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Injective Modules and Amenable Groups

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Injective modules and amenable groups

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Abstract We show that a locally compact group is amenable if and only if it admits a (nontrivial) injective Banach module which is reflexive as a Banach space, generalizing work by H.G. Dales, M. Daws, H.L. Pham, P. Ramsden, and M.E. Polyakov.

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1 Introduction

Let A be a Banach algebra. By a left A-module we shall always mean a Banach left A-module satisfying $\|ax\| \leq \|a\| \, \|x\|$ whenever $a \in A$ and $x \in X$, and a morphism of left A-modules will be a bounded linear map commuting with the respective actions. X will be called injective, cf. [H, III.1.14, p. 136], if for any morphism ι of left A-modules admitting a bounded linear left inverse ℓ, and any morphism λ_0 from Y_0 into X, there is a morphism λ from Y into X satisfying $\lambda_0 = \lambda \circ \iota$,

$$\begin{array}{ccccc} Y_0 & \xrightarrow{\ \iota\ } & Y & \xrightarrow{\ \ell\ } & Y_0 \\ {\scriptstyle\lambda_0}\downarrow & \swarrow {\scriptstyle\lambda} & & & \\ X & & & & \end{array} \qquad \ell \circ \iota = id_{Y_0}, \ \lambda \circ \iota = \lambda_0.$$

Let the essential part, X_e, of a left A-module X be defined as the closed linear hull of the set of products ax, $a \in A$, $x \in X$. We shall call X nontrivial if $X_e \neq 0$, essential if $X = X_e$, and reflexive if X is reflexive as a Banach space. In case that X is reflexive and A has a bounded two-sided approximate unit (of norm $\leq c$), there is an A-module morphism (of norm $\leq c$) projecting X onto X_e. The Banach space dual, X^*, of X becomes a right A-module under the action defined by $\langle x, x^*a \rangle = \langle ax, x^* \rangle$, for $x^* \in X^*$, $a \in A$, and $x \in X$.

Let G be a locally compact group. Choosing a left invariant Haar measure on G, we obtain the Banach algebra $L^1(G)$ whose dual space will be identified with $L^\infty(G)$ by $\langle a, \varphi \rangle = \int a(s)\varphi(s)\, ds$ whenever $a \in L^1(G)$ and $\varphi \in L^\infty(G)$. If G acts on $L^1(G)$ by left translation, $(L_s a)(t) = a(s^{-1}t)$, $s \in G$, $a \in L^1(G)$, its dual action on $L^\infty(G)$ is given by $(L_s^* \varphi)(t) = \varphi(st)$, $s \in G$, $\varphi \in L^\infty(G)$. It is well known that every essential left $L^1(G)$-module is a left G-module such that, for any $x \in X$, the mapping $s \mapsto sx$ is continuous from G into X and $\|sx\| = \|x\|$, $s \in G$, the respective actions being related by the formula $ax = \int a(s)\, sx\, ds$, for $a \in L^1(G)$ and $x \in X$. This same formula defines on any such left G-module an essential left $L^1(G)$-action.

Letting G act by left translation on $L^p(G)$, $1 < p < \infty$, $L^p(G)$ becomes an essential reflexive left $L^1(G)$-module. H.G. Dales, M. Daws, H.L. Pham, and P. Ramsden recently showed the following theorem, [DDPR, Theorem 9.6].

Theorem([DDPR]). – Let G be a locally compact group, and $1 < p < \infty$. If the left $L^1(G)$-module $L^p(G)$ is injective, then G is amenable. ■

Imitating F.J. Yeadon's method, [Y], for establishing the existence of a trace in a finite von Neumann algebra, we show

Proposition. – Let G be a locally compact group. If G admits a nontrivial injective Banach left $L^1(G)$-module which is reflexive as a Banach space, then G is amenable.

Combining this with known results we obtain the following characterization of compact and amenable groups, in good correspondence with Helemskii's philosophy, cf. e.g. [H, p. 262].

Corollary. – Let G be a locally compact group.

a) If G admits a nontrivial projective left $L^1(G)$-module which is reflexive as a Banach space, then G is compact; if, conversely, G is compact then every essential left $L^1(G)$-module is projective.
b) If G admits a nontrivial flat left $L^1(G)$-module which is reflexive as a Banach space, then G is amenable; if, conversely, G is amenable then every left $L^1(G)$-module is flat.

