ESI **The Erwin Schrödinger International Institute for Mathematical Physics**

Boltzmanngasse 9
A-1090 Wien, Austria

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D. Burghelea
L. Friedlander
T. Kappler

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ASYMPTOTIC EXPANSION OF THE WITTEN DEFORMATION OF THE ANALYTIC TORSION

D. Burghelea
L. Friedlander
T. Kappeler

International Erwin Schrödinger Institute for Mathematical Physics

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Abstract. Given a compact Riemannian manifold $(M^{\,d}, g)$, a finite dimensional representation $\rho : \pi_1(M) \to GL(V)$ of the fundamental group $\pi_1(M)$ on a vectorspace V of dimension l and a Hermitian structure μ on the flat vector bundle $\mathcal{E} \xrightarrow{p} M$ associated to ρ, Ray-Singer [RS] have introduced the analytic torsion $T = T(M, \rho, g, \mu) > 0$. Witten's deformation $d_q(t)$ of the exterior derivative $d_q, d_q(t) = e^{-ht} d_q e^{ht}$, with $h : M \to R$ a smooth Morse function, can be used to define a deformation $T(h, t) > 0$ of the analytic torsion T with $T(h, 0) = T$.

The main results of this paper are to provide, assuming that grad $_g h$ is Morse Smale, an asymptotic expansion for $\log T(h, t)$ for $t \to \infty$ of the form $\sum_{j=0}^{d+1} a_j t^j + b \log t + O(\frac{1}{\sqrt{t}})$ and to present two different formulae for a_0. As an application we obtain a shorter derivation of results due to Ray-Singer [RS], Cheeger [Ch], Müller [Mu1,2] and Bismut-Zhang [BZ] which, in increasing generality, concern the equality for odd dimensional manifolds of the analytic torsion with the average of the Reidemeister torsion corresponding to the triangulation $\mathcal{T} = (h, g)$ and the dual triangulation $\mathcal{T}_{\mathcal{D}} = (d - h, g)$.

0. Introduction

Let M be a compact smooth manifold of dimension d without boundary and $\rho : \pi_1(M) \to GL(V)$ a linear representation of the fundamental group $\pi_1(M)$ of M on a vectorspace V of dimension l. The representation ρ induces a smooth vector bundle $\mathcal{E} \to M$ equipped with a flat canonical connection. Let $\Lambda^q(M; \mathcal{E}) = C^\infty(\Lambda^q(T^*M) \otimes \mathcal{E})$ be the space of smooth q-forms with values in $\mathcal{E}$ where T^*M denotes the cotangent bundle of M. The above connection can be interpreted as a first order differential operator ${}_\rho d_q : \Lambda^q(M; \mathcal{E}) \to \Lambda^{q+1}(M; \mathcal{E})$ and its flatness is equivalent to ${}_\rho d_{q+1} \cdot {}_\rho d_q = 0$ for $0 \le q \le d$. Note that if ρ is trivial, ${}_\rho d$ is the usual exterior

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differential d. In case there is no risk of ambiguity we will write d instead of ${}_\rho d$ and continue to call it exterior differential. Let $h : M \to \mathbf{R}$ be a smooth Morse function. For convenience we assume that h is selfindexing i.e. $h(x) =$ index x, for any critical point x of h. Following Witten [Wi] we introduce the deformation $(\Lambda^q(M;\mathcal{E}), d_q(t))$ of the de Rham complex $(\Lambda^q(M;\mathcal{E}), d_q)$ where $d_q(t) = e^{-th} d_q e^{th} = d_q + t dh\wedge$. One verifies that this complex in elliptic. Let g be a Riemannian metric on M and μ a Hermitian structure for $\mathcal{E} \to M$. The Riemannian metric g induces the Hodge operators $J_q : \Lambda^q(T^*M)_x \to \Lambda^{d-q}(T^*M)_x \quad (x \in M)$ and the Hermitian structure μ on $\mathcal{E}$ together with the Hodge operators induce a Hermitian structure on $\Lambda^q(T^*M)\otimes\mathcal{E}$ given by $(w\otimes s, w'\otimes s')(x) = J_d(w(x)\wedge J_q w'(x))\mu(s(x), s'(x))$. The formal adjoint of ${}_\rho d_q(t)$ with respect to this Hermitian structure is a first order differential operator ${}_\rho d_q^*(t) : \Lambda^{q+1}(M;\mathcal{E}) \to \Lambda^q(M;\mathcal{E})$. More explicitly let ρ^* be the dual representation of $\rho, \rho^* : \pi_1(M) \to GL(V^*)$, with V^* denoting the dual of V and let μ^* denote the Hermitian structure μ when viewed as an isomorphism $\mu^* : \mathcal{E} \to \mathcal{E}^*$. Then ${}_\rho d_q^*(t)$ can be written as

$$ {}_\rho d_q^*(t) = (-1)^{dq+1}(J_{d-q} \otimes Id)(Id \otimes \mu^*)^{-1} \cdot {}_{\rho^*} d_{d-(q+1)}(Id \otimes \mu^*)(J_{q+1} \otimes Id). $$

