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ON ASYMPTOTIC ISOPERIMETRIC CONSTANT OF TORI

D. Burago and S. Ivanov

In this note we continue study of asymptotic invariants of Riemannian tori. By asymptotic invariants we mean invariants which do not change under passing to finite covers. In [BuI2] we show that the asymptotic volume growth of a Riemannian torus is at least as fast as that of a flat one. One may ask what are the possible values of "asymptotic isoperimetric constants" for such metrics (see definition below). We show that the asymptotic isoperimetric constant of a conformally flat torus is no less than that of a flat one, while for general metrics (in dimensions higher than 2) this constant may be arbitrarily small.

Let (M,g) be a universal cover of a Riemannian n-torus.

Definition. We define the *asymptotic isoperimetric constant* $\sigma(M,g)$ of (M,g) by

$$\sigma(M,g)=\limsup_{\mathrm{Vol}_n(\Omega)\to\infty}\frac{\mathrm{Vol}_n(\Omega,g)^{1/n}}{\mathrm{Vol}_{n-1}(\partial\Omega,g)^{1/n-1}}$$

where V_n and V_{n-1} are Riemannian measures (for g) of the respective dimensions and Ω ranges over all open bounded subsets of M.

Clearly $\sigma(M,g)$ is finite, positive, and invariant under homotheties of the metric.

We denote by σ_n the isoperimetric constant of the standard Euclidean n-space,

$$\sigma_n=\frac{m_n(D^n)^{1/n}}{m_{n-1}(S^{n-1})^{1/(n-1)}}$$

where m_n and m_{n-1} are the standard (Euclidean) measures of the respective dimensions.

Proposition. *If (M,g) is a universal cover of a conformally flat torus, then $\sigma(M,g)\geq\sigma_n$, and the equality holds if and only if the metric is flat.*

Proof. Our argument is a version of so called Loewner's length-area method (see [G1] for details). Let $M=\mathbf{R}^n$, and let the metric g be given by $g=\lambda g_E$ where g_E is the standard Euclidean metric, λ is a positive smooth function on $\mathbf{R}^n$, λ is periodic with respect to some (co-compact) lattice $\Gamma\subset\mathbf{R}^n$. Without loss of generality we may assume that the volume of $\mathbf{R}^n/\Gamma$ is equal to 1 for both metrics g and g_E, i.e.

$$\int_{\mathbf{R}^n/\Gamma}\lambda^n\,dm_n=m_n(\mathbf{R}^n/\Gamma)=1.$$

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Define $\alpha := \int_{\mathbf{R}^n/\Gamma} \lambda^{n-1}\, dm_n$. By Hölder inequality, the above identity implies that $\alpha < 1$ unless $\lambda \equiv 1$.

For $x \in \mathbf{R}^n$ and $r > 0$ denote by $B(x,r)$ the Euclidean ball of radius r centered at x. To prove that $\sigma(M,g) \geq \sigma_n$ it suffices to construct a sequence of regions Ω_i with volumes growing to infinity and

$$\lim_{i\to\infty} \frac{\mathrm{Vol}_n(\Omega_i,g)^{1/n}}{\mathrm{Vol}_{n-1}(\partial\Omega_i,g)^{1/n-1}} \geq \sigma_n.$$

We will show that such regions can be chosen among Euclidean balls $B(x,r)$.

It is clear that

$$\frac{\mathrm{Vol}(B(x,r),g)}{m_n(B(x,r))} \to 1 \qquad \text{as } r \to \infty,$$

with the convergence being uniform in x. (Indeed, both $\mathrm{Vol}(B(x,r),g)$ and $m_n(B(x,r))$ are asymptotically equal to the number of fundamental domains of Γ contained in $B(x,r)$). Define

$$A_r(x) = \frac{\mathrm{Vol}_{n-1}(\partial B(x,r),g)}{m_{n-1}(\partial B(x,r))},$$

then $A_r(x)$ is the average value of the function λ^{n-1} over the Euclidean sphere $\partial B(x,r)$. Thus the function A_r is Γ-periodic. A standard argument of Euclidean integral geometry shows that the average value of A_r (w.r.t. the standard Euclidean volume m_n) is equal to α. Therefore for every $r > 0$ there is a a point $x_0 = x_0(r) \in \mathbf{R}^n$ such that $A_r(x_0) \leq \alpha$. To complete the proof, observe that

