^*)$ think about when you have equality in the Cauchy–Schwarz inequality. To compute the limit consider the cases $f \in \mathrm{Fix}(A)$ and $f \in \mathrm{Ran}(A-\mathbb{I})$. Also $\mathrm{Ker}(A^*) = \mathrm{Ran}(A)^\perp$ might be useful.)

41. Is the left shift $(a_1, a_2, a_3, \dots) \mapsto (a_2, a_3, \dots)$ compact in $\ell^2(\mathbb{N})$?

42. Is the multiplication operator $M_t : C^k[0,1] \to C[0,1]$ with $M_t f(t) = tf(t)$ compact for $k = 0, 1$? (Hint: Problem 23 and Example 3.2.)

43. Show that the adjoint of the Fredholm integral operator K on $L^2(a,b)$ from Lemma 3.4 is the integral operator with kernel $K(y,x)^*$:

$$(K^* f)(x) = \int_a^b K(y,x)^* f(y) dy.$$

(Hint: Fubini.)

44. Find the eigenvalues and eigenfunctions of the integral operator $K \in \mathfrak{L}(L^2(0,1))$ given by

$$(Kf)(x) := \int_0^1 u(x)v(y)f(y)dy,$$

where $u, v \in C([0,1])$ are some given continuous functions.

45. Find the eigenvalues and eigenfunctions of the integral operator $K \in \mathfrak{L}(L^2(0,1))$ given by

$$(Kf)(x) := 2\int_0^1 (2xy - x - y + 1)f(y)dy.$$

46. Show that the inverse $(A-z)^{-1}$ (provided it exists and is densely defined) of a symmetric operator A is again symmetric for $z \in \mathbb{R}$.

    Show that a densely defined bounded operator is symmetric if and only if its extension, according to Theorem 1.16, is.

    (Hint: $g \in \mathfrak{D}((A-z)^{-1})$ if and only if $g = (A-z)f$ for some $f \in \mathfrak{D}(A)$.)

47. The finite union of nowhere dense sets is nowhere dense and the finite intersections of sets with dense interior has dense interior.

48. Let X be a topological space. Prove that the meager sets form a σ-ideal. Moreover, prove that every superset of a fat set is fat.

49. Consider $X := C[-1,1]$. Show that $M := \{x \in X | x(-t) = x(t)\}$ is meager.

50. Consider $f = \chi_{\mathbb{Q}}$ on $[0,1]$. Show that there cannot be a sequence of continuous functions $f_n \in C[0,1]$ which converges to f pointwise. (Hint: Apply Baire's theorem with $F_n := \{t \in [0,1] | \, |f_n(t) - f_m(t)| \le \frac{1}{3} \, \forall m \ge n\}$.)

51. Let X be a Banach space and Y, Z normed spaces. Show that a bilinear map $B : X \times Y \to Z$ is bounded, $\|B(x,y)\| \le C\|x\|\|y\|$, if and only if it is separately continuous with respect to both arguments. (Hint: Uniform boundedness principle.)

52. Consider a Schauder basis as in (1.31). Show that the coordinate functionals α_n are continuous. (Hint: Denote the set of all possible sequences of Schauder coefficients $\alpha = (\alpha_n)_{n=1}^N$ by $\mathcal{A}$ and equip it with the norm $\|\alpha\| := \sup_n \|\sum_{k=1}^n \alpha_k u_k\|$. By construction the operator $A : \mathcal{A} \to X$, $\alpha \mapsto \sum_k \alpha_k u_k$ has norm one. Now show that $\mathcal{A}$ is complete and apply the inverse mapping theorem.)

53. Show that the operator

$$\mathfrak{D}(A) := \{a \in \ell^p(\mathbb{N}) | j\, a_j \in \ell^p(\mathbb{N})\}, \qquad (Aa)_j := j\, a_j,$$

is a closed operator in $X := \ell^p(\mathbb{N})$.

54. Suppose $A : X \to Y$ and $B : Y \to Z$ are linear operators between Banach spaces X, Y, Z. Show that A is bounded if BA and B are bounded with B injective.