Introduce the deformed Laplacians, acting on q-forms,

$$ \Delta_q(t) = d_q^*(t) d_q(t) + d_{q-1}(t) d_{q-1}^*(t). $$

The operators $\Delta_q(t)$ are elliptic, non negative selfadjoint operators. Denote by $(\lambda_j^q)_{j\geq 1}$ the set of all eigenvalues of $\Delta_q(t)$, counted with multiplicities, and introduce the corresponding zeta function

$$ \zeta_q(s) = \sum_{\lambda_j^q \neq 0} (\lambda_j^q)^{-s} $$

The functions $\zeta_q(s)$ are holomorphic in the half plane $\mathcal{R}es > d/2$ and they can be extended to meromorphic functions in the whole complex plane [Se1] with $s = 0$ being a regular point for all of them. Therefore one can define the regularized determinant of Δ_q

$$ \log\det \Delta_q = -\frac{d}{ds}\zeta_q(0). $$

Consider the function $T(h,t) = T(M,\rho,g,\mu,h,t)$ defined by

$$ \log T(h,t) = \frac{1}{2}\sum_{q=0}^{d}(-1)^{q+1} q \log\det \Delta_q(t). \tag{0.1} $$

Note that $T(h,0)$ is independent of h and is equal to the analytic torsion $T = T(M,\rho,g,\mu)$ as introduced by Ray-Singer [RS]. For $t \neq 0$, $T(h,t)$ will be referred to as Witten's deformation of the analytic torsion. It is well known that the set of eigenvalues of $\Delta_q(t)$, when $t \to \infty$, separates into two parts. The part of small eigenvalues, whose number is equal to $m_q l$ ($l = \lim V, m_q = \#Cr_q(h), Cr_q(h)$ the set of critical points of h of index q), converge to zero at a rate of e^{-2t}/t and all large eigenvalues grow at a rate of $O(t)$. Therefore one can decompose, for sufficiently large t,$(\Lambda^q(M;\mathcal{E}), d_q(t))$ as

$$\Lambda^q(M;\mathcal{E}), d_q(t) = (\Lambda^q(M;\mathcal{E})_{sm}, d_q(t)) \oplus \\ (\Lambda^q(M;\mathcal{E})_{la}, d_q(t))$$

where $\Lambda^q(M;\mathcal{E})_{sm}$ respectively $\Lambda^q(M;\mathcal{E})_{la}$ are the orthogonal subspaces of $\Lambda^q(M;\mathcal{E})$, both depending on t, generated by eigenforms corresponding to small respectively large eigenvalues. Accordingly one decomposes $\zeta_q(s) = \zeta_{q,sm}(s) + \zeta_{q,la}(s)$ and $\log T(h,t) = \log T_{sm}(h,t) + \log T_{la}(h,t)$. The main results of this paper concern the asymptotic expansions of $\log T(h,t)$, $\log T_{sm}(h,t)$ and $\log T_{la}(h,t)$ for $t \to \infty$. Before formulating these results we recall the notion of Reidemeister torsion associated to (M,ρ,g,μ) and to a generalized triangulation of M, which is defined as follows:

Definition.

A pair $\mathcal{T} = (h, g')$ is said to be a generalized triangulation if

(i) $h : M \to \mathbf{R}$ is a smooth Morse function which is selfindexing ($h(x) =$index(x) for any critical point x of h);

(ii) g' is a Riemannian metric so that $\operatorname{grad}_{g'} h$ satisfies the Morse-Smale condition (for any two critical points x and y of h the stable manifold W_x^+ and the unstable manifold W_y^-, with respect to $\operatorname{grad}_{g'} h$, intersect transversely).

The notion of generalized triangulation is justified as it is a straight generalization of the notion of a simplicial triangulation(*

Given a generalized triangulation $\mathcal{T} = (h, g')$ we denote by $\mathcal{T}_{\mathcal{D}}$ the generalized triangulation $\mathcal{T}_{\mathcal{D}} = (d-h, g')$ and extending the definition of dual triangulations in combinatorial topology we call $\mathcal{T}_{\mathcal{D}}$ the generalized triangulation dual to $\mathcal{T}$. The Reidemeister torsion $\tau(\mathcal{T}) = \tau(M,\rho,g,\mu,\mathcal{T})$, associated to (M,ρ,g,μ) and the generalized triangulation $\mathcal{T} = (h, g')$, is a positive number given by

$$\log \tau(\mathcal{T}) = \log \tau_{comb}(\mathcal{T}) + \log \tau_{met}(\mathcal{T})$$

where we refer to $\tau_{comb}(\mathcal{T}) = \tau_{comb}(M,\rho,g,\mu,\mathcal{T})$ as the combinatorial part of $\tau(\mathcal{T})$ and to $\tau_{met}(\mathcal{T}) = \tau_{met}(M,\rho,g,\mu,\mathcal{T})$ as the metric part of $\tau(\mathcal{T})$.

