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Duality in Integrable Systems and Gauge Theories

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We discuss various dualities, relating integrable systems and show that these dualities are explained in the framework of Hamiltonian and Poisson reductions. The dualities we study shed some light on the known integrable systems as well as allow to construct new ones, double elliptic among them. We also discuss applications to the (supersymmetric) gauge theories in various dimensions.

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1. Introduction

Traditionally the physical applications of integrable systems are exhausted by the approximations to real dynamical systems. The configurations of the (classical or quantum) model form the phase space which is a manifold M with the symplectic two-form ω or Poisson bi-vector π.

The phase space might carry a natural complex structure, such that the symplectic form ω is a holomorphic $(2,0)$-form, the Hamiltonians $H(p,q)$ are the holomorphic functions and the vector fields are holomorphic vector fields (see [1] for example). The use of integrable systems as describing the evolution in the physical models is less transparent in this case.

The integrable systems in the holomorphic sense entered physics approximately at the same time as the string theory did. Particle which has a one-real-dimensional worldline is naturally described with the help of a real phase space, its (real) time evolution being generated by the real Hamiltonian. The worldsheet of a string is a complex curve, embedded into the target space. It senses holomorphic geometry in many different ways. In particular, the embeddings of the worldsheet Σ are governed by a two dimensional conformal field theory on Σ. An example of holomorphic integrable system relevant to the latter is the famous Hitchin system. The phase space of this model is the cotangent bundle to the moduli space $\mathcal{M}_G(\Sigma)$ of holomorphic G-bundles over a Riemann surface Σ. One can think of Knizhnik-Zamolodchikov-Bernard [2] equations in two dimensional WZW theories as of the non-stationary quantum version of Hitchin system [3][4][5][6][7][8]. The rôle of times is played by the complex (and hypothetically $\mathcal{W}$)-moduli of Σ. Complex time evolution occurs also in the models of $N=(2,2)$ strings, where space-time may have $(2,2)$ signature [9].

However there exist other possibilities for integrable system to encode the physical information.

In particular, the rich source of holomorphic integrable systems is the combination of supersymmetry and duality. It is known for some time now that the holomorphy of certain quantities (like the superpotential in $\mathcal{N}=1$ or prepotential in $\mathcal{N}=2$) in the supersymmetric theories in three/four dimensions yields powerful predictions for the behavior of the quantum theory even in the presence of the non-perturbative effects [10], [11], [12],[13], [14], [15]. In particular, the complex structure of the moduli space of vacua in $\mathcal{N}=4$ $3d$ gauge theories and $\mathcal{N}=2$ $4d$ gauge theories can be determined revealing the exciting link with the special nature of the geometry of the phase spaces of complex integrable systems [16], [17], [18], [15]. The action variables appear as the central charges in the BPS representations of the susy algebra [16].

There exists an approach to a class of integrable system which allows to uncover the origin of their integrability/solvability. Namely, one realizes the system under investigation as a projection of a simple system on a larger phase space [19],[20]. This idea is actually a counterpart of the main principle behind the gauge theories - the complex dynamics of the actual world (as far as most of the fundamental interactions are concerned) is a projection of a somewhat simpler dynamics of the extended phase space. One of the goals of the present discussion is to use the analogy between the two ideas and explain certain properties of integrable systems as well as gauge theories.

We are going to study the phenomenon of *duality* whose precise definition is presented shortly. Duality is a subject of much recent investigation in the context of (supersymmetric) gauge theories, in which case the duality is an involution, which maps the observables of one theory to those of another. The duality is powerful when the coupling constant in one theory is inverse of that in another (or more generally, when small coupling is mapped to the strong one). For example, a weakly coupled (magnetic) theory can be dual to the strongly coupled (electric) theory thus making possible to understand the strong coupling behavior of the latter. In particular, it was shown by N. Seiberg and E. Witten [12] that using the concept of duality one can find exact low-energy Lagrangian of $\mathcal{N} = 2, d = 4$ $SU(2)$ gauge theory. A more fascinating recent development is that the duality connecting weak and strong coupling regimes of one or different theories may have a geometric origin. The most notorious example of that is provided by M-theory [21],[22]. We are going to study the dualities in integrable systems, related to the gauge theories with the emphasis on their geometric origin.

The study of geometry of integrable systems also allows to understand the origin of certain constructions of separation of variables [23]. The similarity of this construction to the description of the D2-brane moduli space and its rôle in the understanding the string duality makes one hope that both subjects - many-body integrable systems and gauge theories (more generally, D-geometry of M. Douglas [24][25]) will benefit more from each other in the near future.

Topics left beyond the scope of the paper. To keep the size of the paper within reasonable limits we decided to restrict our attention with the pure many-body systems. More or less everything we have said can be carried over to the spin systems both of the 'adjoint' [26][27] and 'fundamental' [28] type. We don't discuss extensively the relation of our dualities in integrable systems to the physics of D-branes [29][30][31]. Some of the results in this direction together with the applications to the theory of separation of variables can be found in [23]. Also, except for the general discussion and two-body examples we don't treat quantum case. For some results related to our main topic see [32][33][34][35]. Realizations of elliptic Ruijsenaars-Schneider models via Hamiltonian and Poisson reductions can be found in [36].

Organization of the paper. Various concepts of duality are discussed in the section 2. The examples of the dual systems are studied in section 3 where mostly two-body case is treated, both classical and quantum one. Many-body systems are studied in the section 4 with the explanation of the dualities between them coming from Hamiltonian/Poisson reductions. The section 5 is devoted to the gauge dynamics and their relation to the integrable systems discussed so far. We discuss the geometry of the moduli spaces of vacua of supersymmetric gauge theories in three, four, five and six dimensions and construct a little dictionary translating the notions of integrable systems to those of gauge theories.

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2. The concepts of Duality:

Let (M,ω) be a symplectic manifold. There exist Darboux local coordinates (cf. [37]) in which the symplectic form looks like a canonical one:

$$\omega = \sum_{i=1}^{m} dp_i \wedge dq^i \tag{2.1}$$

The local canonical coordinates are defined up to the symplectomorphisms. Unlike the general diffeomorphisms, which have N functional degrees of freedom, N being the dimension of the manifold, the symplectomorphisms have only 1 functional degree of freedom.

The evolution of a Hamiltonian system is defined with the help of Hamiltonian $H : M \to \mathbb{R}$. The function H defines a Hamiltonian vector field by the formula

$$\iota_{V_H}\omega = -dH \tag{2.2}$$

The integrable system on M has a maximal collection of the functionally independent commuting Hamiltonians H_i $i = 1, \ldots, m = \frac{1}{2}\mathrm{dim}M$:

$$[V_{H_i}, V_{H_j}] = 0 \tag{2.3}$$

Let $\vec{\mathbf{h}} : M \to B \approx \mathbb{R}^m$ be the map defined as: $\vec{\mathbf{h}} : x \mapsto (H_1(x), \ldots, H_m(x))$. Liouville's theorem states that the integrable system has a normal form locally: there are coordinates (I_i, φ^i), such that

$$\begin{aligned} \omega &= \sum_i dI_i \wedge d\varphi^i \\ H_k &= f_k(\{I\}) \end{aligned} \tag{2.4}$$

i.e. I_k are coordinates on B. For a sufficiently small domain $U \subset B$ the space $\vec{\mathbf{h}}^{-1}(U)$ is the product $U \times \mathbb{R}^{n-m} \times T^{2m-n}$ and φ^k are standard linear coordinates on $\mathbb{R}^{n-m} \times T^{2m-n}$. If the common level set of all Hamiltonians is compact then this set is isomorphic to the torus of the dimension m. In that case one may impose a condition on the coordinates φ^i that the differentials $d\varphi^i$ have periods which are integer multiples of 2π. This fixes the coordinates (I, φ) up to the action of discrete group $\mathrm{PGL}_m(\mathbb{Z})$.

The Liouville theory also has a counterpart in the holomorphic setting where the manifold M is replaced by the complex manifold, the symplectic form is a holomorphic closed $(2,0)$-form, the Hamiltonians H are the holomorphic functions and the vector fields are holomorphic vector fields. The Liouville theorem modifies in this case. In fact the

Liouville real tori are replaced by the complex tori. If we require these tori to be abelian varieties then we get what is called algebraically integrable system [38]. In the family of such varieties the degenerate fibers can appear.

The coordinates I_i are referred to as "action" variables. If $n = 0$ then there is a nice formula for I. Let $b_1, b_2 \in U \subset B$ be sufficiently close to each other. Choose a basis e_b in $\mathbb{H}_1(\vec{\mathbf{h}}^{-1}(b_1), \mathbb{Z})$. Connect the points b_1 and b_2 with a path $\gamma \subset U$. The base e_{b_1} can be transported to $\mathbb{H}_1(\vec{\mathbf{h}}^{-1}(\gamma), \mathbb{Z})$ by means of the Gauß-Manin connection and it defines an element $\vec{\Gamma} \in \mathbb{H}_2(\vec{\mathbf{h}}^{-1}(\gamma), \vec{\mathbf{h}}^{-1}(b_1 \cup b_2); \mathbb{Z})$. Then

$$\vec{I}(b_2) - \vec{I}(b_1) = \int_{\vec{\Gamma}} \omega$$

2.1. $(p, q) \to (I, \varphi)$

Suppose that we have *two integrable systems* $\{H_k\}$ *and* $\{H_k^D\}$ *on the same symplectic manifold* M*. In this situation we say that these two systems are dual to each other.*

Notice that this definition does not make duality an involution.

A pair of integrable systems given on one symplectic manifold (M, ω) is called *self-dual* if there exists a symplectic involution $\sigma : M \to M$ exchanging $\{H_k\}$ and $\{H_k^D\}$, i.e., such that for any $k = 1, \dots, n$

$$\sigma^* H_k = H_k^D$$

Once we have two integrable systems on the same manifold such that both collections of Hamiltonians constitute at least locally a coordinate system on the phase space M, one can write down the equations of motion of the second integrable system in the second order formalism using the action variables I_i of the first system as the coordinates q^i for the second.

The global version of this definition is: *Two Hamiltonian systems are dual to each other in the sense of action-coordinate duality if the action variables* I_i *of the first system coincide with the coordinates* q^i *of the second one and vice versa.*

2.2. $I \to I^D$

In the holomorphic algebraic category there is an interesting complication: the torus has a complex dimension m and therefore $\mathbb{H}_1(\mathbf{T}_b; \mathbb{Z}) = \mathbb{Z}^{2m}$. One can do the following, though: choose a symplectic basis $e_b = (A_\alpha, B^\beta)$ such that $A \cap A = B \cap B = 0, A_\alpha \cap B^\beta = \delta_\alpha^\beta$, where $\cap$ is an intersection form $\mathbb{H}_1 \otimes_{\mathbb{Z}} \mathbb{H}_1 \to \mathbb{Z}$.

Then the action variables are the periods of λ over the A-cycles. The reason for the B-cycles to be discarded is simply the fact that the B-periods of λ are not independent of the A-periods. On the other hand, one can choose as the independent periods the integrals of λ over any lagrangian (in the sense of $\cap$) sublattice in $\mathbb{H}_1(\mathbf{T}_b; \mathbb{Z})$.

This leads to the following structure of the action variables in the holomorphic setting. Locally over a patch in $\mathbf{B}$ one chooses a basis in $\mathbb{H}_1$ of the fiber together with the set of A-cycles. This choice may differ over another patch. Over the intersection of these discs one has a $\mathrm{Sp}_{2m}(\mathbb{Z})$ transformation relating the bases. Altogether they form an $\mathrm{Sp}_{2m}(\mathbb{Z})$ bundle. It is an easy excercise on the properties of the period matrix of abelian varieties that the two form:

$$dI^i \wedge dI_i^D \tag{2.5}$$

vanishes. Therefore one can always locally find a function $\mathcal{F}$ - *prepotential*, such that:

$$I_i^D = \frac{\partial \mathcal{F}}{\partial I^i} \tag{2.6}$$

This duality maps the integrable system to itself. It is called action-action (AA) duality.