55. Let $X := \mathbb{C}^3$ equipped with the norm $|(x,y,z)|_1 := |x| + |y| + |z|$ and $Y := \{(x,y,z) \in X | x + y = 0,\ z = 0\}$. Find at least two extensions of $\ell(x,y,z) := x$ from Y to X which preserve the norm. What if we take $Y := \{(x,y,z) \in X | x + y = 0\}$?

56. Consider $X := C[0,1]$ and let $f_0(x) := 1 - 2x$. Find at least two linear functionals with norm one such that $\ell(f_0) = 1$.

57. Show that the extension from Corollary 4.13 is unique if X^* is strictly convex. (Hint: Problem 11.)

58. Let X be some normed space. Show that

$$\|x\| = \sup_{\ell \in V,\, \|\ell\|=1} |\ell(x)|,$$

where $V \subset X^*$ is some dense subspace. Show that equality is attained if $V = X^*$.

59. Show that every $l \in \ell^p(\mathbb{N})^*$, $1 \le p < \infty$, can be written as

$$l(a) = l_b(a) := \sum_{j\in\mathbb{N}} b_j a_j$$

with some unique $b \in \ell^q(\mathbb{N})$. (Hint: To see $b \in \ell^q(\mathbb{N})$ consider a^N defined such that $a_j^n := |b_j|^q/b_j$ for $j \le n$ with $b_j \neq 0$ and $b_j^n := 0$ else. Now look at $|l(a^n)| \le \|l\| \|a^n\|_p$.)

60. Let X be a Banach space. Show that a subset $U \subseteq X$ is bounded if and only if $\ell(U) \subseteq \mathbb{C}$ is bounded for every $\ell \in X^*$. (Hint: Uniform boundedness principle.)

61. Let X be some normed space. By definition we have

$$\|\ell\| = \sup_{x\in X, \|x\|=1} |\ell(x)|$$

for every $\ell \in X^*$. One calls $\ell \in X^*$ norm-attaining, if the supremum is attained, that is, there is some $x \in X$ such that $\|\ell\| = |\ell(x)|$.

Show that in a reflexive Banach space every linear functional is norm-attaining. (Hint: For the first part apply Problem 58 to X^*.)

62. Suppose X is separable. Show that there exists a countable set $N \subset X^*$ with $N_\perp = \{0\}$.

63. Suppose M_1, M_2 are closed subspaces of X. Show

$$M_1 \cap M_2 = (M_1^\perp + M_2^\perp)_\perp, \qquad M_1^\perp \cap M_2^\perp = (M_1 + M_2)^\perp$$

and

$$(M_1 \cap M_2)^\perp \supseteq \overline{(M_1^\perp + M_2^\perp)}, \qquad (M_1^\perp \cap M_2^\perp)_\perp = \overline{(M_1 + M_2)}.$$

64. Let $X := Y := c_0(\mathbb{N})$ and recall that $X^* \cong \ell^1(\mathbb{N})$ and $X^{**} \cong \ell^\infty(\mathbb{N})$. Consider the operator $A \in \mathfrak{L}(\ell^1(\mathbb{N}))$ given by

$$Aa := (\sum_{j\in\mathbb{N}} a_j, 0, \dots).$$

Show that

$$A'b = (b_1, b_1, \dots).$$

Conclude that A is not the adjoint of an operator from $\mathfrak{L}(c_0(\mathbb{N}))$.

65. Show that that the adjoint of a projection $P \in \mathfrak{L}(X)$ is again a projection.

66. Let X, Y be Banach spaces and $A \in \mathfrak{L}(X, Y)$. Show $\operatorname{Ran}(A)^\perp = \operatorname{Ker}(A')$ and $\operatorname{Ran}(A')_\perp = \operatorname{Ker}(A)$.

67. Consider the multiplication operator $A : \ell^1(\mathbb{N}) \to \ell^1(\mathbb{N})$ with $(Aa)_j := \frac{1}{j} a_j$. Show that $\operatorname{Ran}(A)$ is not closed but dense while $\operatorname{Ran}(A')$ is neither closed nor dense. In particular, show $\operatorname{Ker}(A)^\perp = \{0\}^\perp = \ell^\infty(\mathbb{N}) \supset \overline{\operatorname{Ran}(A')} = c_0(\mathbb{N})$.