In order to define $\tau_{comb}(\mathcal{T})$ and $\tau_{met}(\mathcal{T})$ consider the universal covering $p : \hat{M} \to M$ of M. Choose an orientation of $\hat{M}$ and for each critical point $\hat{x}$ of $\hat{h} = h \cdot p$ an orientation for the unstable manifold $W_{\hat{x}}^-$ of $\operatorname{grad}_{g'} \hat{h}$, the gradient of $\hat{h}$ with respect to the pull back of the metric g' on $\hat{M}$, again denoted by g'. The orientations on $\hat{M}$ and on $W_{\hat{x}}^-$ induce an orientation on $W_{\hat{x}}^+$. Consider the chain complex

$$C_*(M,\mathcal{T}) := \{C_q(\hat{M},\mathcal{T}) | \delta_q : C_q(\hat{M},\mathcal{T}) \to C_{q-1}(\hat{M},\mathcal{T})\}$$

(*Given a smooth simplicial triangulation $\mathcal{T}_{sim}$, one can construct a generalized triangulation $\mathcal{T} = (h, g')$ so that the unstable manifolds W_x^- corresponding to $\operatorname{grad}_{g'} h$, with x a critical point of h, are the open simplexes of $\mathcal{T}_{sim}$ (c.f[Po]). Moreover, one may choose the metric g' in $\mathcal{T}$ in such a way that it agrees with a given Riemannian metric g near all critical points of h.

where $C_q(\hat{M},\mathcal{T})$ denotes the free abelian group generated by the set $Cr_q(\hat{h})$ of critical points of $\hat{h}$ of index q and $\delta_q : C_q(\hat{M},\mathcal{T}) \to C_{q-1}(\hat{M},\mathcal{T})$ is a homomorphism given by

$$\delta_q(\hat{x}) = \sum_{\hat{y}\in Cr_{q-1}} \beta^{\hat{y}}_{q;\hat{x}}\hat{y}.$$

The integers $\beta^{\hat{y}}_{q;\hat{x}}$ are the intersection numbers of $W^-_{\hat{y}}$ and $W^+_{\hat{x}}$ in $h^{-1}(q-1/2)$. The free action of $\pi_1(M)$ on $\hat{M}$ makes $C_*(\hat{M},\mathcal{T})$ a chain complex of free $\mathbf{Z}[\pi_1(M)]$-modules. Let $C^*(M,\rho,\mathcal{T}) := \hom_{\pi_1}(C_*(\hat{M},\mathcal{T}),V)$ be the space of linear maps $C_*(\hat{M},\mathcal{T}) \to V$ which are equivariant with respect to the action provided by $\pi_1(M)$ on $C_*(\hat{M},\mathcal{T})$ and on V, the underlying vector space of the representation ρ. $C_*(\hat{M},\mathcal{T})$ is a cochain complex of finite dimensional vector spaces. The vector space $C^q(M,\rho,\mathcal{T})$ has dimension $m_q l$, where $l = \lim V$ and $m_q = \# Cr_q(h)$; it can be identified with the space of sections of the restriction of $\mathcal{E}$ to the discrete space $Cr_q(h) \subseteq M$ of critical points of h of index q. Therefore it is equipped with the scalar product induced by the Hermitian structure μ. An alternative description of this scalar product is the following: For each critical point $x_{q;j} \in Cr_q(h)$ choose a μ-orthonormal basis $e_{q;j1},\dots,e_{q;jl}$ of $\mathcal{E}_{x_{q;j}}$ and select a lift $\hat{x}_{q;j}$ of $x_{q;j}$ in $\hat{M}$. Define $E_{q;jr}(1 \le j \le m_q, 1 \le r \le l)$ to be the unique element in $\hom_\pi(C_q(\hat{M},\mathcal{T}),V)$ which satisfies $E_{q;jr}(\hat{x}_{q;i}) = \delta_{ji}\bar{e}_{q;jr}$ where $\bar{e}_{q;jr} \in V$ corresponds to the element $e_{q;jr}$ uniquely determined by the commutative diagram

$$\begin{array}{ccc} \hat{M}\times V & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ \hat{M} & \longrightarrow & M \end{array}$$