2.3. Quantum duality

There exists a clear quantum counterpart of this picture. Consider the eigenvalue problem for the Schrödinger operators and the issue of the normalization of the wave-functions.

The quantum integrable system is a complete collection of "independent" (in the appropriate sense) commuting operators $\{\hat{H}_i\}, i = 1, \ldots, m$, acting in the Hilbert space $\mathcal{H}$ of the model. By completeness we mean that these operators have simple common spectrum $U \subset \mathbb{R}^m$.The commuting operators have common eigenfunctions. Generically the eigen-value problem:

$$\hat{H}_k|\vec{\lambda}\rangle = e_k(\lambda)|\vec{\lambda}\rangle \tag{2.7}$$

has the unique (up to normalization) solution. Here e_i is the corresponding eigenvalue and $\vec{\lambda}$ is a label, which takes values in some set Λ. Altogether, e_k form an imbedding $e_* : \Lambda \to U$

Typically one has another set of commuting operators ("position operators") $\{\hat{H}_k^D\}$ and the eigen-states $|\vec{x}\rangle$ with eigen-values $e_k(\vec{x})$ in the Hilbert space of the model are represented as the appropriate functionals on the space of the eigen-values of the operators

$\hat{H}_k^D$. Here $\vec{x}$ is another label, which takes values in the set Λ^D, which is mapped by e_*^D to $U^D \in \mathbb{R}^m$. The familiar case is $M = T^*\mathcal{M}$, $\mathcal{H}_\mathcal{M} = L^2(\mathcal{M})$, the operators $\hat{H}_i$ are represented as commuting differential operators, and $\hat{H}_k^D$ are represented as operators of multiplication by a function $e_i^D(\vec{x})$. Suppose that we are given two classically AC dual Hamiltonian systems. Let I_i and I_i^D be their action variables (recall that I_i^D are the coordinates for the first system). Assume that there exist quantum integrals of motion for both systems. Let us denote by $\hat{I}_i$ the quantum integrals of motion of the first system and by $\hat{I}_i^D$ the quantum integrals of motion of the second one.

Once we have a quantum integrable system we can identify $\mathcal{H}$ with the space $L^2(\Lambda)$ of square integrable functions on Λ (w.r.t. the spectral measure $d\mu$). Indeed, choose a basis in $\mathcal{H}$ consisting of the common eigenvectors (2.7) $|\vec{\lambda}\rangle$, where $\vec{\lambda} = \{\lambda_1, \ldots, \lambda_m\} \in \Lambda$. Then for any $|\psi\rangle \in \mathcal{H}$ one can associate the function $\langle\vec{\lambda}|\psi\rangle$ on Λ. Of course this mapping $\mathcal{H} \to L^2(\Lambda)$ depends on the normalization of the eigenvectors we have chosen.

In particular any operator A acting on $\mathcal{H}$ can be expressed as an operator acting on $L^2(\Lambda)$ as

$$\hat{A} : \psi(\vec{\lambda}) \mapsto \int_\Lambda \langle\vec{\lambda}|A|\vec{\lambda}'\rangle\psi(\vec{\lambda}')d\mu(\vec{\lambda}')$$

Now suppose that we have two integrable systems $\hat{H}_1, \ldots, \hat{H}_n$ and $\hat{H}_1^D, \ldots, \hat{H}_n^D$ on the same Hilbert space $\mathcal{H}$. We can use the first one to identify the Hilbert space with the space of functions on its spectrum $L^2(\Lambda)$ and write down the operators of the second integrable system as acting on these functions and not on some abstract Hilbert space vectors. Consider the function $\langle\vec{\lambda}|\vec{\lambda}^D\rangle \in L^2(\Lambda) \otimes L^2(\Lambda^D) = L^2(\Lambda \times \Lambda^D)$, where $|\vec{\lambda}\rangle$ and $|\vec{\lambda^D}\rangle$ are the eigenvectors of the first and the second integrable system respectively. This function by definition satisfy for any $k = 1, \ldots, m$ the equations:

$$\begin{aligned} &\int_\Lambda \langle\vec{\lambda}|\hat{H}_i^D|\vec{\lambda}'\rangle\langle\vec{\lambda}'|\vec{\lambda}^D\rangle d\mu(\vec{\lambda}') = e_i^D(\lambda^D)\langle\vec{\lambda}|\vec{\lambda}^D\rangle \\ &\int_{\Lambda^D} \langle\vec{\lambda}|\vec{\lambda}^{D\prime}\rangle\langle\vec{\lambda}^{D\prime}|\hat{H}_i|\vec{\lambda}^D\rangle d\mu(\lambda^{D\prime}) = e_i(\lambda)\langle\vec{\lambda}|\vec{\lambda}^D\rangle \end{aligned} \tag{2.8}$$

We see that the function $\langle\vec{\lambda}|\vec{\lambda}^D\rangle$ turns out to be an eigenfunction for two commuting set of operators acting on two groups of variables. Otherwise the function $\langle\vec{\lambda}|\vec{\lambda}^D\rangle$ is not uniquely defined by the two dual integrable systems since the arbitrary change of the normalizations of the eigenvectors $|\vec{\lambda}\rangle \mapsto F(\vec{\lambda})|\vec{\lambda}\rangle$ and $|\vec{\lambda}^D\rangle \mapsto F^D(\vec{\lambda}^D)|\vec{\lambda}^D\rangle$ results in

$$\langle\vec{\lambda}|\vec{\lambda}^D\rangle \mapsto \overline{F(\vec{\lambda})}F^D(\vec{\lambda}^D)\langle\vec{\lambda}|\vec{\lambda}^D\rangle$$

Note also that though we have written the equations (2.8) as integral equation with the kernel being a generalized function for particular dual systems these equations may be differential or difference ones. (In fact it happens in examples considered in the sequel.)

Analogously to the classical case a pair of quantum integrable systems $\{H_i\}$ and $\{H_i^D\}$ is called self-dual if there exists a unitary involution $\sigma : \mathcal{H} \to \mathcal{H}$ exchanging the two collections of operators, i.e. such that for any $k = 1, \ldots, m$

$$\sigma \cdot H_k = H_k^D \cdot \sigma$$

Though the two integrable systems of a self-dual pair are completely equivalent, the corresponding function $\langle \vec{\lambda} | \vec{\lambda}^D \rangle$ does not necessarily satisfy the condition $\langle \vec{\lambda} | \vec{\lambda}^D \rangle = \overline{\langle \vec{\lambda}^D | \vec{\lambda} \rangle}$. One can make it obey this equation after a suitable normalization of the eigenvector bases.

In some cases it is natural to choose λ among the action variables of the classical integrable system.

3. Examples of dual systems. One degree of freedom:

In this section we work out explicitly a few examples of the dual systems.

3.1. Classical systems

Two-particle systems which we are going to consider reduce (after exclusion of the center of mass motion) to a one-dimensional problem. The action-angle variables can be written explicitly and the dual system emerges immediately once the natural Hamiltonians are chosen. The problem is the following. Suppose the phase space is coordinatized by (p, q). The dual Hamiltonian (in the sense of AC duality) is a function of q expressed in terms of I, φ, where I, φ are the action-angle variables of the original system : $H_D(I, \varphi) = H_D(q)$. In all the cases below there is a natural choice of $H_D(q)$.

Calogero oscillator. The Hamiltonian in the center of mass frame reads as:

$$H(p, q) = \frac{p^2}{2} + \frac{\omega^2 q^2}{2} + \frac{\nu^2}{2q^2} \tag{3.1}$$

where ω and ν are the parameters. In the limiting cases $\nu = 0$ and $\omega = 0$ one gets the usual oscillator and rational Calogero-Moser system respectively. The action-angle variables I, φ can be found by the standard procedure:

$$\begin{aligned} I = \frac{1}{2\pi} \oint p dq = \frac{1}{2\pi} \oint \sqrt{2(E - \frac{\omega^2 q^2}{2} - \frac{\nu^2}{2q^2})} dq \\ d\varphi = \frac{dq}{p} \Big(\frac{\partial I}{\partial E} \Big)_I^{-1} \end{aligned} \tag{3.2}$$

with the result:

$$\begin{aligned} I = \frac{E-\omega\nu}{2\omega} &= \frac{1}{4\omega}\left[p^2+(\omega q-\frac{\nu}{q})^2\right] \\ H_D(I,\varphi) = \frac{q^2}{2} &= \frac{I}{\omega}\left[1+\frac{\nu}{2I}+\sqrt{1+\frac{\nu}{I}}\cos\varphi\right] \end{aligned} \tag{3.3}$$

The limit $\nu \to 0$ is straightforward, yet tricky. We must rescale $\varphi \to 2\varphi$ sinc e the period of motion jumps as ν approaches zero. We get:

$$\begin{aligned} I = \frac{E}{2\omega} \\ H_D(I,\varphi) = q = 2\sqrt{I}\cos(\varphi). \end{aligned} \tag{3.4}$$

The limit $\omega \to 0$ is more subtle as the classical motion become infinite. For the system with the Hamiltonian

$$H(p,q) = \frac{p^2}{2} + \frac{\nu^2}{2q^2} \tag{3.5}$$

the action variable could be defined as the asymptotic value of the momentum: $I = \sqrt{2E}$. This choice gives rise to the evolution, linear in the "angle"-like variable,

$$\begin{aligned} \varphi &= \sqrt{q^2 - \frac{\nu^2}{2E}} \\ H_D(I,\varphi) &= \frac{q^2}{2} = \frac{\varphi^2}{2} + \frac{\nu^2}{2I^2} \end{aligned} \tag{3.6}$$

<u>Sutherland model.</u> The Hamiltonian is:

$$H(p,q) = \frac{1}{2}p^2 + \frac{\nu^2}{2\sin^2(q)}. \tag{3.7}$$

The action variable I can be chosen to be:

$$I = \sqrt{2E} \tag{3.8}$$

To prove that one might go to the coordinate $t = \cos(q)$ and compute the integral $\frac{1}{2\pi}\oint pdq$ by residues. The angle variable φ can be determined from the condition $dp \wedge dq = d\tilde{I} \wedge d\varphi$. We get:

$$d\varphi = \frac{Idq}{\sqrt{I^2 - \frac{2\nu^2}{\sin^2(q)}}} \tag{3.9}$$

$$H_D(I,\varphi) = \cos(q) = \cos\varphi\sqrt{1-\frac{2\nu^2}{I^2}} \tag{3.10}$$

Notice, that (3.10) coincides with the Hamiltonian of the rational Ruijsenaars model (see below).