These results are equally valid for uniformly bounded, left or right Banach $L^1(G)$-modules. For the notions of the injective tensor product, $\check{\otimes}$, of Banach spaces and integral operators to be used we refer to the monograph of J. Cigler, V. Losert, and P. Michor, [CLM]. The proof of the Proposition starts right after this Introduction.

2 Some Preparations

The $L^1(G)$-module action on $K(L^1(G), X)$ and the morphism ι described below were introduced by P. Ramsden, [Ra, Ch. 5, p. 21]; cf. also[DDPR, Ch. 9].

2.1 The $L^1(G)$-module $K(L^1(G), X)$

Let G be a locally compact group, and X an essential Banach left $L^1(G)$-module. We denote by $K(L^1(G), X)$ the Banach space of compact linear mappings from $L^1(G)$ into X. For any $s \in G$ and $T \in K(L^1(G), X)$, the operator sT, defined by $(sT)(b) = sT(L_{s^{-1}}b)$, $b \in L^1(G)$, belongs to $K(L^1(G), X)$. Since for any $b \in L^1(G)$, the function $s \mapsto (sT)(b)$ is continuous and bounded, from G into X, the integral

$$(aT)(b) = \int a(s)\ (sT)(b)\, ds \qquad\qquad (b \in L^1(G))$$

defines, for any $a \in L^1(G)$, a bounded linear operator aT from $L^1(G)$ into X, of norm $\|aT\| \leq \|a\|\ \|T\|$. To show that it is compact, we may assume a nonnegative, of integral one, and of compact support, K. But then the image of the unit ball, $OL^1(G)$, of $L^1(G)$ under aT is contained in the closed convex hull of $K \cdot T(OL^1(G))$. Since this is compact, the compactness of aT follows.

2.2 The morphism $\iota : X \longrightarrow K(L^1(G), X)$

As in [Ra, p. 21], we define an isometric linear embedding ι of X into $K(L^1(G), X)$ by $(\iota x)(b) = \langle b, 1_G\rangle\, x$, for $x \in X$ and $b \in L^1(G)$, 1_G denoting the constant function one on G. Since for any $a \in L^1(G)$ and $x \in X$, we have

$$\begin{aligned}
(a(\iota x))(b) &= \int a(s)\ s(\iota x)(b)\, ds \\
&= \int a(s)\ s(\iota x(L_{s^{-1}}b))\, ds \\
&= \int a(s)\ \langle L_{s^{-1}}b, 1_G\rangle\ sx\, ds \\
&= \langle b, 1_G\rangle \int a(s)\, sx\, ds \\
&= \langle b, 1_G\rangle\ ax \\
&= \iota(ax)(b) \qquad\qquad (b \in L^1(G)),
\end{aligned}$$

ι is a morphism: $\iota(ax) = a(\iota x)$ whenever $a \in L^1(G)$, $x \in X$. For any $b \in L^1(G)$ of integral one, the bounded linear operator $\ell : K(L^1(G), X) \longrightarrow X$, $\ell(T) = T(b)$, $T \in K(L^1(G), X)$, satisfies $\ell(\iota x) = \iota x(b) = \langle b, 1_G\rangle\ x = x$, so that ℓ is a left inverse of ι.

2.3 Let now the essential left $L^1(G)$-module X be injective. Setting $Y_0 = X$, $Y = K(L^1(G), X)$, and $\lambda_0 = id_X$ in the definition, we obtain a morphism λ,

$$X \xrightarrow{\ \iota\ } K(L^1(G), X) \xrightarrow{\ \lambda\ } X,$$

satisfying the following properties:

(i) λ is bounded and linear;
(ii) $\lambda(aT) = a(\lambda T)$;
(iii) $\lambda(\iota x) = x$,

whenever $a \in L^1(G)$, $T \in K(L^1(G),\, X)$, and $x \in X$. To show that λ commutes also with the respective G-actions, we proceed as in [L, Lemma 2, p. 354].