The elements $E_{q;jr}$ $(1 \le j \le m_q, 1 \le r \le l)$ form a basis of $C^q(M;\rho,\mathcal{T})$ and the scalar product which makes this basis orthonormal is identical with the one defined previously. When equipped with this scalar product $C^*(M;\rho,\mathcal{T})$ will be denoted by $C^*(M;\rho,\mathcal{T},\mu)$. Denote by $\delta^*_q : C^{q+1}(M;\rho,\mathcal{T},\mu) \to C^q(M;\rho,\mathcal{T},\mu)$ the adjoint of $\delta_q : C^q(M;\rho,\mathcal{T},\mu) \to C^{q+1}(M;\rho,\mathcal{T},\mu)$ and form the Laplacians

$$\underline{\Delta}_q := \delta^*_q\delta_q + \delta_{q-1}\delta^*_{q-1}, \underline{\Delta}_q : C^q(M;\rho,\mathcal{T},\mu) \to C^q(M;\rho,\mathcal{T},\mu).$$

Denote by $\det'\underline{\Delta}_q$ the product of all non zero eigenvalues of $\underline{\Delta}_q$. Notice that $\det'\underline{\Delta}_q$ is always a positive number and therefore we may introduce $\tau_{comb}(\mathcal{T}) = \tau_{comb}(M;\rho,\mathcal{T},\mu)$ by

$$\log\tau_{comb}(\mathcal{T}) = 1/2\sum_{q=0}^{d}(-1)^{q+1}q\log\det{}'\underline{\Delta}_q. \tag{0,2}$$

It remains to define τ_{met}. De Rham theory provides a canonically defined isomorphism between the de Rham cohomology $H^q(M;\mathcal{E}) = H^q(\Lambda^*(M;\mathcal{E}),d)$ and the cohomology $H^q(C^*(M,\rho,\mathcal{T}),\delta)$

$$R_q : H^q(M;\mathcal{E}) \to H^q(C^*(M,\rho,\mathcal{T}),\delta).$$

Let A_q respectively B_q be the canonical isomorphisms between $Ker\Delta_q$ and $H^q(M;\mathcal{E})$ respectively between $Ker\underline{\Delta}_q$ and $H^q(C^*(M,\rho,\mathcal{T}),\delta)$. Denote by $V_q = V_q(\rho,\mathcal{T})$ the volume $V_q = |\det(r_q^{-1})|$,where $r_q^{-1} = A_q^{-1}R_q^{-1}B_q$. The metric part of the Reidemeister torsion $\tau_{met}(\mathcal{T}) = \tau_{met}(M,\rho,\mathcal{T},g,\mu)$ is defined by

$$\log \tau_{met}(\mathcal{T}) = \sum_{q=0}^{d} (-1)^q \log V_q \tag{0.3}$$

To state the first result we introduce the following definitions:

Definition.

A system (M,ρ,h,g,μ) satisfies hypothesis (H) if:

(H.1) $\mathcal{T} = (h,g)$ is a generalized triangulation.

(H.2) In a neighborhood of any critical point of index q of the function h one can introduce local coordinates such that

$$h(x) = q - (x_1^2 + ...x_q^2)/2 + (x_{q+1}^2 + ...x_d^2)/2,$$

and the metric g is Euclidean in these coordinates.

(H.3) μ is parallel with respect to the canonical connection in a neighbourhood of any critical point of h.

Definition.

A continuous function $a : \mathbf{R} \to \mathbf{R}$ is said to have a complete asymptotic expansion for $t \to \infty$ if there are sequences $i_1 > i_2 > ...$ with $\lim_{k\to\infty} i_k = -\infty$, $(a_k)_{k\geq 1}$ and $(b_k)_{k\geq 1}$ such that, for any $L \geq 1$,

$$a(t) = \sum_{k=1}^{L-1} a_k t^{i_k} + \sum_{k=1}^{L-1} b_k t^{i_k} \log t + O(t^{i_L} \log t). \tag{0.4}$$

In that case we write

$$a(t) \sim \sum_{k\geq 1} a_k t^{i_k} + \sum_{k\geq 1} b_k t^{i_k} \log t.$$

The function $a(t)$ is said to have an asymptotic expansion for $t \to \infty$ if there exists $i_1 > \cdots > i_N = 0 > i_{N+1}$ and sequences of numbers $(a_k)_{k\geq 1}$ and $(b_k)_{k\geq 1}$ such that

$$a(t) = \sum_{1}^{N} a_k t^{i_k} + \sum_{1}^{N} b_k t^{i_k} \log t + O(t^{i_{N+1}}). \tag{0.5}$$

For convenience we denote by $FT(a(t))$ the coefficient of the asymptotic expansion of $a(t)$ corresponding to t^0.