Elliptic Calogero – Moser system. The Hamiltonian is:

$$H(p,q)=\frac{p^2}{2}+\nu^2\wp_\tau(q) \tag{3.11}$$

Here p,q are complex, $\wp_\tau(q)$ is the Weierstrass function on the elliptic curve E_τ:

$$\wp_\tau(q)=\frac{1}{q^2}+\sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m,n)\neq(0,0)}}\frac{1}{(q+m\pi+n\tau\pi)^2}-\frac{1}{(m\pi+n\tau\pi)^2} \tag{3.12}$$

Let us introduce the Weierstrass notations: $x=\wp_\tau(q)$, $y=\wp_\tau(q)'$. We have an equation defining the curve E_τ:

$$y^2=4x^3-g_2(\tau)x-g_3(\tau)=4\prod_{i=1}^{3}(x-e_i),\quad \sum_{i=1}^{3}e_i=0 \tag{3.13}$$

The holomorphic differential dq on E_τ equals $dq=dx/y$. Introduce the variable $e_0=2E/\nu^2$. The action variable is one of the periods of the differential $\frac{pdq}{2\pi}$ on the curve $E=H(p,q)$:

$$I=\frac{1}{2\pi}\oint_A\sqrt{2(E-\nu^2\wp_\tau(q))}=\frac{1}{4\pi i}\oint_A\frac{dx\sqrt{x-e_0}}{\sqrt{(x-e_1)(x-e_2)(x-e_3)}} \tag{3.14}$$

The angle variable can be determined from the condition $dp\wedge dq=dI\wedge d\varphi$:

$$d\varphi=\frac{1}{2iT(E)}\frac{dx}{\sqrt{\prod_{i=0}^{3}(x-e_i)}} \tag{3.15}$$

where $T(E)$ normalizes $d\varphi$ in such a way that the A period of $d\varphi$ is equal to 2π:

$$T(E)=\frac{1}{4\pi i}\oint_A\frac{dx}{\sqrt{\prod_{i=0}^{3}(x-e_i)}} \tag{3.16}$$

Thus:

$$\begin{aligned}2iT(E)d\varphi&=\frac{dx}{\sqrt{4\prod_{i=0}^{3}(x-e_i)}}\\ \omega d\varphi&=\frac{dt}{\sqrt{4\prod_{i=1}^{3}(t-t_i)}}\end{aligned} \tag{3.17}$$

where

$$\omega = -2iT(E)\sqrt{e_{01}e_{02}e_{03}} = \frac{1}{2\pi}\oint_A \frac{dt}{\sqrt{4\prod_{i=1}^3 (t-t_i)}}$$

$$t = \frac{1}{x-e_0} + \frac{1}{3}\sum_{i=1}^3 \frac{1}{e_{0i}} \quad t_i = \frac{1}{3}\sum_{j=1}^3 \frac{e_{ji}}{e_{0i}e_{0j}} \tag{3.18}$$

$$e_{ij} = e_i - e_j$$

Introduce a meromorphic function on E_τ:

$$\widehat{cn}_\tau(z) = \sqrt{\frac{x-e_1}{x-e_3}} \tag{3.19}$$

where z has periods 2π and $2\pi\tau$. It is an elliptic analogue of the cosine (in fact, up to a rescaling of z it coincides with the Jacobi elliptic cosine). Then we have:

$$H_D(I,\varphi) = \widehat{cn}_\tau(z) = \widehat{cn}_{\tau_E}(\varphi)\sqrt{1 - \frac{\nu^2 e_{13}}{2E - \nu^2 e_3}} \tag{3.20}$$

where τ_E is the modular parameter of the relevant spectral curve $v^2 = 4\prod_{i=1}^3 (t-t_i)$:

$$\tau_E = \Big(\oint_B \frac{dt}{\sqrt{4\prod_{i=1}^3 (t-t_i)}}\Big)\Big/\Big(\oint_A \frac{dt}{\sqrt{4\prod_{i=1}^3 (t-t_i)}}\Big). \tag{3.21}$$

For large I, $2E(I) \sim I^2$

Elliptic Ruijsenaars model. The Hamiltonian is:

$$H(p,q) = \cos(\beta p)\sqrt{1 - 2(\beta\nu)^2 \wp_\tau(q)}. \tag{3.22}$$

As the curve E_τ degenerates one flows down to the trigonometric ($\wp_\tau(q) \to \frac{1}{\sin^2(q)}$) or rational ($\wp_\tau(q) \to \frac{1}{q^2}$) Ruijsenaars system. The spectral curve $H(p,q) = E$ helps to define the action variable I:

$$I = \frac{1}{2\pi}\oint_A pdq, \tag{3.23}$$

up to the transformations $I \to n_1 I^D + n_2 I + \frac{2\pi}{\beta} n_3$ where $n_1, n_2, n_3 \in \mathbb{Z}$ and $(n_1, n_2) = 1$ (choice of a cycle. The appearence of n_3 twas used in [39]). We can write an explicit formula for the quantity which is better defined:

$$\frac{\partial I}{\partial E} = \frac{1}{2\sqrt{2}\pi\beta^2\nu}\oint_A \frac{dx}{\sqrt{\prod_{i=0}^3 (x-e_i)}} \tag{3.24}$$

where now $e_0 = \frac{1-E^2}{2(\beta\nu)^2}$. Under $A \to B$ transformation $\frac{\partial I}{\partial E}$ gets multiplied by τ_E, where τ_E is defined as in (3.21) . Quite similarly to (3.17) we get:

$$d\varphi = \frac{1}{T(E)} \frac{dx}{\sqrt{\prod_{i=0}^{3}(x - e_i)}} \tag{3.25}$$

with

$$T(E) = \frac{1}{2\pi} \oint_A \frac{dx}{\sqrt{\prod_{i=0}^{3}(x - e_i)}} \tag{3.26}$$

Finally, for H_D given by (3.19) we get:

$$H_D(I, \varphi) = \widehat{cn}_\tau(z) = \widehat{cn}_{\tau_E}(\varphi)\sqrt{1 - \frac{2(\beta\nu)^2 e_{13}}{E(I)^2 - 1 - 2(\beta\nu)^2 e_3}} \tag{3.27}$$

Asymptotically, for large I, $E(I) \sim \cos(\beta I)$.

General elliptic model. In the general case one modifies the formula (3.13) in such a way that the coefficients g_2 and g_3 are the sections of the line bundles $\mathcal{O}(4n)$ and $\mathcal{O}(6n)$ respectively over $\mathbb{B} \approx \mathbb{P}^1$. The elliptic curve E_z defined by the modified (3.13) degenerates over the divisor of zeroes of its discriminant:

$$\Delta = g_2^3 - 27g_3^2 \tag{3.28}$$

which is a section of $\mathcal{O}(12n)$. The latter has generically $12n$ zeroes. To make the total space of fibration isomorphic to the $K3$ surface (compact simply-connected symplectic surface) we need 24 singular fibers, which fixes $n = 2$. The Hamiltonian of the integrable system we consider is any function on $\mathbb{B}$. It gives rise to a meromorphic vector field on M, which linearizes along the elliptic fibers. The symplectic form is given by:

$$\omega = \frac{dx \wedge dz}{y} \tag{3.29}$$

where z is the projective coordinate on $\mathbb{B}$. Under change of the variables: $\tilde{z} = \frac{1}{z}$, $\tilde{y} = -\frac{y}{z^6}$, $\tilde{x} = \frac{x}{z^4}$ the form (3.29) goes over to $\frac{d\tilde{x} \wedge d\tilde{z}}{\tilde{y}}$ and the equation (3.13) is mapped to

$$\tilde{y}^2 = 4\tilde{x}^3 - \tilde{g}_2(\tilde{z})\tilde{x} - \tilde{g}_3(\tilde{z})$$

where the polynomials $\tilde{g}_k$ are defined through the relation:

$$\tilde{g}_k(\tilde{z}) = \tilde{z}^{4k} g_k(1/\tilde{z}).$$

Over a simply connected region $U \subset \mathbb{P}^1 \backslash \Delta^{-1}(0)$ one can trivialize the bundle of the first homologies $\mathbb{H}_1(E_z, \mathbb{Z})$, in particular to make a well-defined choice of the A-cycle of the ellitpic fibre. The local action variable $I = I(z)$ is defined over U by the equation:

$$\frac{dI(z)}{dz} = T(z) = \frac{1}{2\pi} \oint_A \frac{dx}{y} \tag{3.30}$$

where the integral is taken over a chosen A-cycle. The fibration of $\mathbb{H}_1(E_z, \mathbb{Z})$ over $\mathbb{B} \backslash \Delta^{-1}(0)$ is non-trivial and there is no global monodromy invariant choice of A-cycles. So the action variable is defined by (3.30) only locally. The monodromies around the degenerate fibers corresponding to various singularities has been worked out by Kodaira and their physical interpretation can be found in [40]. For generic polynomials $g_2(z), g_3(z)$ the singularities are of the type A_1.

The angle variable dual to $I(z)$ is nothing but the linear coordinate on the Jacobian of the fiber elliptic curve (3.13). In particular it is periodic with the periods 2π and $2\pi\tau(z)$. It is to be found from the relation:

$$d\varphi = \frac{1}{T(z)} \frac{dx}{y}. \tag{3.31}$$

We can get a dual system by treating x as the Hamiltonian. Since x is not a meromorphic function on $K3$ (it changes under the $z \to \frac{1}{z}$ transformation) this is only possible if we delete the elliptic fiber E_∞.

Let us see what will be the action-angle variables. First of all, generically the fiber C_x over $x \in \mathbb{P}^1$ is an incomplete hyperelliptic curve of genus 5. The holomorphic differentials on this curve are:

$$\omega_k = \frac{z^k dz}{\sqrt{4x^3 - g_2(z)x - g_3(z)}}, \qquad k = 0, \ldots 4$$

The action variable $I^D = I^D(x)$ obeys the equation:

$$\frac{dI^D}{dx} = T^D(x) = \frac{1}{2\pi} \oint_L \frac{dz}{y} \tag{3.32}$$

where L is a one-cycle in $\mathbb{H}_1(C_x, \mathbb{Z})$. Here we face AA duality in its extreme form: the freedom to choose L is much bigger then in the case of original system, since the corresponding duality group is $\mathrm{Sp}_{10}(\mathbb{Z})$. We can partially integrate (3.32) to get

$$I^D = \frac{1}{2\pi} \oint_L \varphi T(z) dz \tag{3.33}$$

where

$$x = \frac{1}{T(z)^2}\wp(\varphi; \tau(z))$$

The angle variable is one of the linear coordinates on the Jacobian variety of C_x, which is 5 dimensional abelian variety:

$$d\varphi^D = \frac{1}{T^D(x)}\frac{dz}{y}$$

The embedding of the Liouville tori into the abelian varieties of higher rank originating from hyperelliptic curves is a well-known phenomenon in the theory of integrable systems, going back to the original work of S. Novikov and A. Veselov [41].

3.2. Quantum systems.

Here we work out a few examples of quantum dual systems.

Harmonic oscillator. The Hamiltonian (3.1) in the limit $\nu = 0$ quantizes to:

$$\hat{H} = -\frac{1}{2}\frac{\partial^2}{\partial q^2} + \frac{\omega^2 q^2}{2} \tag{3.34}$$

Its normalized eigen-functions are [42]:

$$\begin{aligned}\hat{H}\psi_n =& \omega(n+\frac{1}{2})\psi_n \\ \psi_n(q) = (\frac{\omega}{\pi})^{1/4} & \frac{e^{-\frac{\omega q^2}{2}}}{2^{n/2}\sqrt{n!}}H_n(q\sqrt{\omega})\end{aligned} \tag{3.35}$$

where $H_n(\xi)$ is the Hermite polynomial: $H_n(\xi) = e^{\xi^2}(-\partial_\xi)^n e^{-\xi^2}$. Using this representation of the wave-function one can easily obtain a reccurence relation (details are in the appendix):

$$\sqrt{n+1}\psi_{n+1}(q) + \sqrt{n}\psi_{n-1}(x) = \sqrt{2\omega}q\psi_n(x) \tag{3.36}$$

It means that $\psi_n(q)$ is an eigen-function of the following difference operator:

$$\hat{H}_D = T_+\sqrt{n} + \sqrt{n}T_-, \quad T_\pm = e^{\pm\frac{\partial}{\partial n}} \tag{3.37}$$

acting on the subscript n. It is easy to recognize in (3.37) the quantized version of (3.4).