Proof Let us fix $s \in G$ and $T \in K(L^1(G), X)$. Setting, for $a \in L^1(G)$, $(R_s a)(t) = a(ts^{-1})\Delta(s^{-1})$, $t \in G$, Δ the Haar modulus of G, we first compute, for $a \in L^1(G)$ and $x \in X$,

$$\begin{aligned} a(sT) &= \int a(t)\; t(sT)\, dt \\ &= \int a(t)\; (ts)T\; dt \\ &= \int a(ts^{-1})\Delta(s^{-1})\; tT\; dt \\ \\ &= (R_s a)\, T, \end{aligned}$$

and

$$\begin{aligned} (R_s a)x &= \int (R_s a)(t)\; tx\, dt \\ &= \int a(ts^{-1})\Delta(s^{-1})\; tx\, dt \\ &= \int a(t)\Delta(s^{-1})\Delta(s)\; (ts)x\, dt \\ &= \int a(t)\; t(sx)\, dt \\ \\ &= a(sx). \end{aligned}$$

Since X is essential, these formulae yield, (a_i) being an approximate unit in $L^1(G)$,

$$\begin{aligned} \lambda(sT) &= \lim\; a_i \lambda(sT) \\ &= \lim\; \lambda(a_i(sT)) \\ &= \lim\; \lambda((R_s a_i)T) \\ &= \lim\; (R_s a_i)(\lambda T) \\ &= \lim\; a_i(s(\lambda T)) \\ &= s(\lambda T), \end{aligned}$$

i.e. $\lambda(sT) = s(\lambda T)$, for $s \in G$ and $T \in K(L^1(G), X)$. ∎

2.4 Since $L^\infty(G) = L^1(G)^*$ enjoys Grothendieck's approximation property, the mapping

$$L^\infty(G)\check{\otimes}X \longrightarrow K(L^1(G), X), \qquad (\varphi \otimes x)^\sim(b) = \langle b, \varphi\rangle\, x \qquad (b \in L^1(G)),$$

$\varphi\otimes x \in L^\infty(G)\check{\otimes}X$, defines an isometric linear isomorphism from the injective tensor product of $L^\infty(G)$ and X onto $K(L^1(G), X)$, cf. [CLM, 3.5. Corollary, p. 85]. Defining the G-action on $L^\infty(G)\check{\otimes}X$ by $s(\varphi \otimes x) = (L_{s^{-1}})^*\varphi \otimes sx$, for $s \in G$, $\varphi \otimes x \in L^\infty(G)\check{\otimes}X$, this isomorphism becomes a G-morphism, since for any $b \in L^1(G)$,

$$\begin{aligned}(s(\varphi \otimes x)^\sim)(b) &= s((\varphi \otimes x)^\sim(L_{s^{-1}}b)) = s\langle L_{s^{-1}}b, \varphi\rangle x\\ &= \langle b, (L_{s^{-1}})^*\varphi\rangle sx = ((L_{s^{-1}})^*\varphi \otimes sx)^\sim(b),\end{aligned}$$

i.e. $(s(\varphi \otimes x))^\sim = s(\varphi \otimes x)^\sim$ whenever $s \in G$ and $\varphi \otimes x \in L^\infty(G)\check{\otimes}X$.

2.5 Summarizing the above, we have: if the essential left $L^1(G)$-module X is injective, the morphism ι, $\iota x = 1_G \otimes x$, $x \in X$, possesses a left inverse λ,

$$X \xrightarrow{\iota} L^\infty(G)\check{\otimes}X \xrightarrow{\lambda} X$$

enjoying the following properties:

(i) λ is bounded and linear;
(ii) $\lambda((L_{s^{-1}})^*\varphi \otimes sx) = s\lambda(\varphi \otimes x)$;
(iii) $\lambda(1_G \otimes x) = x$,

whenever $s \in G$, $\varphi \in L^\infty(G)$, and $x \in X$.