Both Theorem A and Theorem B concern the asymptotic expansions of $\log T(h,t)$, $\log T_{sm}(h,t)$ and $\log T_{la}(h,t)$. There are two different methods to analyze these expansions. Theorem A contains results which can be obtained by applying the analysis of Helffer-Sjöstrand [HS] of Wittens's deformation of the de Rham complex. Theorem B contains results which can be obtained from the theory of families of pseudodifferential operators elliptic with parameter. As usual $\beta_q = \beta_q(M,\rho)$ denotes the qth Betti number, $\beta_q = \dim H^q(M;\mathcal{E})$.

Theorem A. *Assume that the system (M, ρ, h, g, μ) satisfies (H) and let $\mathcal{T}$ denote the generalized triangulation (h, g). Then the following statements are true:*

(i) The function $\log T(h,t)$, $\log T_{sm}(h,t)$ and $\log T_{la}(h,t)$ admit asymptotic expansions for $t \to \infty$.

(ii) The asymptotic expansion of $\log T(h,t)$ is of the form

$$\begin{aligned}\log T(h,t) = \log T - \log \tau_{met}(\mathcal{T}) + \left(\log \pi \sum_{q=0}^{d} (-1)^q \frac{d-2q}{4} \beta_q\right) + \\ + \left(\sum_{q=0}^{d} (-1)^{q+1} \frac{d-2q}{4} \beta_q\right) \log t + \left(\sum_{q=0}^{d} (-1)^{q+1} q \beta_q\right) t \\ + \sum_{j=1}^{d+1} \left(\sum_{q=0}^{d} (-1)^q p_{q;j}\right) t^j + O\left(\frac{1}{\sqrt{t}}\right)\end{aligned} \tag{0.6}$$

where the $p_{q;j}$'s are local terms (of Lemma 2.3) which all vanish in the case M is of odd dimension.

(iii) The asymptotic expansion of $\log T_{sm}(h,t)$ is of the form

$$\log \tau_{comb}(\tau) + \frac{1}{2}\left(\sum_{q=0}^{d} (-1)^q q (mql - \beta_q)\right)(2t - \log t + \log \pi) + O\left(\frac{1}{\sqrt{t}}\right). \tag{0.7}$$

Theorem A is proved in section 2 using, as already mentioned, results due to Helffer-Sjöstrand [HS] concerning the asymptotic analysis of the complex $(\Lambda^*(M)_{sm},\ d(t))$. Actually, we use a generalization of their results to the case of vector valued differential forms. It has been verified by Bismut-Zhang [BZ] that all the arguments of Helffer-Sjöstrand carry over for the asymptotic analysis of the complex $(\Lambda^*(M, \mathcal{E})_{sm}, d(t))$. For the convenience of the reader the auxiliary results needed for the proof of Theorem A are reviewed in section 1. However the paper can be well understood without reading section 1.

Concerning Theorem B we first remark that the family of operators $\Delta_q(t)$ is a family with parameter of order 2 and weight 1 (see [Sh] [BFK] or section 1). If $\Delta_q(t)$ were a family of operators elliptic with parameter, then $\log \det \Delta_q(t)$ would admit a complete asymptotic expansion for $t \to \infty$, whose coefficients would be given by local expressions, involving the symbol of the resolvent of $\Delta_q(t)$ ([BFK], Theorem A.(iii)). This would lead to the false conclusion that the analytic torsion itself is a local expression.

However, the family $\Delta_q(t)$ fails to be elliptic with parameter precisely at the critical points of the Morse function h. We therefore can use the Mayer-Vietoris type formula for determinants [BFK], reviewed in section 1, to localize the failure of the family $\Delta_q(t)$ to be elliptic with parameter and to obtain a relative result, comparing the asymptotic expansions corresponding to two different systems (M, ρ, h, g, μ) and $(\tilde{M}, \tilde{\rho}, \tilde{h}, \tilde{g}, \tilde{\mu})$. In view of the application in section 4 we only present a result concerning the free term of the asymptotic expansion for $\log T_{la}(h,t) - \log T_{la}(\tilde{h}, t)$. In section 3 we prove

Theorem B. *Suppose* (M^d, ρ, h, g, μ) *and* $(\tilde{M}^d, \tilde{\rho}, \tilde{h}, \tilde{g}, \tilde{\mu})$ *satisfy both hypothesis (H) and further assume that* $l(\tilde{\rho}) = l(\rho)$ *and* $Cr_q(h) = Cr_q(\tilde{h})(0 \le q \le d)$. *Then the following statements hold:*