$$\begin{aligned}\sigma(M,g) &\geq \limsup_{r\to\infty} \frac{\mathrm{Vol}_n(B(x_0(r)),g)^{1/n}}{A_r(x_0(r))^{1/(n-1)}} \\ &\geq \frac{1}{\alpha^{1/(n-1)}} \cdot \frac{m_n(B(x_0(r)))^{1/n}}{m_{n-1}(\partial B(x_0(r)))^{1/(n-1)}} = \frac{\sigma_n}{\alpha^{1/(n-1)}}. \qquad \square\end{aligned}$$

Remark. The Proposition implies, in particular, that the asymptotic isoperimetric constant of an arbitrary 2-dimensional torus is at least σ_2, since all 2-tori are conformally flat. There is a direct 2-dimensional argument, which may give better geometric insight and suggests different candidates for “optimal” regions Ω_i. We give an outline of the argument below.

In the notations of the Proposition, denote by E_i the ellipse of maximum area inscribed in the ball of radius i in the stable norm of g (see [BuI2] for definitions). The proof of the volume growth theorem [BuI2] implies that the area enclosed in E_r in g is at least $\pi i^2 + o(i^2)$. Choose $\sqrt{i}$ points on E_i such that all distances between neighboring points are equal, an let Ω_i be the interior of a geodesic polygon (w.r.t. g) with these vertices. One easily sees that $\lim_{i\to\infty} \frac{\mathrm{Vol}_2(\Omega_i)}{\mathrm{Vol}_2(E_i)} = 1$. On the other hand, the length of each side of Ω_i differs from the distance between its endpoints in the stable norm by no more than a constant c. This distance, in its turn, is no greater than the distance between these endpoints in the Euclidean metric for which E_i is a ball of radius i. Combining these inequalities, one concludes that $\mathrm{Vol}_1(\partial\Omega_i) \leq 2\pi i + c\sqrt{i}$, that completes the argument.

Remark. Reasoning as above, one easily sees that, for given asymptotic volume growth, in dimension 2 the exact value of $\sigma(M,g)$ can be recovered from the stable norm (this observation was made independently by P. Pansu.) This is no longer true in higher dimensions, as one can observe from the examples in the proof of the next theorem.

Theorem. *If $n \geq 3$, then there exists $\mathbf{Z}^n$-periodic Riemannian metrics on $\mathbf{R}^n$ with arbitrarily small asymptotic isoperimetric constants.*

Proof. Let $\mathrm{pr}_i \colon \mathbf{R}^n \to \mathbf{R}^{n-1}$ denote the projection onto the ith coordinate hyperplane. Fix n congruent open balls $U_1, \dots U_n$ in I^{n-1} (where $I = [0,1]$), such that the "tubes" $G_i := \mathrm{pr}_i^{-1}(U_i)$ in $\mathbf{R}^n$ are disjoint. This is possible since $n \geq 3$. Let $v = m_{n-1}(U_1) = \dots = m_{n-1}(U_n)$. Now pick a small $\varepsilon > 0$ and construct a $\mathbf{Z}^n$-periodic Riemannian metric g_ε on $\mathbf{R}^n$ such that

(1) The volume form determined by g_ε is equal to m_n.
(2) In every tube G_j, $j = 1, \dots, n$, the line element of g_ε is given by

$$ds^2 = \varepsilon^{2n-2} dx_j^2 + \sum_{i \neq j} \frac{dx_i^2}{\varepsilon^2}.$$

We will prove that $\sigma(\mathbf{R}^n, g_\varepsilon) \to 0$ as $\varepsilon \to 0$. Consider a domain $\Omega \subset \mathbf{R}^n$ and let $V = \mathrm{Vol}_m(\Omega, g_\varepsilon) = m_n(\Omega)$. We think of $\mathbf{R}^n$ as the union of "cells" $I^n + k$, $k \in \mathbf{Z}^n$, each cell carrying a copy of the same metric. Let Ω' denote the union of cells $I^n + k$ for which $m_n(\Omega \cap (I^n + k)) > 1 - v/2$. To complete the proof, we will derive a lower bound for $\mathrm{Vol}_{m-1}(\Omega, g_\varepsilon)$ in terms of V and ε. We argue separately for the two following cases.