3 A Lemma

3.1 Let K be a compact Hausdorff space, and Y be a Banach space. It is well known that the dual space of the injective tensor product $C(K)\check{\otimes}Y = C(K, Y)$ is isometrically isomorphic to the Banach space, $I(C(K), Y^*)$, of integral operators v from $C(K)$ into Y^*, and that this again is isometrically isomorphic to the Banach space, $bvrca(B(K), Y^*)$, of regular countably additive vector measures m of bounded variation on the Borel σ-algebra, $B(K)$, of K with values in Y^*,

$$(C(K)\check{\otimes}Y)^* = I(C(K), Y^*) = bvrca(B(K), Y^*),$$

the correspondence between v and m being given by $m(A) = \tilde{v}(c_A)$, $A \in B(K)$, where $\tilde{v} : C(K)^{**} \longrightarrow Y^*$ denotes the unique weak*-weak* continuous extension of v, and c_A the characteristic function of A. The variation, $|m|$, of $m \in bvrca(B(K), Y^*)$, defined as

$$|m|(A) = \sup \sum \|m(A_i)\| \qquad (A \in B(K)),$$

the supremum being taken over all finite Borel partitions $\{A_i\}$ of A, is a finite positive regular Borel measure on K. Defining the norm of $m \in bvrca(B(K), Y^*)$ by $\|m\| = |m|(K)$, we have $\|m\| = I(v)$, the integral norm of $v \in I(C(K), Y^*)$ corresponding to m. – The theorems involved in this discussion are due to I. Singer, [S]; cf. also VI.3.Theorem 3, p. 162, and VI.3.Theorem 12, p. 169, in [DU], and, in particular, Satz 1 in Losert's Thesis, [Lo, p.7].

3.2 **Lemma.** – Let K be a compact Hausdorff space, X and Y be Banach spaces, and u a weakly compact linear map,

$$u : C(K) \check{\otimes} Y \longrightarrow X.$$

For any sequence (A_n) of pairwise disjoint Borel subsets of K, and any bounded sequences (y_n) in Y and (x_n^*) in X^*, we have

$$\lim \langle x_n^*, u^{**}(c_{A_n} \otimes y_n)\rangle = 0.$$

Proof Let (A_n) be a sequence of disjoint sets in $B(K)$, (x_n^*) a bounded sequence in X^*, (y_n) a sequence in Y of bound c, and $\varepsilon > 0$. Using the dual operator $u^* : X^* \longrightarrow (C(K)\check{\otimes}Y)^*$, we define, for any n, $v_n \in (C(K)\check{\otimes}Y)^* = I(C(K), Y^*)$ and $m_n \in bvrca(B(K), Y^*)$ by

$$v_n = u^*(x_n^*), \qquad m_n(A) = \tilde{v}_n(c_A) \qquad\qquad (A \in B(K), n \geq 1).$$

Since, by the weak compactness of u^*, the set $\{m_n\}$ is relatively weakly compact in $bvrca(B(K), Y^*)$, Theorem 1 in [B, p. 288] – or IV.2.Theorem 5 in [DU, p. 105], in case Y happens to be, for instance, reflexive – yields a finite positive Borel measure μ on K such that the set, $\{|m_n|\}$, of variations of the m_n's is equicontinuous with respect to μ. Hence, for some $\delta = \delta(\varepsilon) > 0$, we have $|m_n|(A) < \varepsilon$ for all n whenever $A \in B(K)$ satisfies $\mu(A) < \delta$. From $\lim \mu(A_n) = 0$ we obtain an $n(\delta)$ such that $\mu(A_n) < \delta$, and therefore $|m_n|(A_n) < \varepsilon$, for all $n \geq n(\delta)$. This implies

$$\begin{aligned} |\langle x_n^*,\, u^{**}(c_{A_n} \otimes y_n)\rangle| &= |\langle u^*(x_n^*),\, c_{A_n} \otimes y_n\rangle| \\ &= |\langle y_n, \tilde{v}_n(c_{A_n})\rangle| \\ &= |\langle y_n, m_n(A_n)\rangle| \\ &\leq \|y_n\|\, \|m_n(A_n)\| \\ &\leq \|y_n\|\, |m_n|(A_n) \\ &\leq c\,\varepsilon, \end{aligned}$$

for all $n \geq n(\delta)$, proving the lemma. ■

Any of the following conditions on X and Y assures the weak compactness of a bounded linear map u from $C(K)\check{\otimes}Y$ into X:

(a) X is reflexive, and Y arbitrary;
(b) X is weakly sequentially complete, and Y^* has the Radon-Nikodým property, cf. [G];
(c) X is weakly sequentially complete, and Y is a C^*-algebra, cf. [ADG, Theorem 4.2, p. 449].