68. Let X be a Banach space. Suppose $\ell_n \to \ell$ in X^* and $x_n \rightharpoonup x$ in X. Prove that $\ell_n(x_n) \to \ell(x)$. Similarly, suppose s-lim $\ell_n = \ell$ and $x_n \to x$. Prove that $\ell_n(x_n) \to \ell(x)$. Does this still hold if s-lim $\ell_n = \ell$ and $x_n \rightharpoonup x$?

69. An element $p \in X$ from a unital Banach algebra satisfying $p^2 = p$ is called a projection. Show that $\sigma(p) \subseteq \{0, 1\}$. Compute the resolvent and the spectrum of a projection.

70. Consider the rank-one operator

$$Ax := x_0'(x)x_0, \qquad x_0 \in X,\ x_0' \in X^*,$$

on some Banach space X. Compute the resolvent, $\sigma(A)$, $\|A\|$, and $r(A)$. When do you have $\|A\| = r(A)$?

71. Let X be a Banach algebra and $x, y \in X$. Show $\sigma(xy)\setminus\{0\} = \sigma(yx)\setminus\{0\}$. (Hint: Find a relation between $(xy - \alpha)^{-1}$ and $(yx - \alpha)^{-1}$.)

72. Recall that a Laurent polynomial is an expression of the form $q(\alpha) := \sum_{j=-m}^{n} q_j \alpha^j$ with $m, n \in \mathbb{N}_0$ and $q_j \in \mathbb{C}$ (i.e. in contradistinction to a polynomial we also allow negative powers of α).

    Let X be a Banach algebra. Show that for any Laurent polynomial q and any invertible $x \in X$ we have

$$\sigma(q(x)) = q(\sigma(x)).$$

73. Let X be a C^* algebra and suppose $x \in X$ is invertible. Show the polar decomposition $x = u|x|$, where u is unitary and $|x| := \sqrt{x^* x}$ is positive. (Hint: You do not need to prove that x^*x is positive.)

74. Let X be a unital C^* algebra and $x \in X$ be self-adjoint. Show that the following are equivalent:

    (i) $\sigma(x) \subseteq [0, \infty)$.
    (ii) x is positive.
    (iii) $\|\lambda - x\| \le \lambda$ for all $\lambda \ge \|x\|$.
    (iv) $\|\lambda - x\| \le \lambda$ for one $\lambda \ge \|x\|$.

75. Show that an operator $A \in \mathfrak{L}(\mathfrak{H})$ is positive as defined in Section 2.3 (i.e., $\langle f, Af\rangle \ge 0$) if and only if it is positive in the sense of C^* algebras. (Hint: One direction is easy. For the other direction use Problem 74 and Theorem 2.17.)

76. Show that a C^* algebra (with identity) is one-dimensional if and only if the spectrum of every self-adjoint element consists of one point. Conclude that the norm of a C^* algebra (with identity) can only stem from a scalar product if it is one-dimensional. (Hint: Look at the $*$-subalgebra of a suitable self-adjoint element.)

77. Let X be a topological space and let $U \subseteq X$. Show that the closure and interior operators are dual in the sense that

$$X \setminus \overline{U} = (X \setminus U)^\circ \qquad \text{and} \qquad X \setminus U^\circ = \overline{(X \setminus U)}.$$

    In particular, the closure is the set of all points which are not interior points of the complement. (Hint: De Morgan's laws.)

78. Let X be a set and $\mathcal{B}$ a family of subsets satisfying

    (B1) $\bigcup \mathcal{B} = X$.

    (B2) For every $B_1, B_2 \in \mathcal{B}$ and every $x \in B_1 \cap B_2$ there is some $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$.

    Show that the family of sets $\mathcal{O} := \{\bigcup_\alpha B_\alpha | B_\alpha \in \mathcal{B}\}$ which can be written as a union of sets from $\mathcal{B}$ is a topology which has $\mathcal{B}$ as a base.