Sutherland model. Here we deal with the Hamiltonian:

$$\hat{H} = -\frac{1}{2}\frac{\partial^2}{\partial q^2} + \frac{\nu(\nu-1)}{2\sin^2(q)} \tag{3.38}$$

Its normalized eigen-functions are [42]:

$$\hat{H}\psi_n = \frac{n^2}{2}\psi_n$$
$$\psi_n(q) = \sin^\nu(q)\sqrt{n\frac{(n-\nu)!}{(n+\nu-1)!}}\Pi^{\nu-\frac{1}{2}}_{n-\frac{1}{2}}(\cos(q)) \tag{3.39}$$
$$\Pi^m_l(x) = \frac{1}{l!}\partial^{l+m}_x(\frac{x^2-1}{2})^l$$

For simplicity we take ν and n to be half-integers. One can change $\nu \to -\nu - 1$ to get another eigen-function with the same eigenvalue. Using the fact that the generating function for Π^0_l's is

$$Z(y,x) = \sum_{l=0}^{\infty} y^l \Pi^0_l = \frac{1}{\sqrt{1-2xy+y^2}} \tag{3.40}$$

one derives the recurrence relations (details are in the appendix):

$$x\Pi^m_l = \frac{l+1-m}{2l+1}\Pi^m_{l+1} + \frac{l+m}{2l+1}\Pi^m_{l-1}$$
$$\cos(q)\psi_n = \frac{1}{2}\Big(\sqrt{\frac{1-\nu(\nu-1)}{n(n+1)}}\psi_{n+1} + \sqrt{\frac{1-\nu(\nu-1)}{n(n-1)}}\psi_{n-1}\Big) \tag{3.41}$$

that is ψ_n is an eigen-function of the finite-difference operator acting on the n subscript:

$$\hat{H}_D\psi(q) = \cos(q)\psi(q)$$
$$\hat{H}_D = T_+\sqrt{1-\frac{\nu(\nu-1)}{n(n-1)}} + \sqrt{1-\frac{\nu(\nu-1)}{n(n-1)}}T_- \tag{3.42}$$

which is a quantum version of (3.10).

Moral of the story. The moral of the previous discussion is that the polynomial dependence on momenta of the hamiltonian is traded with the rational potential of the dual system. The trigonometric potential is mapped to the trigonometric (= relativistic) dependence on momenta of dual system. The elliptic potential gives rise to elliptic (="double-relativistic") dependence on momentum of the dual system Hamiltonian. When the system with trigonometric dependence on momentum is quantized its Hamiltonian becomes a finite- difference operator. The wave-functions become the functions of the discrete variables. The origin of this is in the Bohr-Sommerfeld quantization condition. Indeed, since the trigonometric dependence of momenta implies that the leaves of the polarization are compact and moreover non-simply connected the covariantly constant sections of the prequantization connection along the polarization fiber generically seases to exist. It is only for special "quantized" values of the action variables that the section exists. In the elliptic case the quantum dual Hamiltonian is going to be a difference operator of inifinite order. The self-dual elliptic many-body system is still to be constructed. It seems that to achieve this goal one needs a notion of the Heisenberg double for the central extension of the two dimensional current group [43].

Example of the prepotential. To illustrate the meaning of the AA duality we look at the two-body system, relevant for the $SU(2)$ $\mathcal{N}=2$ supersymmetric gauge theory [16]:

$$H = \frac{p^2}{2} + \Lambda^2 \cos(q) \tag{3.43}$$

with Λ^2 being a complex number - the coupling constant of a two-body problem and at the same time a dynamically generated scale of the gauge theory. The action variable is given by one of the periods of the differential pdq. Let us introduce more notations: $x = \cos(q)$, $y = \frac{p\sin(q)}{\sqrt{-2\Lambda}}$, $u = \frac{H}{\Lambda^2}$. Then the spectral curve, associated to the system (3.43) which is also a level set of the Hamiltonian can be written as follows:

$$y^2 = (x-u)(x^2-1) \tag{3.44}$$

which is exactly Seiberg-Witten curve [12] as it was first observed in [16]. The periods are:

$$\begin{aligned} I &= \int_{-1}^{1} \sqrt{\frac{x-u}{x^2-1}}dx, \\ I^D &= \int_{1}^{u} \sqrt{\frac{x-u}{x^2-1}}dx \end{aligned} \tag{3.45}$$

They obey Picard-Fuchs equation:

$$\left(\frac{d^2}{du^2} + \frac{1}{4(u^2-1)}\right)\begin{pmatrix} I \\ I^D \end{pmatrix} = 0$$

which can be used to write down an asymptotic expansion of the action variable near $u=\infty$ or $u=\pm 1$ as well as that of prepotential (2.6). The AA duality is manifested in the fact that near $u=\infty$ (which corresponds to the high energy scattering in the two-body problem and also a perturbative regime of $SU(2)$ gauge theory) the appropriate action variable is I (it experiences a monodromy $I \to -I$ as u goes around ∞), while near $u=1$ (which corresponds to the dynamics of the two-body system near the top of the potential and to the strongly coupled $SU(2)$ gauge theory) the appropriate variable is I^D

(which corresponds to a weakly coupled magnetic $U(1)$ gauge theory and is actually well defined near $u=1$ point) [12]. The monodromy invariant combination of the periods:

$$II^D - 2\mathcal{F} = u \tag{3.46}$$

(whose origin is in the periods of Calabi-Yau manifolds on the one hand and in the properties of anomaly in t heory on the other) can be chosen as a global coordinate on the space of integrals of motion **B**. At $u \to \infty$ the prepotential has an expansion of the form:

$$\mathcal{F} \sim \frac{1}{2} u \log u + \ldots \sim I^2 \log I + \sum_n \frac{f_n}{n} I^{2-4n}$$

3.3. Appendix.

To derive the recurrence relation for the oscillator wave-functions we use the creation operator representation: $\psi_n = \frac{1}{\sqrt{2n}}(-\partial_\xi + \xi)\psi_n$. Applying this relation twice and using the fact that ψ_n is an eigen-function of $\hat{H}$ one arrives at (3.36). For the Sutherland model we use two obvious relations:

$$(x-y)\partial_x Z = y\partial_y Z \tag{3.47}$$

$$(1-2xy+y^2)\partial_y Z = (x-y)Z \tag{3.48}$$

Next, (3.47) implies:

$$(y\partial_y - m)\partial_x^m Z = (x-y)\partial_x^{m+1} Z \tag{3.49}$$

and (3.48) yields:

$$\big((1-2xy+y^2)\partial_y + y - x\big)\partial_x^m Z = m(1+2y\partial_y)\partial_x^{m-1} Z \tag{3.50}$$

Combination of those two gives rise to (3.41).

4. Duality in Many-Body Systems:

In the previous sections we discussed the concepts of duality and worked out explicitly several examples of dual two-body systems in both classical and quantum cases. We now turn to a study of many-body systems. The many-body systems can be divided into three classes: rational, trigonometric and elliptic one. The Hamiltonian of the model may depend on momenta/coordinates in any one of these three fashions. The duality transformation exchanges them.

4.1. Examples.

We summarize the systems and their duals in the following table:

$$\begin{array}{ccccc} & \text{rat.CM} & \leftrightarrow & \text{rat.CM} & \\ R\to 0 & \uparrow & & \uparrow & \beta\to 0 \\ & \text{trig.CM} & \leftrightarrow & \text{rat.RS} & \\ \beta\to 0 & \uparrow & & \uparrow & R\to 0 \\ & \text{trig.RS} & \leftrightarrow & \text{trig.RS} & \end{array} \tag{4.1}$$

Here CM denotes *Calogero-Moser* models [44][45][46] and RS stands for *Ruijsenaars-Schneider* [47][48][49][50][51]. The parameters R and β here are the radius of the circle the coordinates of the particles take values in and the inverse speed of light respectively. The horizontal arrows in this table are the dualities, relating the systems on the both sides. Most of them were discussed by Simon Ruijsenaars [51],[48]. We notice that the duality transformations form a group which in the case of self-dual systems listed here contains $\mathrm{SL}_2(\mathbb{Z})$. The generator S is the gorizontal arrow described below, while the T generator is in fact a certain finite time evolution of the original system (which is always a symplectomorphism, which maps the integrable system to the dual one). We begin with recalling the Hamiltonians of these systems. Throughout this section q_{ij} denotes $q_i - q_j$.

Rational CM model. The phase space is $(T^*V)/\Gamma$, where V is a line ar space acted on by a Coxeter group Γ. We consider the simplest case $V = \mathbb{R}^N$, $\Gamma = \mathcal{S}_{N+1}$. Let (p_i, q_i) be the set of coordinates, $i = 1, \ldots, N+1$ with the constraint $\sum q_i = \sum p_i = 0$. The Hamiltonians can be conveniently packaged using the Lax operator:

$$\begin{aligned} H_k &= \frac{1}{k}\mathrm{Tr}L^k \\ L_{ij} &= p_i\delta_{ij} + \frac{i\nu(1-\delta_{ij})}{q_i - q_j} \end{aligned} \tag{4.2}$$

In particular, the quadratic Hamiltonian reads:

$$H_2 = \sum_i \frac{1}{2} p_i^2 + \sum_{i<j} \frac{\nu^2}{q_{ij}^2}. \tag{4.3}$$

Trigonometric CM = Sutherland model. The phase space is $(T^*V)/\hat{\Gamma}$, where V is a linear space acted on by an affine Coxeter group $\hat{\Gamma}$. We consider the simplest case $V = \mathbb{R}^{N-1}$, $\hat{\Gamma} = \mathcal{S}_N \times \frac{2\pi}{R}\mathbb{Z}^N$. Let (p_i, q_i) be the set of coordinates, $i = 1, \ldots, N$ with the constraint $\sum q_i = \sum p_i = 0$, and the identifications $q_i \sim q_i + \frac{2\pi}{R} n_i, n_i \in \mathbb{Z}$. The Hamiltonians can be conveniently packaged using the Lax operator:

$$\begin{aligned} H_k &= \frac{1}{k} \mathrm{Tr} L^k \\ L_{ij} &= p_i \delta_{ij} + \frac{iR\nu(1-\delta_{ij})}{2\sin(\frac{R(q_i - q_j)}{2})} \end{aligned} \tag{4.4}$$

In particular, the quadratic Hamiltonian equals:

$$H_2 = \sum_i p_i^2 + \sum_{i<j} \frac{R^2 \nu^2}{4\sin^2(\frac{R(q_i-q_j)}{2})}. \tag{4.5}$$

Rational RS = Relativistic rational CM model. The phase space is $(T^*V)/\hat{\Gamma}$, where V is a linear space acted on by an affine Coxeter group $\hat{\Gamma}$. We consider the simplest case $V = \mathbb{R}^{N-1}$, $\hat{\Gamma} = \mathcal{S}_N \times \frac{2\pi}{\beta}\mathbb{Z}^N$. Let (p_i, q_i) be the set of coordinates, $i = 1, \ldots, N$ with the constraint $\sum q_i = \sum p_i = 0$, and the identifications $p_i \sim p_i + \frac{2\pi}{\beta} n_i, n_i \in \mathbb{Z}$. The Ha miltonians can be conveniently packaged using the Lax operator:

$$\begin{aligned} H_k &= \frac{1}{k} \mathrm{Tr} L^k \\ L_{ij} &= e^{-i\beta p_i} \frac{\beta\nu}{q_{ij} + \beta\nu} \prod_{k \neq j} \sqrt{1 - \frac{(\beta\nu)^2}{q_{jk}^2}} \end{aligned} \tag{4.6}$$

In particular, the Hamiltonian $\frac{1}{2}(H_1 - H_{-1})$ equals:

$$H = \frac{1}{2} \mathrm{Tr}(L + L^{-1}) = \sum_i \cos(\beta p_i) \prod_{j \neq i} \sqrt{1 - \frac{(\beta\nu)^2}{q_{ij}^2}}. \tag{4.7}$$

The Lax operator (4.6) is gauge equivalent to the operator

$$\begin{aligned} \mathcal{L}_{ij} &= e^{-i\beta p_i} \frac{\beta\nu}{q_{ij} + \beta\nu} \sqrt{\phi_i^+ \phi_j^-} \\ \phi_i^\pm &= \pm \frac{\Pi(q_i \pm \beta\nu)}{\beta\nu \Pi'(q_i)}, \quad \Pi(q) = \prod_i (q - q_i) \end{aligned} \tag{4.8}$$

In the limit $\beta \to 0$ both $L, \mathcal{L}$ of (4.6),(4.8) behave as $\mathrm{Id} - i\beta$ (Lax operator in (4.2)) $+ o(\beta)$.