3.3 **Corollary.** – Let G be a locally compact group, X a reflexive Banach space, and u a bounded linear map,

$$u : L^\infty(G) \check{\otimes} X \longrightarrow X.$$

For any sequence (A_n) of pairwise disjoint measurable subsets of G, and any bounded sequences (x_n) in X and (x_n^*) in X^*, we have

$$\lim_n \langle\, u(c_{A_n} \otimes x_n), x_n^* \,\rangle = 0.$$

Proof $L^\infty(G)$ being an abelian von Neumann algebra, there exist an (extremely disconnected) compact Hausdorff space K and a *-isomorphism from $L^\infty(G)$ onto $C(K)$, mapping (c_{A_n}) onto a sequence of pairwise orthogonal projections in $C(K)$, so that (3.2) applies. ■

4 Proof of Proposition

4.1 We shall use the following lemma for the algebra $\mathcal{M} = L^\infty(G)$ whose projections are of the form c_A for some measurable subset A of G. It is due to Grothendieck, [Gro, Théorème 2, p. 146]; cf. also [A, Theorem II.2.(2), p. 288].

Lemma. – Let $\mathcal{M}$ be an abelian von Neumann algebra. A bounded subset K of the Banach space dual, $\mathcal{M}^*$, of $\mathcal{M}$ is relatively weakly compact if and only if every sequence (p_n) of orthogonal projections in $\mathcal{M}$ converges uniformly on K to zero, i.e., for any $\varepsilon > 0$ exists $n(\varepsilon)$, such that

$$\sup_{f \in K} |f(p_n)| < \varepsilon \qquad (n \geq n(\varepsilon)).$$

■

4.2 **Lemma.** – Let G be a locally compact group, X an essential reflexive injective Banach left $L^1(G)$-module, and $\lambda : L^\infty(G) \check{\otimes} X \longrightarrow X$ a map satisfying (2.5. i, ii). Defining, for any $x \in X$ and $x^* \in X^*$, $M \in L^\infty(G)^*$ by

$$M(\varphi) = \langle\, \lambda(\varphi \otimes x), x^* \rangle \qquad (\varphi \in L^\infty(G)),$$

the set, $\{L_s^{**} M : s \in G\}$, of left translates of M is relatively weakly compact in $L^\infty(G)^*$.

Proof If this were not the case, there would exist, by 4.1, an $\varepsilon > 0$, a sequence (A_n) of pairwise disjoint measurable subsets of G, and a sequence (s_n) of points in G, such that

$$|L_{s_n}^{**} M(c_{A_n})| \geq \varepsilon \qquad (n \geq 1).$$

Setting $\lambda = u,\ s_n x = x_n,\ x^* s_n^{-1} = x_n^*,$ we obtain, from (3.3), that

$$\begin{aligned} L_{s_n}^{**} M(c_{A_n}) &= M(L_{s_n}^{*} c_{A_n}) \\ &= \langle\, \lambda(L_{s_n}^{*} c_{A_n} \otimes x), x^* \rangle \\ &= \langle\, \lambda(L_{s_n}^{*} c_{A_n} \otimes s_n^{-1}(s_n x)), x^* \rangle \\ &= \langle\, s_n^{-1} \lambda(c_{A_n} \otimes s_n x), x^* \rangle \\ &= \langle\, \lambda(c_{A_n} \otimes s_n x),\ x^* s_n^{-1} \rangle \end{aligned}$$

tends to zero with n tending to infinity, contradicting the assumption. ■

4.3 **Proof of Proposition**

Let G be a locally compact group, and X a nontrivial injective left $L^1(G)$-module, reflexive as a Banach space. Since $L^1(G)$ possesses a bounded two-sided approximate unit, the essential part of X – possessing a module complement in X – is equally injective, so that we may assume from the outset X to be essential itself. Let then $\lambda : L^\infty(G) \check{\otimes} X \longrightarrow X$ be a map satisfying (2.5. i, ii, iii). For any pair $(x, x^*) \in X \times X^*$ such that $\langle x, x^* \rangle = 1$, the element $M \in L^\infty(G)^*$,

$$M(\varphi) = \langle\, \lambda(\varphi \otimes x), x^* \rangle \qquad (\varphi \in L^\infty(G)),$$

enjoys, by (2.5. iii) and (4.2), the following properties,
(i) $M(1_G) = \langle\, \lambda(1_G \otimes x), x^* \rangle = \langle x, x^* \rangle = 1;$
(ii) $\{L_s^{**} M : s \in G\}$ is relatively weakly compact in $L^\infty(G)^*$.