(i) The free term $FT(\log T_{la}(h,t) - \log T_{la}(\tilde{h},t))$ *of the asymptotic expansion of* $\log T_{la}(h,t) - \log T_{la}(\tilde{h},t)$ *is given by*

$$\begin{aligned} FT(\log T_{la}(h,t) - \log T_{la}(\tilde{h},t)) = &\int_{M\setminus Cr(h)} a_0(h, \varepsilon = 0, x)dx \\ &- \int_{\tilde{M}\setminus Cr(\tilde{h})} a_0(\tilde{h}, \varepsilon = 0, \tilde{x})d\tilde{x} \end{aligned} \tag{0.8}$$

where the densities $a_0(h, \varepsilon, x)$ *and* $a_0(\tilde{h}, \varepsilon, \tilde{x})$ *are forms of degree d and are given by explicit local formulae (see (3.5)); the difference is taken in the sense explained below.*

(ii)If $h' : M \to \mathbf{R}$ *is another self indexing Morse function with the same critical points as h, which is equal to h in a neighborhood of the critical points, and* $grad_g h'$ *generates the same cochain complex as* $grad_g h$ *then* $a_0(h, \varepsilon = 0, x) - a_0(h', \varepsilon = 0, x) = db(x)$ *with b(x) a differential form of degree d-1 which vanishes in a neighborhood of Cr(h).*

(iii)If d is odd then $a_0(h, \varepsilon = 0, x) + a_0(d - h, \varepsilon = 0, x) = 0$

The integral $\int_{M\setminus Cr(h)} a_0(h, \varepsilon = 0, x)dx$ is not convergent so the difference on the right hand side of (0.8) should be understood in the following way: in view of the hypothesis (H) there exist neighborhoods V of $Cr(h)$ and $\tilde{V}$ of $Cr(\tilde{h})$, a diffeomorphism $\tau : V \to \tilde{V}$ and a bundle isomorphism $\chi : \mathcal{E}|_V \to \tilde{\mathcal{E}}|_{\tilde{V}}$ so that τ and χ intertwine the functions h and $\tilde{h}$,the metrics g and $\tilde{g}$,the hermitian structures μ and $\tilde{\mu}$ and the differential operators ${}_{\rho}d$ and ${}_{\tilde{\rho}}d$. Define

$$\begin{aligned} &\int_{M\setminus Cr(h)} a_0(h, \varepsilon = 0, x) - \int_{\tilde{M}\setminus Cr(\tilde{h})} a_0(\tilde{h}, \varepsilon = 0, \tilde{x}) \\ &\quad := \int_{M\setminus V} a_0(h, \varepsilon = 0, x) - \int_{\tilde{M}\setminus \tilde{V}} a_0(\tilde{h}, \varepsilon = 0, \tilde{x}) \end{aligned} \tag{0.9}$$

Clearly, this definition is independent of the choice of V and $\tilde{V}$.

As an immediate application of Theorem A and Theorem B we obtain the following Corollary which will be proved in section 3 as well:

CorollaryC. *Suppose* (M^d, ρ, h, g, μ) *and* $(\tilde{M}^d, \tilde{\rho}, \tilde{h}, \tilde{g}, \tilde{\mu})$ *both satisfy (H). Moreover assume that d is odd,* $l(\rho) = l(\tilde{\rho})$ *and* $\#Cr_q(h) = \#Cr_q(\tilde{h})$ *for* $0 \le q \le d$. *Denote by* $\mathcal{T}$ *respectively* $\tilde{\mathcal{T}}$ *the generalized triangulation* (h, g) *respectively* $(\tilde{h}, \tilde{g})$. *Then, with* $\tilde{T} := T(\tilde{M}^d, \tilde{\rho}, \tilde{h}, \tilde{g}, \tilde{\mu})$

$$\begin{aligned} \log T - \log \tilde{T} = &(\log \tau(\mathcal{T}) + \log \tau(\mathcal{T}_{\mathcal{D}}))/2 \\ &- (\log \tau(\tilde{\mathcal{T}}) + \log \tau(\tilde{\mathcal{T}}_{\mathcal{D}}))/2. \end{aligned} \tag{0.9}$$

Section 4 contains an application of Corollary C. We prove a theorem due to, in increasing generality, Ray-Singer, Cheeger, Müller, Bismut-Zhang.

Theorem 4.1. *(*[RS], [Ch], [Mü 1.2], [BZ]*) Assume that (M^d, g) is a Riemannian manifold of odd dimension, ρ a representation $\rho : \pi_1(M) \to GL(V)$, μ a Hermitian structure of $\mathcal{E} \to M$ and $\mathcal{T} = (h, g')$ a generalized triangulation. Then*

$$\log T = (\log \tau(\mathcal{T}) + \log \tau(\mathcal{T}_{\mathcal{D}}))/2. \tag{0.10}$$

The formula (0.10) is not true for n even.However in view of Theorem B,i) the same arguments as in the odd dimensional case imply that $\log T - \log \tau$ can be expressed by a local formula. Such a formula has been obtained by Bismut-Zhang [BZ]. We can also obtain such a formula by explicitely evaluating the density $a_0(h, \varepsilon = 0, x)$,a calculation which might be of independent interest.