Trigonometric RS = Relativistic Sutherland model. The phase space is $(T^*V)/\Gamma_E$, where V is a linear space acted on by a double affine Coxeter group Γ_E, E being an elliptic curve. We consider the simplest case $V = \mathbb{R}^{N-1}$, $\Gamma = \mathcal{S}_N \times \left(\frac{2\pi}{\beta}\mathbb{Z}^N \oplus \frac{2\pi}{R}\mathbb{Z}^N\right)$. Let (p_i, q_i) be the set of coordinates, $i = 1, \ldots, N$ with the constraint $\sum q_i = \sum p_i = 0$, and the identifications $p_i \sim p_i + \frac{2\pi}{\beta} n_i, \quad q_i \sim q_i + \frac{2\pi}{R} m_i, \qquad n_i, m_i \in \mathbb{Z}$. The Hamiltonians can be conveniently packaged using the Lax operator:

$$H_k = \frac{1}{k}\mathrm{Tr} L^k$$
$$L_{ij} = e^{-i\beta p_i} \frac{\sin\left(\frac{R\beta\nu}{2}\right)}{\sin\left(\frac{R}{2}(q_{ij} + \beta\nu)\right)} \prod_{k \neq j} \sqrt{1 - \frac{\sin^2\left(\frac{R\beta\nu}{2}\right)}{\sin^2\left(\frac{Rq_{jk}}{2}\right)}} \tag{4.9}$$

In particular, the Hamiltonian $\frac{1}{2}(H_1 - H_{-1})$ equals:

$$H = \frac{1}{2}\mathrm{Tr}(L + L^{-1}) = \sum_i \cos(\beta p_i) \prod_{j \neq i} \sqrt{1 - \frac{\sin^2(R\beta\nu)}{\sin^2\left(\frac{R(q_{ij})}{2}\right)}}. \tag{4.10}$$

The Lax operator (4.6) is gauge equivalent to the operator

$$\mathcal{L}_{ij} = e^{-i\beta p_i} \frac{\sin\left(\frac{NR\beta\nu}{2}\right)}{N\sin\left(\frac{R}{2}(q_{ij} + \beta\nu)\right)} \sqrt{\Phi_i^+ \Phi_j^-}$$
$$\Phi_i^{\pm} = \pm \frac{NR}{2\sin\left(\frac{NR\beta\nu}{2}\right)} \frac{P(q_i \pm \beta\nu)}{P'(q_i)}, \quad P(q) = \prod_{i=1}^{N} \sin\left(\frac{R}{2}(q - q_i)\right) \tag{4.11}$$

In the limit $R \to 0$, with β fixed the expressions (4.9),(4.10),(4.11) naturally go over to (4.6), (4.7), (4.8) respectively. In the limit $\beta \to 0$, R fixed both $L, \mathcal{L}$ behave as $\mathrm{Id} - i\beta$(Lax operator in (4.4)) $+ o(\beta)$.

4.2. Explanations: Hamiltonian and/or Poisson reduction

Suppose we are given a symplectic manifold (X, ω_X) with the Hamiltonian action of a Lie group G with equivariant moment map $\mu : X \to \underline{\mathbf{g}}^*$. The symplectic quotient of X with respect to G is the symplectic manifold M, denoted as $X//G$ and defined as:

$$M = \mu^{-1}(0)/G$$

Its symplectic form ω_M is defined through the relation:

$$p^*\omega_M = i^*\omega_X$$

where $p : \mu^{-1}(0) \to M$ is the projection and $i : \mu^{-1}(0) \to X$ is the inclusion.

Let us assume that an integrable Hamiltonian system is defined on X. Let $\bar{K} = \{K_1, \dots, K_x\}$, $x = \frac{1}{2}\mathrm{dim}X$ denote the set of its integrals of motion. Suppose that this system is equivariant with respect to the action of G. This is equivalent to the statement, that K_i and μ^a form a closed algebra $\mathcal{K}$ with respect to the Poisson brackets. Let us assume that on the zero level of the moment map μ the center $Z(\mathcal{K})$ of the algebra $\mathcal{K}$ is sufficiently big, i.e. the dimension of its spectrum equals half the dimension of M. Then the integrable system on X descends to the integrable system on M, $\mathcal{K}$ being replaced by $Z(\mathcal{K})$.

Now let us impose one further restriction. Suppose that X possesses another G-equivariant integrable Hamiltonian system, with integrals $\bar{Q} = \{Q_1, \dots, Q_x\}$, which is dual to the system $\bar{K}$ (algebraically it means that $\bar{K}$ and $\bar{Q}$ generate all functions on X). We also assume that $\bar{Q}$ descends to M.

On the original manifold X the evolution of the system $\bar{K}$ looks non-trivially in the action-angle variables for the system $\bar{Q}$ and vice versa. The same is true for the reduced systems. The advantage of the consideration of X is that the systems on X can be much simpler then those on M. In the following sections we shall consider various examples of this situation.

The similar statements hold in the case of Poisson manifolds, the relevant reduction being the Poisson one (one first takes a quotient with respect to the group and then picks out a symplectic leaf). We leave the details to the interested readers.

Now we proceed to the explicit constructions. We will discuss the models introduced in the previous section on case-by-case basis and show how the reduction which yields these systems also explains the dualities between different systems.

Rational CM model. This model can be obtained as a result of Hamiltonian reduction applied to $T^*\underline{\mathbf{g}} \times \mathcal{O}$ [52] for $\underline{\mathbf{g}} = su(N)$, $\mathcal{O} = \mathbb{CP}^{N-1}$. The symplectic form on this manifold is the sum of Liouville form on $T^*\underline{\mathbf{g}}$ and $-N\nu\times$ Fubini-Study form on $\mathcal{O}$. Let $(e_1 : \dots : e_N)$ be the homogeneous coordinates on $\mathcal{O}$. The group $G = SU(N)$ acts on $T^*\underline{\mathbf{g}}$ via conjugation and on $\mathcal{O}$ in a standard way ($\mathcal{O} = G/H$, $H = S(U(N-1) \times U(1))$). Then the moment map for the action of $\mathcal{G}$ on $T^*\underline{\mathbf{g}} \times \mathcal{O}$ is

$$\mu = ad^*_Q(P) - J \quad J_{ij} = \nu(N\delta_{ij} - e_i e^*_j) \tag{4.12}$$

where $Q \in \underline{\mathbf{g}}, P \in \underline{\mathbf{g}}^*$. Now we choose two sets of Hamiltonians:

$$H_k = \frac{1}{k}\mathrm{Tr}P^k \quad \text{and} \quad H_k^D = \frac{1}{k}\mathrm{Tr}Q^k \tag{4.13}$$

If we identify $\underline{\mathbf{g}}^*$ and $\underline{\mathbf{g}}$ with the help of Tr then the equation $\mu = 0$ has the form:

$$[P, Q] = J \tag{4.14}$$

which is obviously preserved by the involution: $P \to Q, Q \to -P$. So we are guaranteed to get a self-dual system. Now we have to find suitable coordinates and action variables. Let us choose the gauge (remember that we have to mod out (4.14) by the action of G):

$$Q = \mathrm{diag}(q_1, \ldots, q_N) \tag{4.15}$$

This gauge is preserved by the action of the maximal torus $T = U(1)^{N-1}$ which turns out to be sufficient to set all e_i to be equal: $e_i = 1$ [53]. Then the equation (4.14) fixes P which turns out to be nothing but L in (4.2). As it is obvious that the reduced symplectic form equals $\sum_i dp_i \wedge dq_i$ (with the constraint $\sum q_i = \sum p_i = 0$) one concludes that q_i's are the action variables for the system generated by H_k^D's. Therefore eigenvalues of P are the action variables for the flows generated by H_k's. We therefore proved the following

Statement. *Consider the map:*

$$\sigma : \{(p_i, q_i)\} \to \{(\xi_i, -\eta_i)\} \tag{4.16}$$

where η_i's are the eigenvalues of $L \equiv P$ and ξ_i are the diagonal entries of Q in the eigenbasis of P. It is an involution

Let us go back to the systems (4.13). The moment map equation (4.14) is obviously preserved by the transformations of the form

$$(P, Q) \mapsto (aP + bQ, cP + dQ) \quad ad - bc = 1 \tag{4.17}$$

which form $SL_2(\mathbb{R})$ group. The transformed Hamiltonians

$$g \cdot H_k = \frac{1}{k}\mathrm{Tr}(aP + bQ)^k$$

are easy to express through the original Hamiltonians (4.13) in the coordinates (p_i, q_i):

$$g \cdot H_k(p_i, q_i)|_\nu = H_k(ap_i + bq_i, q_i)|_{a\nu}$$

Let us restrict our attention to the $SL_2(\mathbb{Z})$ subgroup of the group (4.17). It is generated by the transformations

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{4.18}$$

It is clear that S coincides with the involution leading to (4.16) while T is the unit time evolution with respect to the Hamiltonian H_2^D.

Trigonometric CM, Rational RS. The trigonometric CM system can be obtained as Hamiltonian reduction applied to either $T^*G \times \mathcal{O}$ [52] or $T^*\underline{\hat{\mathbf{g}}} \times \mathcal{O}$ [54] where $\underline{\hat{\mathbf{g}}}$ is the central extension of the loop algebra. In the latter case one has to specify the action of the gauge group LG on the orbit $\mathcal{O}$. The correct choice is the most natural one: since the orbit is finite-dimensional, the only sensible way the loop group can act on it is through the evaluation at some point. The elements of $T^*\underline{\hat{\mathbf{g}}}$ of our interest are the pairs:

$$P(x), k\partial_x + Q(x) \tag{4.19}$$

where k is a fixed number, $P(x)$ is a $\underline{\mathbf{g}}$-valued function on a circle $\mathbf{S}^1$ and $Q(x)$ is a gauge field on a circle. The phase space is acted on by the gauge group:

$$P(x) \mapsto g(x)^{-1}P(x)g(x), \quad Q(x) \mapsto g(x)^{-1}Q(x)g(x) + kg^{-1}(x)\partial_x g(x) \tag{4.20}$$

The moment equation has the form:

$$k\partial_x P + [Q, P] = J\delta(x) \tag{4.21}$$

where J is the one from (4.12). The number k can be rescaled by the choice of the radius of a circle $\mathbf{S^1}$. Instead we choose the circle of unit radius and keep k. To solve the equation (4.21) we fix a gauge (4.20). We can either decide that Q is a constant diagonal matrix

$$Q = \mathrm{diag}(q_1, \dots, q_N)$$

and then the solution for $P(x)$ will produce the Lax operator (4.4) of the Sutherland model with $R = \frac{2\pi}{k}$ [54],[55].