Case 1: $m_n(\Omega') \geq V/2$. By a known Loomis–Whitney inequality [LW] the volume of Ω' can be estimated from above in terms of volumes of its projections:

$$\prod_{i=1}^{n} m_{n-1}(\mathrm{pr}_i(\Omega')) \geq m_n(\Omega')^{n-1}.$$

Hence there exists j, $1 \leq j \leq n$, for which $m_{n-1}(\mathrm{pr}_j(\Omega')) \geq m_n(\Omega')^{1-1/n}$. The projection $\mathrm{pr}_j(\Omega')$ is a union of $(n-1)$-dimensional cells of the form $I^{n-1} + k$, $k \in \mathbf{Z}^{n-1}$. For every cell $I^{n-1} + k \subset \mathrm{pr}_j(\Omega')$ we have

$$m_{n-1}(\mathrm{pr}_j(\Omega) \cap (I^{n-1} + k)) > 1 - v/2$$

and therefore

$$m_{n-1}(\mathrm{pr}_j(\Omega) \cap (U_j + k)) > v/2.$$

Note that $\mathrm{pr}_j(\partial\Omega) = \mathrm{pr}_j(\Omega)$ since Ω is bounded. The definition of g_ε implies that $m_{n-1}(\mathrm{pr}_j(X)) \leq \varepsilon^{n-1} \mathrm{Vol}_{n-1}(X, g_\varepsilon)$ for any set $X \subset \mathrm{pr}_j^{-1}(U_j + k)$, so

$$\mathrm{Vol}_{n-1}(\partial\Omega \cap \mathrm{pr}_j^{-1}(U_j + k), g_\varepsilon) \geq \frac{v}{2} \cdot \varepsilon^{-(n-1)}.$$

Therefore

$$\mathrm{Vol}_{n-1}(\partial\Omega, g_\varepsilon) \geq \frac{v}{2}\cdot\varepsilon^{-(n-1)}\cdot m_{n-1}(\mathrm{pr}_j(\Omega')) \geq c(n)\cdot\varepsilon^{-(n-1)}\cdot m_n(\Omega')^{1-1/n}.$$

Replacing $m_n(\Omega')$ by V in the last expression will only affect the constant.

Case 2: $m_n(\Omega') < V/2$. Consider the cells of the form I^n+k, $k\in\mathbf{Z}^n$, intersecting with Ω with $m_n(\Omega\cap(I^n+k)) \leq 1-v/2$. It is easy to check (e.g., by applying the Loomis–Whitney inequality to the set $\Omega\cap\mathrm{Int}(I^n+k)$) that, for these cells, the values

$$\frac{m_{n-1}(\partial\Omega\cap\mathrm{Int}(I^n+k))}{m_n(\Omega\cap\mathrm{Int}(I^n+k))}$$

have a positive lower bound depending only on v and the dimension n. Since the terms $m_n(\Omega\cap\mathrm{Int}(I^n+k))$ adds up to a value of at least $m_n(\Omega\setminus\Omega') \geq V/2$. This gives us the following estimation

$$\mathrm{Vol}_{n-1}(\partial\Omega, g_\varepsilon) \geq c(n,\varepsilon)\cdot V$$

for case 2.

As $V\to\infty$, the inequalities obtained for the two cases take the form

$$\limsup \frac{V^{1/n}}{\mathrm{Vol}_{n-1}(\partial\Omega, g_\varepsilon)^{1/(n-1)}} \leq c(n)\cdot\varepsilon$$

and

$$\limsup \frac{V^{1/n}}{\mathrm{Vol}_{n-1}(\partial\Omega, g_\varepsilon)^{1/(n-1)}} = 0,$$

respectively. Thus we obtain that $\sigma(\mathbf{R}^n, g_\varepsilon) \leq c(n)\cdot\varepsilon$. □

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