In forthcoming papers we will provide further extensions and applications of Theorem A, B and Corollary C to analyse the G-torsion of a compact G-manifold (G a compact Lie group) and to treat the case (in collaboration with P.Macdonald) where ρ is a finite type Hilbert module. As a particular case this extension includes the L^2-torsion.

A few historical comments concerning Theorem 4.1 are in place. The above result was conjectured by Ray-Singer [RS] and proved independently by Cheeger [Ch] and Müller [Mü 1] in the case where the Hermitian structure μ is parallel with respect to the canonical connection induced by ρ. We point out that in this case $\tau(\mathcal{T}) = \tau(\mathcal{T}_{\mathcal{D}})$.

The result in the generality stated is implicit in the work of Bismut-Zhang ([BZ], Theorem 2) (see also [Mü 2] for a less general version). Their proof is rather involved and lengthy while our proof is an application of Theorem A and Theorem B and is considerably shorter. This paper was preceeded by [BFK-2](unpublished) where a new proof of the Cheeger Müller theorem was given on the lines presented in this paper.

1. Auxiliary results

In this section, for the convenience of the reader and to the extent needed in this paper we review in the first part Seeley's work on the value of the zeta function of an elliptic operator at zero [Se 1,2], previous results of ours concerning the asymptotic expansion of log det of an elliptic family of pseudodifferential operators with parameter and a Mayer-Vietoris type formula for determinants [BFK].

In the second part we review results due to Helffer-Sjöstrand [HS] concerning the analysis of Witten's de Rham complex.

Let M be a compact smooth manifold of dimension d, possibly with nonempty boundary ∂M and let $E \xrightarrow{p} M$ be a smooth complex vector bundle of rank N. A pair (φ, ψ) of smooth maps $\varphi : X \to \varphi(X) = U \subset M$, U an open set and $\psi : X \times \mathbf{C}^N \to E/U$, with $X = \mathbf{R}^d$ or $\mathbf{R}^d_+ = \{(x_1, \dots, x_d)/x_d \geq 0\}$, is said to be a coordinate chart of $(M, E \xrightarrow{p} M)$ if φ is a chart of M and ψ is a trivialization of $E \to M$ above U, i.e. $p\psi = \varphi p_1$ where $p_1 : X \times \mathbf{C}^N \to X$ is the canonical projection.

Let $Q : C^\infty(E) \to C^\infty(E)$ be a classical pseudodifferential operator of order m; this means that with respect to any chart of $(M, E \xrightarrow{p} M)$, the symbol q of

Q, $q : X \times \mathbf{R}^d \to End(\mathbf{C}^N)$, $(x, \xi) \to q(x, \xi)$, admits an asymptotic expansion $\sum_{j=0}^{\infty} q_{m-j}(x, \xi)$, where $q_{m-j}(x, \xi)$ is positive homogeneous of degree $m - j$ in ξ. Any differential operator is a classical pseudodifferential operator. The principal symbol $\sigma(Q) = q_m(x, \xi)$ of a classical pseudodifferential operator Q of order m is invariantly defined as a map $T_x^* M \to End(E_x)$.

Definition.

Q is called elliptic if $q_m(x, \xi)$ is invertible for all $(x, \xi) \in T_x^* M \backslash 0$.

We first review the case $\partial M = \emptyset$. One can construct a parametrix $R(\mu)$ of the resolvent for an elliptic classical pseudodifferential operator Q. This is a family of elliptic classical pseudodifferential operators depending on $\mu \in \mathbf{C} \backslash \cup_{x,\xi}$ Spec $(q_m(x, \xi))$, $(x, \xi) \in T^* M \backslash 0$ representing an inverse of $\mu - Q$ up to smoothing operators. The symbol of $R(\mu)$ in a chart (φ, ψ) has an asymptotic expansion determined inductively as follows

$$r_{-m}(x, \xi, \mu) = (\mu - q_m(x, \xi))^{-1}, \tag{1.1}$$

and, for $j \geq 1$,

(1.2)

$$r_{-m-j}(x, \xi, \mu) = r_{-m}(x, \xi, \mu) \sum_{k=0}^{j-1} \sum_{|\alpha|+l+k=j} \frac{1}{\alpha!} \partial_\xi^\alpha q_{m-l}(x, \xi) \cdot (\frac{1}{i} \partial_x)^\alpha r_{-m-k}(x, \xi, \mu)$$

where $\alpha = (\alpha_1, \ldots, \alpha_d)$ denotes a multiindex, $\alpha! = \alpha_1! \ldots \alpha_d!$, and $\partial_\xi^\alpha = \partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} \ldots \partial_{\xi_d}^{\alpha_d}$. The component $r_{-m-j}(x, \xi, \mu)$ is positive homogeneous of degree $-m - j$ in $(\xi, \mu^{1/m})$, i.e. $r_{-m-j}(x, \lambda\xi, \lambda^m \mu) = \lambda^{-m-j} r_{-m-j}(x, \xi, \mu)$ for any $\xi \neq 0$, $\lambda > 0$.