79. Equip $X := \mathbb{R}$ with the lower limit topology generated by half-open intervals of the form $[a, b)$. Show that this topology is finer than the metric topology. Also show that the sets $[a, b)$ are both open and closed with respect to this topology, and the same holds true for the intervals $[a, \infty)$ and $(-\infty, b)$.

80. Show that singeltons (one-point sets) are closed if and only if X is a T_1 space, that is, a topological space where for any two different points $x, y \in X$ there is a neighborhood of x disjoint from y. Evidently every Hausdorff space is T_1.

81. Show that in a Hausdorff space limits are unique. Show that the converse is true for a first countable space.

82. Let X be a first countable topological space. Show that x is a limit point of $U \subseteq X$ if and only if there is a sequence $x_n \in U \setminus \{x\}$ converging to x.

83. Let X, Y be topological spaces and $\{U_\alpha\}_{\alpha\in A}$ a cover of X (i.e. $\bigcup_\alpha U_\alpha = X$). Suppose $f_\alpha : U_\alpha \to Y$ are continuous functions with respect to the relative topology on U_α such that $f_\alpha = f_\beta$ on $U_\alpha \cap U_\beta$ and hence $f : X \to Y$ is well defined by $f(x) := f_\alpha(x)$ if $x \in U_\alpha$. Show that f is continuous if either all sets U_α are open or if the index set A is finite and all sets U_α are closed.

84. Let X, Y be topological spaces and let $f : X \to Y$ be continuous. Show that $f(\overline{A}) \subseteq \overline{f(A)}$ for any $A \subset X$. Show that equality fails in general.

85. Let X, Y be topological spaces and $A \subseteq X$, $B \subseteq Y$. Show that $(A\times B)^\circ = A^\circ \times B^\circ$ and $\overline{A \times B} = \overline{A} \times \overline{B}$.

86. Consider $X := \mathbb{R}$ with the lower semicontinuity topology $\mathcal{O} := \{(a, \infty)|a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$. Show that this is indeed a topology. Show that a nonempty set $C \subset X$ is compact if and only if it has a minimum.

87. (Alexandroff one-point compactification) Suppose X is a locally compact Hausdorff space which is not compact. Introduce a new point ∞, set $\hat{X} = X \cup \{\infty\}$ and make it into a topological space by calling $O \subseteq \hat{X}$ open if either $\infty \notin O$ and O is open in X or if $\infty \in O$ and $\hat{X} \setminus O$ is compact. Note that $\hat{C} \subseteq \hat{X}$ is closed if either $\hat{C} = C \cup \{\infty\}$ where C is a closed subset of X or $\hat{C} = K$ where K is a compact subset of X.

    Show that $\hat{X}$ is a compact Hausdorff space which contains X as a dense subset.

88. Let X be a compact metric space and $f : X \to X$ isometric, $d(f(x), f(y)) = d(x, y)$ for $x, y \in X$. Show that f is bijective. Show that this fails without the compactness assumption. (Hint: If $x_0 \notin f(X)$, consider the sequence $x_{n+1} := f(x_n)$.)

89. Let X be a topological space and suppose $A, B \subseteq X$ are connected and not separated. Show that $A \cup B$ is connected.

90. Let X be a topological space. Show that X is normal if and only if for every open set $O \subseteq X$ and every closed set $C \subseteq O$ there is an open set $U \subseteq X$ such that $C \subseteq U \subseteq \overline{U} \subseteq O$.

91. Let X_α be topological spaces and let $X := \bigtimes_{\alpha\in A} X_\alpha$ with the product topology. Show that the product $\bigtimes_{\alpha\in A} C_\alpha$ of closed sets $C_\alpha \subseteq X_\alpha$ is closed.

92. Let X_α be topological spaces. Show that $X := \bigtimes_{\alpha\in A} X_\alpha$ has the following properties if all X_α have.

    (i) Hausdorff

    (ii) separable provided A is countable

    (iii) first/second countable provided A is countable