(ii) implies that the closed convex hull, K, of $\{L_s^{**} M : s \in G\}$ is a weakly compact convex subset of $L^\infty(G)^*$. Since it is invariant under the group of linear isometries $L_s^{**}, s \in G$, the fixed point theorem of Ryll-Nardzewski yields an element $M_0 \in K$ satisfying $L_s^{**} M_0 = M_0,\ s \in G$, and, in virtue of (i), $M_0(1_G) = 1$. Decomposing M_0 into its selfadjoint parts and these into their positive parts, we obtain, possibly after some rescaling, a positive linear functional on $L^\infty(G)$, left invariant and taking the value one at the constant function 1_G, thus establishing the amenability of G. ■

5 Proof of Corollary

For the definition of projective and flat Banach modules over a Banach algebra we refer to [H, III.1.14, p. 136] and [H, VII.1.2, p. 239], respectively. Rather than reproducing them here, we note only that every projective module is flat, and that a module X is flat if and only if its dual module, X^*, is injective, cf. [H, VII.1.14, p. 243].

5.1 **Proof of Corollary a**

Let X be a nontrivial projective left $L^1(G)$-module which is reflexive as a Banach space. Since X_e is module complemented in X, X_e is also projective, and reflexive, so that G is compact, by [R1, 1.4, p. 316]. (It is shown there that a locally compact group is already compact, if it admits a nonzero essential projective left $L^1(G)$-module X whose dual Banach space, X^*, is weakly sequentially complete or norm separable.) The second statement is also proved there, [R1, 1.2, p. 316]. ■

The second part of Corollary b is equally well-known. In [H, VII.2.29, p. 257], it is deduced from the vanishing of the Tor functor over an amenable algebra, or can be seen, more directly, from B.E. Johnson's original definition, [J, p. 60], as follows.

5.2 **Lemma** ([H]) Let A be an amenable Banach algebra. Then every Banach left or right module over A is flat.

Proof Let us show that the dual right module, X^*, of a left A-module X is injective. Replacing X with X^* in the diagram defining injectivity, (1), and taking ι and λ_0 as morphisms of right A-modules, we will consider $\lambda_0 \circ \ell$ as element of the Banach space, $L(Y, X^*)$, of bounded linear maps from Y into X^*. Making it an A-bimodule by $(aT)(y) = T(ya)$ and $(Ta)(y) = (Ty)a$, for $a \in A$, $T \in L(Y, X^*)$, and $y \in Y$, we obtain a bounded linear map $D : A \longrightarrow L(Y, X^*)$, $Da = a(\lambda_0 \circ \ell) - (\lambda_0 \circ \ell)a$, $a \in A$, whose values vanish on the closed submodule ιY_0 of Y, thus defining a new map, $D_0 : A \longrightarrow L(Y/\iota Y_0, X^*)$, by the formula $(D_0 a)(\pi y) = (Da)(y)$, $a \in A$, $y \in Y$, and π denoting the canonical morphism of Y onto $Y/\iota Y_0$. Endowing the projective tensor product $Y/\iota Y_0 \hat{\otimes} X$ with A-actions $a(\pi y \otimes x) = \pi y \otimes ax$ and $(\pi y \otimes x)a = \pi ya \otimes x$, the Banach space $L(Y/\iota Y_0, X^*) = (Y/\iota Y_0 \hat{\otimes} X)^*$, cf [CLM, II.1.7, p. 54], becomes a dual A-bimodule and D_0 a derivation, such that $D_0 a = aS - Sa$, $a \in A$, for some $S \in L(Y/\iota Y_0, X^*)$. Comparing with the definition of D_0 yields

$$a(\lambda_0 \circ \ell - S \circ \pi) = (\lambda_0 \circ \ell - S \circ \pi)a \qquad (a \in A),$$

so that $\lambda = \lambda_0 \circ \ell - S \circ \pi$ becomes a morphism extending λ_0 along ι. Hence X^* is injective, and X flat. ■