It is quite amusing that the same reduction yields the rational RS model as well. In order to see that choose the gauge

$$P(x) = \mathrm{diag}\left(p_1\left(x\right), \dots, p_N\left(x\right)\right) \tag{4.22}$$

Then the moment equation (4.21) implies that $Q(x)$ is diagonal everywhere except $x=0$ where it has an off-diagonal part proportional to the delta-function. At the same time $P(x)$ is forced to be constant $p_i(x) = q_i$.

$$Q(x)_{ij} = \theta_i(x)\delta_{ij} + \delta(x)\frac{i\nu}{q_i - q_j} \tag{4.23}$$

The natural candidate for a Hamiltonian in this setting would be a gauge invariant function of $Q(x)$. Since $Q(x)$ is actually a gauge field the gauge invariant function is a trace in some representation R of a Wilson loop:

$$H_R = \mathrm{Tr}_R P \exp \oint \frac{1}{k} Q(x) dx \tag{4.24}$$

which is easy to evaluate provided we assume the following structure of the diagonal piece of $Q(x)$ (which supported by the alternative derivation of the solution to the moment equation below):

$$\theta_i(x) = \varphi_i(x) + \delta(x) \sum_{k \neq i} \frac{i\nu}{q_k - q_i}$$

which makes the Wilson loop

$$B = P \exp \oint \frac{1}{k} Q(x) dx = \mathrm{diag}\left(e^{\frac{1}{k}\oint \varphi_i(x) dx}\right) \exp\left(\frac{i\nu}{k} \mathrm{r}\right) \tag{4.25}$$

with r being the matrix:

$$\mathrm{r}_{ij} = \frac{1}{q_{ij}}, \quad i \neq j, \quad \mathrm{r}_{ii} = -\sum_{j \neq i} \mathrm{r}_{ij} \tag{4.26}$$

It is shown in the Appendix (in the trigonometric case from which this one follows as well) that the matrix B is gauge equivalent to (4.8) with the identification $\beta = \frac{1}{k}$, and (cf. (4.8)):

$$p_i + \frac{1}{2i\beta} \log(-\phi_i^+ / \phi_i^-) = \oint \varphi_i(x) dx.$$

One can also get the same matrix (without the assumptions like (4.23)) B by performing a reduction of T^*G under the adjoint action of G at the same level J of the moment map:

$$\mu = B^{-1} P B - P = J$$

Trigonometric RS = Relativistic trigonometric CM model. There are three different approaches, all leading to the same finite-dimensional Hamitlonian system. There are two

Hamiltonian reductions and one Poisson reduction. The advantage of Hamiltonian one is the simplicity and geometric clarity. The advantage of Poisson one is finite-dimensionality at each step and considerable simplicity of the proof of the canonical commutation relations. We try to outline all three approaches with the emphasis on the Poisson reduction, as the relevant Hamiltonian reduction was described in some details in [55]. We keep in mind a sequence of contractions:

$$\mathcal{A}_{\mathbf{T}^2} \to T^*\hat{G} \to G \times G \tag{4.27}$$

where the first entry is the space of G-valued gauge fields on a two-torus $\mathbf{T}^2$, the second entry is the cotangent bundle to the central extension of the loop group LG and the last one is the space of lattice (for the simplest graph, representing a two-torus) connections, described below.

Hamiltonian approach. Consider the space $\mathcal{A}_{\mathbf{T}^2}$ of $SU(N)$ gauge fields A on a two-torus $\mathbf{T}^2 = \mathbf{S}^1 \times \mathbf{S}^1$. Let the circumferences of the circles be R and β. The space $\mathcal{A}_{\mathbf{T}^2}$ is acted on by a gauge group $\mathcal{G}$, which preserves a symplectic form

$$\Omega = \frac{k}{4\pi^2} \int \mathrm{Tr} \delta A \wedge \delta A, \tag{4.28}$$

with k being an arbitrary real number for now. The gauge group acts via evaluation at some point $p \in \mathbf{T}^2$ on any coadjoint orbit $\mathcal{O}$ of G, in particular, on $\mathcal{O} = \mathbb{CP}^{N-1}$. Let $\mathbb{CP}^{N-1}$ have a $-N\nu\times$ Fubini-Study symplectic form. Let $(e_1 : \ldots : e_N)$ be the homogeneous coordinates on $\mathcal{O}$. Then the moment map for the action of $\mathcal{G}$ on $\mathcal{A}_{\mathbf{T}^2} \times \mathcal{O}$ is

$$kF_A + J\delta^2(p), \quad J_{ij} = i\nu(\delta_{ij} - e_i e_j^*) \tag{4.29}$$

F_A being the curvature two-form. Here we think of e_i as being the coordinates on $\mathbb{C}^N$ constrained so that $\sum_i |e_i|^2 = N$ and considered up to the multiplication by a common phase factor.

Let us provide a certain amount of commuting Hamiltonians. Obviously, the eigenvalues of the monodromy of A along any fixed loop on $\mathbf{T}^2$ commute with themselves. We consider the reduction at the zero level of the moment map. We have at least $N-1$ functionally independent commuting functions on the reduced phase space $\mathcal{M}_\nu$.

Let us estimate the dimension of $\mathcal{M}_\nu$. If $\nu = 0$ then the moment equation forces the connection to be flat and therefore its gauge orbits are parameterized by the conjugacy classes of the monodromies around two non-contractible cycles on $\mathbf{T}^2$: A and B. Since the fundamental group $\pi_1(\mathbf{T}^2)$ of $\mathbf{T}^2$ is abelian A and B are to commute. Hence they are simultaneously diagonalizable, which makes $\mathcal{M}_0$ a $2(N-1)$ dimensional manifold. Notice that the generic point on the quotient space has a non-trivial stabilizer, isomorphic to the maximal torus T of $SU(N)$. Now, in the presence of $\mathcal{O}$ the moment equation implies that the connection A is flat outside of p and has a non-trivial monodromy around p. Thus:

$$ABA^{-1}B^{-1} = \exp(R\beta J) \tag{4.30}$$

(the factor $R\beta$ comes from the normalization of the delta-function in (4.29)). If we diagonalize A, then B is uniquely reconstructed up to the right multiplication by the elements of T. The potential degrees of freedom in J are "eaten" up by the former stabilizer T of a flat connection: if we conjugate both A and B by an element $t \in T$ then J gets conjugated. Now, it is important that $\mathcal{O}$ has dimension $2(N-1)$. The reduction of $\mathcal{O}$ with respect to T consists of a point and does not contribute to the dimension of $\mathcal{M}_\nu$. Thereby we expect to get an integrable system. Without doing any computations we already know that we get a pair of dual systems. Indeed, we may choose as the set of coordinates the eigen-values of A or the eigen-values e the action variables for the system generated by $\mathrm{Tr}B^k$.

The two-dimensional picture has the advantage that the geometry of the problem suggest the $SL_2(\mathbb{Z})$-like duality. Consider the operations S and T realized as:

$$S : (A, B) \mapsto (ABA^{-1}, A^{-1}); \quad T : (A, B) \mapsto (A, BA) \tag{4.31}$$

which correspond to the freedom of choice of generators in the fundamental group of a two-torus. Notice that both S and T preserve the commutator $ABA^{-1}B^{-1}$ and commute with the action of the gauge group. The group Γ generated by S and T (it is a subrogup of the group $\mathbf{OutFree}(2)$ of the outer authomorphismes of the free group with two generators) seems to be larger then $\mathrm{SL}_2(\mathbb{Z})$. However in the limit $\beta, R \to 0$ it contracts to $\mathrm{SL}_2(\mathbb{Z})$ in a sense that we get the transformations (4.18) by expanding

$$A = 1 + \beta P + \ldots, \quad B = 1 + RQ + \ldots$$

for $R, \beta \to 0$.

The disadvantage of the two-dimensional picture us the necessity to keep to many redundant degrees of freedom. The first of the contractions (4.27) actually allows to replace the space of two dimensional gauge fields by the cotangent space to the (central extension of) loop group:

$$T^*\hat{G} = \{(g(x), k\partial_x + P(x))\}$$

which is a "deformation" of the phase space of the previous example ($Q(x)$ got promoted to a group-valued field). The relation to the two dimensional construction is the following. Choose a non-contractible circle $\mathbf{S^1}$ on the two-torus which does not pass through the marked point p. Let x, y be the coordinates on the torus and $y = 0$ is the equation of the $\mathbf{S^1}$. The periodicity of x is β and that of y is R. Then

$$P(x) = A_x(x, 0), g(x) = P\exp\int_0^R A_y(x, y)dy.$$

The gauge transformations on $\mathbf{S^1}$ transform on $(g(x), P(x))$ is a way, similar to (4.20). The moment map equation (4.29) goes over to the moment map equation [55]:

$$kg^{-1}\partial_x g + g^{-1}Pg - P = J\delta(x), \tag{4.32}$$

with $k = \frac{1}{R\beta}$. The solution of this equation in the gauge $P = \mathrm{diag}(q_1, \ldots, q_N)$ leads to the Lax operator $A = g(0)$ of the form (4.11) with R, β exchanged [55]. On the other hand, if we follow (4.22) and diagonalize $g(x)$:

$$g(x) = \mathrm{diag}\left(z_1 = e^{iRq_1}, \ldots, z_N = e^{iRq_N}\right) \tag{4.33}$$

then a similar calculation leads to the Lax operator

$$B = P\exp\oint\frac{1}{k}P(x)dx = \mathrm{diag}(e^{i\theta_i})\exp iR\beta\nu\mathrm{r}$$

with

$$\mathrm{r}_{ij} = \frac{1}{1 - e^{iRq_{ji}}}, i \neq j; \quad \mathrm{r}_{ii} = -\sum_{j\neq i}\mathrm{r}_{ij}$$

thereby establishing the duality $A \leftrightarrow B$ explicitly.

Poisson description. Here we introduce a set of commuting functions on the space of graph connection on a graph, corresponding to a moduli space of flat connections on a torus with one hole and describe the flow generated by this set. Being reduced to a particular symplectic leaf of the moduli space of flat connections on the torus , this set of functions turns out to be a full set of commuting Hamiltonians. We introduce another full set of commuting variables and write down the Hamiltonians taking the latter set as a set of coordinates thus recovering the Ruijsenaars integrable system. Consider a graph, consisting of two edges and one vertex with the fat graph structure corresponding to a punctured torus [56]. The space of graph connections $\mathcal{A}^L$ for such graph is just a product of two copies of the group G: $\mathcal{A}^L = G \times G = \{(A,B)|A,B \in G\}$, where A and B are assigned to the edges of the graph. For a choice of ciliation on $\mathcal{A}^L$ the Poisson bracket on $\mathcal{A}^L$ is given by the relations, following from the general rules [56].

$$\begin{aligned}
\{A \overset{\otimes}{,} A\} &= r^a A \otimes A + A \otimes A r^a - 2(A \otimes 1) r^a (1 \otimes A) \\
\{B \overset{\otimes}{,} B\} &= r^a B \otimes B + B \otimes B r^a - 2(B \otimes 1) r^a (1 \otimes B) \\
\{A \overset{\otimes}{,} B\} &= r(A \otimes B) + A \otimes B r + (1 \otimes B) r_{21} (A \otimes 1) - (A \otimes 1) r (1 \otimes B),
\end{aligned} \tag{4.34}$$

where $r^a = \frac{1}{2}(r - r_{21})$.

Now let us restrict ourselves to the case $G = \mathrm{SL}_N$ and the standard r-matrix:

$$r = \sum_{\alpha>0} E_\alpha \otimes E_{-\alpha} + \frac{1}{2} \sum_i H_i \otimes H_i, \quad r^a = \frac{1}{2} \sum_{\alpha>0} E_\alpha \wedge E_{-\alpha} \tag{4.35}$$

In this case one can easily derive the following commutaion relations

$$\{\mathrm{Tr} A^n, A\} = 0 \quad \{\mathrm{Tr} B^n, B\} = 0 \tag{4.36}$$

$$\{\mathrm{Tr} A^n, B\} = n(A^n)_0 \quad \{\mathrm{Tr} B^n, A\} = nA(B^n)_0 \tag{4.37}$$

where $(X)_0$ denotes the traceless part of the matrix X. Therefore, the functions $\mathrm{Tr} B^n$ for $n = 1 \ldots N-1$ considered as Hamiltonians generate commuting flows on $\mathcal{A}^L$.

$$\begin{aligned}
B(t_1, \ldots, t_{N-1}) &= B(0, \ldots, 0) \\
A(t_1, \ldots, t_{N-1}) &= A(0, \ldots, 0)\, e^{\left(t_1 B + \cdots + t_{N-1} B^{N-1}\right)_0}
\end{aligned} \tag{4.38}$$

As it was shown in [56] the lattice gauge group $\mathcal{G}^L$ acts on $\mathcal{G}^L$ in a Poisson way, and the quotient Poisson manifold coincides with the moduli space $\mathcal{M}$ of smooth flat connection on the Riemann surface, corresponding to the fat graph L. In our case the group $\mathcal{G}^L$ is G itself (for the graph has just one vertex) which acts on A and B by simultaneous conjugation.

$$g : (A, B) \mapsto (gAg^{-1}, gBg^{-1}). \tag{4.39}$$

The functions $\mathrm{Tr} A^k$ and $\mathrm{Tr} B^k$ are invariant under this action, and therefore their pull-downs on the moduli space $\mathcal{M}$ generate commuting flows there, which trajectories are just projections of (4.38).

However the moduli space $\mathcal{M}$ in our case is a Poisson manifold with degenerate Poisson bracket. The Casimir functions of this Poisson structure are the functions of conjugacy classes of monodromies around holes and constant value levels of such functions are just the symplectic leaves of $\mathcal{M}$. In our case such Casimir functions are $\mathrm{Tr}\big(ABA^{-1}B^{-1}\big)^k$, pulled down to $\mathcal{M}$.

Different symplectic leaves have different dimensions and the lowest dimension of them is $2(N-1)$. These leaves correspond to the monodromy around the hole conjugated to a matrix

$$e^{-iR\beta\nu}\mathrm{Id} + P,$$

$\mathrm{rk}P \le 1$, ν is a numerical constant from the previous section parameterizing the set of symplectic leaves of lowest dimension. Let $t = e^{-iR\beta\nu}$. On the leaf $\mathcal{M}_\nu$ the family of functions $\mathrm{Tr}A^k, k = 1, \dots, N-1$ forms a full set of Poisson-commuting variables.

Introduce local coordinates on these symplectic leaves in the following way. Let $z_1 = e^{iRq_1}, \dots, z_N = e^{iRq_N}$ be the eigenvalues of the operator A and $\mu_1, \dots, \mu_N$ are the corresponding diagonal matrix elements of B (in the basis, diagonalizing A). One can check that in this basis

$$B^i_j = \sqrt{\mu_i\mu_j}\frac{(1-t)}{z_i/z_j - t}. \tag{4.40}$$

The functions z_i and μ_j are well-defined locally on the symplectic leaf $\mathcal{M}_\nu$. Their Poisson brackets are equal to:

$$\begin{aligned}
&\{z_i, z_j\} = 0 \\
&\{\mu_i, \mu_j\} = \mu_i\mu_j\frac{(z_i + z_j)}{(z_i/z_j - t)(z_j/z_i - t)(z_i - z_j)} \quad i \ne j \\
&\{z_i, \mu_j\} = z_i\mu_j\delta_{i,j}.
\end{aligned} \tag{4.41}$$

To define the variables, canonically conjugated to z_i we can just multiply μ_i by factors independent on μ_i. For example one can take:

$$s_i = \mu_i t^{\frac{N-1}{2}} \prod_{k,k\ne i} \sqrt{\frac{(z_k - z_i)(z_i - z_k)}{(z_k - tz_i)(z_i - tz_k)}} \tag{4.42}$$

One can check, that these new variables s_i have the Poisson brackets

$$\{s_i, s_j\} = 0 \quad \{z_i, s_j\} = z_i s_j \delta_{i,j}. \tag{4.43}$$

Substituting this back to the formula (4.40) we get:

$$B_{ij} = \frac{1-t}{z_i/z_j - t}\left(\Phi^+_i\Phi^-_i\Phi^+_j\Phi^-_j\right)^{1/4} \tag{4.44}$$

which is gauge equivalent to (4.11).

Moral revisited. We have seen in all the previous examples that the origin of the dual system is connected with the existence of transversal G-invariant foliations on the original space, which become Lagrangian foliations when pulled down to the quotient. The simplicity of the operating with dual systems in the advocated framework in the classical case allows one to hope that the duality can systematically elevated to the quantum case as well. See [57][34].

4.3. Appendix. Computation of the Poisson brackets

The bivector defining the Poisson structure on $\mathcal{A}^L$ can be rewritten in the form

$$\pi = \frac{1}{2}\sum_{i,j,u,v} E_j^{i(u)} \otimes E_i^{j(v)}(\epsilon(u,v)+\epsilon(i,j)), \tag{4.45}$$

where $\epsilon(i,j)$ is $-1, 0$ or 1 depending on whether i is less, equal or greater than j respectively and $E_j^{i(u)}$ are the standard GL_N generators acting on the u-th end of the edge. (In our case $E_j^{i(1)}$ acts on A from the left, $E_j^{i(2)}$ acts on B from the left, $E_j^{i(3)}$ acts on A from the right and $E_j^{i(4)}$ acts on B from the right.)

It is not convenient to compute the Poisson brackets between z_i and μ_j using this bivector directly, for it does not preserve the diagonality of the matrix A. However, to compute the Poisson brackets of the gauge invariant functions we are allowed to add to this bivector any term vanishing on such functions i.e. any terms of the type $\sum_u E_j^{i(u)} \otimes X$ or $X \otimes \sum_u E_j^{i(u)}$, where X is an arbitrary vector field. Using this and also the fact that for the diagonal A[1]:

$$\begin{aligned} E_j^{i(1)} &= \frac{z_i}{z_j - z_i}(E_j^{i(2)} + E_j^{i(4)}) \\ E_j^{i(3)} &= \frac{z_j}{z_i - z_j}(E_j^{i(2)} + E_j^{i(4)}) \end{aligned} \tag{4.46}$$

the bivector π can be transformed to the form:

$$\pi' = \sum_{i>j} E_j^{i(2)} \wedge E_i^{j(4)} \frac{z_i + z_j}{2(z_i - z_j)} + \frac{1}{2}\sum_i E_i^{i(2)} \wedge E_i^{i(1)} + E_i^{i(3)} \wedge E_i^{i(4)} \tag{4.47}$$

Applying this bivector for the chosen A and B we get the desired Poisson brackets. ♭

---

[1] These equations are the infinitesimal forms of the statements that (i) up to the gauge transformation the conjugation of A by g is equivalent to the conjugation of B by g^{-1} thanks to (4.39) ;
(ii) $g_L A = A^{-1}(A g_R^{-1})A$, with $g_L = g_R^{-1}$

4.4. Appendix. Solution of the moment equation

Here we solve the equation (4.30):

$$
\begin{aligned}
&A^{-1}BAB^{-1} = \exp R\beta J \\
&J = -i\nu(\mathrm{Id} - e\otimes e^{\dagger}), \quad \langle e^{\dagger}, e\rangle = N
\end{aligned}
\tag{4.48}
$$

with A, B - $N\times N$ unitary matrices defined up to the gauge transformations (4.39). We use the notation: $\alpha = R\beta\nu$. We partially fix a gauge:

$$
A = \mathrm{diag}\left(e^{iRq_1}, \ldots, e^{iRq_N}\right) \tag{4.49}
$$

which leaves gauge transformations of the form

$$
h = \exp\left(i\,\mathrm{diag}(l_1, \ldots, l_N)\right). \tag{4.50}
$$

which preserve A, conjugate B and map e to $h^{-1}e$. The exponent $\exp R\beta J$ is easy to compute:

$$
\exp J = e^{-i\alpha}\left(\mathrm{Id} + \frac{e^{iN\alpha}-1}{N}e\otimes e^{\dagger}\right)
$$

Let $f = B^{-1}e$, $z_i = e^{iRq_i}$, $\Phi_i^+ := |e_i|^2$, $\Phi_i^- := |f_i|^2$. Then:

$$
\begin{aligned}
&B_{ij} = e^{-i\alpha}\frac{e^{iN\alpha}-1}{N}\frac{e_i f_j^*}{e^{iRq_{ji}} - e^{-i\alpha}} \\
&f = B^{-1}e \Rightarrow \frac{Ne^{i\alpha}}{e^{iN\alpha}-1} = \sum_{i=1}^{N}\frac{z_i\Phi_i^+}{z_j - e^{-i\alpha}z_i}
\end{aligned}
\tag{4.51}
$$

The last equation implies (see below):

$$
\Phi_i^+ = \frac{N}{e^{-iN\alpha}-1}\frac{P(e^{-i\alpha}z_i)}{z_iP'(z_i)}, \quad P(z) = \prod_{i=1}^{N}(z - z_i) \tag{4.52}
$$

Now the unitarity of B implies, that

$$
\delta_{ik} = f_i f_k^* \frac{e^{iN\alpha}-1}{N}\sum_{j=1}\frac{P(e^{-i\alpha}z_j)}{z_jP'(z_j)(z_j/z_i - e^{i\alpha})(z_k/z_j - e^{-i\alpha})} \tag{4.53}
$$

Hence

$$
\Phi_i^- = \frac{N}{e^{iN\alpha}-1}\frac{P(e^{i\alpha}z_i)}{z_iP'(z_i)} \tag{4.54}
$$

To prove (4.52) consider the contour integral

$$\frac{1}{2\pi i}\oint_{\Gamma}\frac{P(e^{-i\alpha}z)dz}{P(z)(z-e^{i\alpha}z_j)}=\mathrm{Res}_\infty=e^{-iN\alpha}$$

To prove (4.54) consider the integral:

$$\frac{1}{2\pi i}\oint_{\Gamma}\frac{P(e^{-i\alpha}z)dz}{P(z)(z-e^{i\alpha}z_i)(z_k-e^{-i\alpha}z)}=\delta_{ik}\mathrm{Res}_{e^{i\alpha}z_k}=\delta_{ik}\frac{P'(z_k)}{P(e^{i\alpha}z_k)}$$

In both cases the contour Γ surrounds the roots of $P(z)$.

Notice that both $\Phi_i^{\pm}$ are real:

$$\Phi_i^{\pm}=\frac{N\sin(\alpha/2)}{\sin(N\alpha/2)}\prod_{j\neq i}\frac{\sin\left(\frac{Rq_{ij}\pm\alpha}{2}\right)}{\sin\left(\frac{Rq_{ij}}{2}\right)} \tag{4.55}$$

Substituting this back to (4.51) we get:

$$B_{ij}=e^{i((1-N)\alpha/2+Rq_{ij}/2+\varepsilon_i-\varphi_j)}\frac{\sin\left(\frac{N\alpha}{2}\right)}{N\sin\left(\frac{Rq_{ij}+\alpha}{2}\right)}\sqrt{\Phi_i^+\Phi_j^-} \tag{4.56}$$

where $e_i =: |e_i|e^{i\varepsilon_i}$, $f_j =: |f_j|e^{i\varphi_j}$. The gauge transformations (4.50) allow us to set $\varphi_i+Rq_i/2=0$. Then define

$$p_i=-\frac{1}{\beta}\left((1-N)\alpha/2+Rq_i/2+\varepsilon_i\right) \tag{4.57}$$

Finally, the matrix B can also be written as:

$$B=\left(\Phi^-\right)^{-\frac{1}{2}}\left(e^{-i\beta\tilde{p}}e^{-i\alpha\mathrm{r}}\right)\left(\Phi^-\right)^{\frac{1}{2}} \tag{4.58}$$

where $\Phi^-=\mathrm{diag}(\Phi_i^-)$, $\tilde{p}=\mathrm{diag}(\tilde{p}_i)$,

$$\tilde{p}_i=p_i-\frac{\alpha+\pi}{2\beta}+\frac{i}{2\beta}\log\left(\frac{\Phi_i^-}{\Phi_i^+}\right) \tag{4.59}$$

and

$$\begin{aligned}\mathrm{r}_{ij}&=\frac{z_i}{z_i-z_j},\quad i\neq j\\ \mathrm{r}_{ii}&=\frac{1}{2}\frac{z_iP''(z_i)}{P'(z_i)}\end{aligned} \tag{4.60}$$

To prove the last statement consider the matrix

$$R_{ij}(\alpha) = \frac{\sin\left(\frac{N\alpha}{2}\right)}{N\sin\left(\frac{Rq_{ij}+\alpha}{2}\right)}\phi_i^+ = e^{\frac{i(N-1)\alpha}{2}}\sqrt{\frac{z_j}{z_i}}\frac{P(e^{-i\alpha}z_i)}{(e^{-i\alpha}z_i - z_j)P'(z_i)}$$

We have:

$$B = \left(\Phi^-\right)^{-\frac{1}{2}}\left(e^{-i\beta\tilde{p}}R(\alpha)\right)\left(\Phi^-\right)^{\frac{1}{2}}$$

A simple contour integral calculation shows that

$$R(\alpha_1)R(\alpha_2) = R(\alpha_1 + \alpha_2)$$

The rest follows by expanding near $\alpha = 0$. If one performs an expansion near $R = 0$ one gets the statement that the rational Lax operator (4.8) is conjugated to the operator of the form announced in (4.25).