5.3 **Proof of Corollary b**

Let X be a nontrivial flat left $L^1(G)$-module, reflexive as a Banach space. Then X^* is a nontrivial injective right $L^1(G)$-module and equally reflexive, implying the amenability of G by the Proposition. If, conversely, the group G is amenable, then the Banach algebra $L^1(G)$ is amenable, [J, Theorem 2.5, p. 32], so that every left $L^1(G)$-module is flat by the Lemma above. ■

6 An open problem

Let $L^p(\mathcal{M})$, $1 < p < \infty$, be the L^p-space associated with a von Neumann algebra $\mathcal{M}$, a reflexive normal Banach left $\mathcal{M}$-module. Does its injectivity imply the injectivity of the von Neumann algebra $\mathcal{M}$? – The answer seems to be yes in case $p = 2$ and the injectivity constant being one, cf. [R2, 2.6 Corollary].

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References

[A] Akemann, C.A. The dual space of an operator algebra. Trans. Amer. Math. Soc. 126 (1967), 286-302.

[ADG] Akemann, C.A., Dodds, P.G. and Gamlen, J.L.B. Weak compactness in the dual space of a C^*-algebra. J. Funct. Anal. 10 (1972), 446-450.

[B] Batt, J. On weak compactness in spaces of vector-valued measures and Bochner-integrable functions in connection with the Radon-Nikodým property of Banach spaces. Rev. Roum. Math. Pures et Appl. 19 (1974), 285-304.

[CLM] Cigler, J., Losert, V., Michor, P. Banach modules and functors on categories of Banach spaces. Marcel Dekker, New York, 1979.

[DDPR] Dales, H.G., Daws, M., Pham, H.L. and Ramsden, P. Multi-norms and the injectivity of $L^p(G)$. Lond. Math. Soc., to appear.

[DP] Dales, H.G. and Polyakov, M.E. Homological properties of modules over group algebras. Proc. London Math. Soc. (3) 89 (2004), 390-426.

[DU] Diestel, J. and Uhl, J.J. Vector measures. Mathematical Surveys 15. Amer. Math. Soc., Providence, Rhode Island, 1977.

[G] Gamlen, J.L.B. On a theorem of A. Pełczyński. Proc. Amer. Math. Soc. 44 (1974), 283-285.

[Gro] Grothendieck, A. Sur les applications linéaires faiblement compactes d'espaces du type $C(K)$. Can.J.Math. 5 (1953), 129-173.

[H] Helemskii, A.Ya. The homology of Banach and topological algebras. Izdat. Moskov. Gos. Univ., Moskva, 1986; English transl. Kluwer, Dordrecht, 1989.

[J] Johnson, B.E. Cohomology in Banach algebras. Mém. Amer. Math. Soc. 127 (1972).

[L] Lau, A.T-M. Operators which commute with convolutions on subspaces of $L_\infty(G)$. Colloq. Math. 39 (1978), 351-359.

[Lo] Losert, V. Dualität von Funktoren und Operatorenideale. Dissertation, Phil. Fak. Universität Wien, 1975.

[R1] Racher, G. On the projectivity and flatness of some group modules. Banach Center Publications 91 (2010), 315-325.

[R2] Racher, G. On injective von Neumann algebras. Proc. Amer. Math. Soc., to appear.

[Ra] Ramsden, P. Multi-norms and modules over group algebras. Manuscript, August 2, 2009.

[S] Singer, I. Lineinye funktsionali na prostranstve nepreryvnykh otobrazhenii bikompaktnogo khausdorfogo prostranstva v prostranstvo Banakha. Rev. Math. Pures Appl. 2 (1957), 301-315.

[Y] Yeadon, F.J. A new proof of the existence of a trace in a finite von Neumann algebra. Bull. Amer. Math. Soc. 77 (1971), 257-260.