It is amusing that the expression $\frac{1}{2i\beta}\log(\Phi_i^+/\Phi_i^-)$ appears quite often in Bethe Ansatz Equations for XXX magnets and their field theoretic limits [58].

5. Gauge theories and duality in integrable systems

5.1. Old approach: many-body systems as low-dimensional gauge theories

It is a fruitful approach to think of the many-body system as of the gauge theory of a certain kind. Namely, the particles of the model can be identified (sometimes) with the eigenvalues of the Wilson loops in the theory and the gauge dynamics becomes a dynamics of the particles. Of course, in the real four dimensional world the gauge field has infinitely many degrees of freedom and we don't expect to see any tractable quantum mechanical system unless we have a principle which allows us to restrict the dynamical problem to a finite number of degrees of freedom. The simplest case is the case of low-dimensional gauge theory, where the gauge field simply doesn't have propagating degrees of freedom. Consider, for example, two dimensional Yang-Mills theory with a gauge group $G = U(N)$. When formulated on a circle of radius $\mathbf{R}$ in the Hamiltonian formalism the theory has as a phase space the space of gauge fields $A(x)$ on the circle and their duals - chromoelectric fields $E(x)$. The gauge group acts on (E, A) as follows:

$$(E, A) \to (g^{-1}Eg, g^{-1}\partial_x g + g^{-1}Ag) \tag{5.1}$$

leading to the Gauss law $\partial_x E + [A, E]$, which is nothing but the moment map from the section 4.2. We can go to the gauge where A is a constant (w.r.t. x) diagonal matrix

$$A = \mathrm{diag}(q_1, \dots, q_N) \tag{5.2}$$

Here are our particles. The time evolution makes q_i to move and depending on the circumstances such as the presence of the sources like J (which correspond to the time-like Wilson lines) one gets the Hamiltionian system of the kind we described and studied. The large gauge transformations shift q_i's by integer multiples of $\frac{1}{\mathbf{R}}$ making them live on a circle of radius $\frac{1}{2\pi\mathbf{R}}$. One can get more complicated examples by deforming the model as follows. Replace $\mathbf{S^1}$ by $\mathbf{T^2}$, E by the second component of the gauge field along the torus, the symplectic form being $\int_{\mathbf{T^2}} \mathrm{Tr}\delta A \wedge \delta A$. Then the Gauss law becomes $F_A = \partial_x A_y - \partial_y A_x + [A_x, A_y]$. Setting it to zero allows to diagonalize A_x, A_y simultaneously:

$$\begin{pmatrix} A_x \\ A_y \end{pmatrix} = \mathrm{diag}\left(\begin{pmatrix} p_1 \\ q_1 \end{pmatrix}, \dots, \begin{pmatrix} p_N \\ q_N \end{pmatrix} \right) \tag{5.3}$$

Here, x_i and y_i do not Poisson-commute, although both live on circle. One gets, therefore a system of relativistic partcles on a circle. The radius of the circle is $\frac{1}{2\pi\mathbf{R}_x}$, the speed of light is $\mathbf{R}_y$. The gauge theory this model corresponds to is known as Chern-Simons theory on a torus (perhaps with punctures).

One can go higher in dimensions with some care. For example, by considering supersymmetric $\mathcal{N}=2$ theory in $d=4$ with compact space M^3 one gets a quantum mechanics on the moduli space of monopoles in $\mathbb{R}^3$.

5.2. *New approach: many-body systems in supersymmetric gauge theories*

Recent progress in the understanding of non-perturbative phenomena emerged after the work of Seiberg and Witten on four dimensional $\mathcal{N}=2$ SYM [12] and works of Seiberg and his collaborators on $\mathcal{N}=1$ $d=4$ theories. The major tool in these studies is the low energy effective Lagrangian which is constrained by two priniciples - the holomorphy of chiral objects and electric-magnetic duality. It is the electric-magnetic duality which makes the integrable systems to appear in the solutions to the gauge theories.

In particular, one can argue on the general grounds [18] that any $\mathcal{N}=2$ supersymmetric gauge theory in four dimensions corresponds to a certain integrable system in the holomorphic sense. The point is that the Coulomb branch of the theory parameterizes the family of abelian varieties (whose period matrix coincides with the matrix of coupling constants of the effective low-energy abelian theory). Moreover the total space must carry a holomorphic symplectic form ω, whose integral along the cycle in the fiber gives rise to a derivative of the central charge of a BPS representation of $\mathcal{N}=2$ susy algebra along the base. Moreover the abelian varieties must be Lagrangian with respect to ω.

The integrable systems corresponding to a large number of field theories are identified. In particular, the low-energy theory of the pure $N=2$ $SU(N_c)$ SYM is governed by the A_{N_c-1} periodic (or affine) Toda system. The $\mathcal{N}=2$ theory with a massive adjoint hypermultiplet corresponds to the elliptic Calogero-Moser system, where the mass (which is naturally a complex parameter in the $\mathcal{N}=2$ theory) is identified with the coupling constant. The theory is UV finite (in fact, it is softly broken $\mathcal{N}=4$ theory) and therefore has as another modulus – the ultra-violet coupling τ which enters the integrable model as the modulus of the curve. Another theories which were mentioned so far are the relativistic generalizations of those two. These correspond to five dimensional gauge theories with the same number of supercharges, compactified on a circle of a finite radius R. The speed of light of the relativistic model is proportional to the inverse radius $\frac{1}{R}$ of the circle. For the theories with fundamental matter the firm identification with the integrable systems has been made in four [28][59] as well as in five and six dimensions [59].

In some cases the dualities suggested by the integrable systems are not obvious on the field theory side. We plan to return to more detailed treatment of these cases (which involve six dimensional theories) in the future.

5.3. Dualities in field theories vs. dualities in many-body systems

Dualities in the old approach. Let us start with the two-dimensional Yang-Mills theory with the gauge coupling g^2 formulated on a Riemann surface of area A. It was shown by E. Witten in [60] that the perturbative in g^2A part of the correlation functions in this (non-supersymmetric) theory coincides with the correlation functions of certain observables in twisted $\mathcal{N}=2$ supersymmetric two-dimensional Yang-Mills theory.

Among the twisted supercharges of the latter theory one finds a scalar Q which annihilates the complex scalar ϕ in the vector multiplet. The observables constructed out of the gauge invariant functions of ϕ and their descendants can be mapped to certain observables in non-supersymmetric theory.

As we discussed above, when Yang-Mills theory is formulated on a cylinder with the insertion of an appropriate time-like Wilson line, it is equivalent to the Sutherland model describing a collection of N particles on a circle. The observables $\mathrm{Tr}\phi^k$ of the previous paragraph are precisely the integrals of motion of this system.

One can look at other supercharges as well. In particular, when the theory is formulated on a cylinder there is another class of observables annihilated by a supercharge. One can arrange the combination of supercharges which will annihilate the Wilson loop operator. By repeating the procedure similar to the one in [60] one arrives at the quantum mechanical theory whose Hamiltonians are generated by the spatial Wilson loops. This model is nothing but the rational Ruijsenaars-Schneider many-body system.

The duality between these two systems is a consequence of the fact that when lifted to the supersymmetric model both field theories become equivalent to the same $\mathcal{N}=2$ super-Yang-Mills theory in two dimensions.

The self-duality of trigonometric Ruijsenaars system has even more transparent physical meaning. Namely, the field theory whose quantum mechanical avatar is the Ruijsenaars system is three dimensional Chern-Simons theory on $\mathbf{T}^2\times\mathbf{R}^1$ with the insertion of an appropriate temporal Wilson line and spatial Wilson loop. It is the freedom to place the latter which leads to several equivalent theories. The group of (self-)dualities of this model is very big and is generated by the transformations S and T (4.31).

In short, the duality reveals here itself as a consequence of Lorentz invariance of the underlying field theory.

Duality in the new approach. The new approach deals with supersymmetric gauge theories in three, four, five and six dimensions. Perhaps the richest case is the six dimensional theory compactified on a three dimensional torus $\mathbf{T}^3$ down to three dimensions.

As was discussed extensively in [15] in case where two out of three radii of $\mathbf{T}^3$ are much smaller then the third one $\mathbf{R}$ the effective three dimensional theory is a sigma model with the target space $\mathcal{X}$ being the hyper-kahler manifold (in particular, holomorphic symplectic) which is a total space of algebraic integrable system. The complex structure in which $\mathcal{X}$ is the algebraic integrable system is independent of the radius $\mathbf{R}$ while the Kähler structure depends on $\mathbf{R}$ in such a way that the Kähler class of the abelian fiber is proportional to $1/\mathbf{R}$.

The duality of the integrable systems shows up in the gauge theories in the several ways.

First of all AA duality the well-known phenomenon in the four dimensional $\mathcal{N} = 2$ gauge theory which was observed and exploited in [12] and then later on in the plenty of works. The low-energy effective theory has different sets of relevant degrees of freedom over different regions of the moduli space of vacua. The transformations between different descriptions go through the electric-magnetic duality on the gauge field side which is accompanied by supersymmetry by a AA-type duality on the scalar side. Although this duality is connected with the electric-magnetic symmetry which is not realized geometrically in four dimensions, it does become geometric when the theory is lifted to a tensor theory in six dimensions [61].

The duality of the AC type is also present and is rather interesting. As one varies the moduli of $\mathbf{T}^3$ the geometry of $\mathcal{X}$ varies as well. In particular, different four dimensional theories can flow to the same three dimensional theory. This is where the AC duality in the integrable systems shows up.

For example, a certain scaling limit of the five dimensional $SU(N)$ theory with massive adjoint hypermultiplet, compactified on a circle seems to be equivalent/dual to four dimensional $SU(N)$ theory with massive adjoint hypermultiplet when instanton corrections are turned off in both theories.

The theory in three dimensions which came from four dimensions upon a compactification on a circle whose low-energy effective action describes only abelian degrees of freedom can be always dualized to the theory of scalars/spinors only, due to the vector-scalar duality in three dimensions. In this way different sets of vector and hypermultiplets in four dimensions can lead to the same three dimensional theory (one of the examples of such symmetries is provided by the three dimensional mirror symmetry [62][63]).

One can also use the AC duality to establish the following fact. Consider two AC dual integrable systems. Consider three dimensional $\mathcal{N} = 4$ supersymmetric gauge theory whose Higgs branch is $\mathcal{X}$ - the phase space of these systems. Then two $\mathcal{N} = 2$ supersymmetric theories whose spaces of scalars are both $\mathcal{X}$ but the superpotentials are taken from the sets of Hamiltonians of the first and the second systems respectively are dual to each other in the sense that both flow in the UV/IR (depending on the whether these Hamiltonians correspond to the relevant or irrelevant operators) to the same theory.

We are certain that there are more applications of the notion of duality in integrable systems both in the theory of integrability itself and in the physics, gauge theories being the arena for the most immediate ones.

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