Assume that the angle π is a principal angle for a classical elliptic pseudodifferential operator Q of positive order, i.e. there exists $\varepsilon > 0$ so that for any $(x, \xi) \in T^* M \backslash 0$ the spectrum of the principal symbol $\sigma(Q)(x, \xi)$ does not intersect the solid angle $V_{\pi,\varepsilon} = \{z \in \mathbf{C} : Rez \leq 0, |Imz| \leq \varepsilon |Rez|\}$. It is a well known fact that at most a finite number of eigenvalues of the operator Q may be in $V_{\pi,\varepsilon}$. Therefore, if the operator Q is invertible, one can find a closed solid angle

$$V_{\theta,\varepsilon_1} = \{z \in \mathbf{C} : \theta - \varepsilon_1 \leq \arg(z) \leq \theta + \varepsilon_1\}, \quad \pi - \varepsilon < \theta < \pi + \varepsilon$$

that does not intersect the spectrum of Q. To simplify notations, without loss of generality, we assume that the angle $V_{\pi,\varepsilon}$ itself and the ball $B_\varepsilon(0) = \{z \in \mathbf{C} ||z| < \varepsilon\}$ do not intersect the spectrum of Q. In this case we say that π is an Agmon angle. In the case when π is not an Agmon angle, one can choose an Agmon angle θ from $V_{\pi,\varepsilon}$, and make all constructions with π replaced by θ.[1]

According to Seeley [Se 1] these conditions are more than sufficient to define the complex powers of Q. To do this denote by Γ the contour $\Gamma_1 \sqcup \Gamma_2 \sqcup \Gamma_3$ with

$$\Gamma_1 := \{z = \rho e^{i\pi} : \infty > \rho \geq \varepsilon_0\} \text{ (ray towards origin)}$$

[1]Note that the zeta-function itself depends on the choice of θ but the determinant is independent of this choice as long as $\pi - \varepsilon < \theta < \pi + \varepsilon$.

$$\Gamma_2 := \{z = \varepsilon_0 e^{i\theta} : \pi \geq \theta \geq -\pi\} \text{ (circle with clockwise orientation)}$$
$$\Gamma_3 := \{z = \rho e^{i\pi} : \varepsilon_0 \leq \rho < \infty\} \text{ (ray towards infinity)}$$

and for $s \in \mathbf{C}$, with $Res > d/m$ define $Q^{-s} : C^\infty(E) \to C^\infty(E)$ by the formula

$$Q^{-s} = \frac{1}{2\pi i} \int_\Gamma \mu^{-s}(\mu - Q)^{-1} d\mu. \tag{1.3}$$

Seeley [Se 1] proved that Q^{-s} is a pseudodifferential operator depending holomorphically on s, and it is of trace class when $Res > d/m$. For $\alpha \in C^\infty(M; \mathbf{C})$ and $Res > d/m$ one defines the generalized zeta function

$$\zeta_{\alpha,Q}(s) := \frac{1}{2\pi i} \int_\Gamma \mu^{-s} Tr(\alpha(\mu - Q)^{-1}) d\mu. \tag{1.4}$$

It is easy to see that $\zeta_q(s)$ defined in the introduction is equal to $\zeta_{1,\Delta_q}(s)$. The following result is implicit in [Se 1].

Theorem 1.1[Se 1]**.**
(1) Assume Q is a classical pseudodifferential operator which is elliptic and has π as an Agmon angle. If $\alpha \in C^\infty(M, \mathbf{C})$ then $\zeta_{\alpha,Q}(s)$ admits a meromorphic continuation to the entire s-plane. It has at most simple poles, and $s = 0$ is its regular point. The value of $\zeta_{\alpha,Q}(s)$ at $s = 0$ is given by

$$\zeta_{\alpha,Q}(0) = \int_M \alpha(x) I_d(x)$$

where $I_d(x)$ is a density on M. In a coordinate chart (φ, ψ), $I_d(x)$ is given by

$$I_d(x) = \frac{1}{m} \frac{1}{(2\pi)^d} \int_{|\xi|=1} d\xi \int_0^\infty Tr r_{-m-d}(x, \xi, -\mu) d\mu. \tag{1.5}$$

If Q is differential operator and $\dim M$ *is odd, then $I_d(x) \equiv 0$.*

(2) Assume $Q(t) : H^m(E) \to L^2(E)$ is a family of classical pseudodifferential operators of order m depending in C^r-fashion on a parameter t varying in an open set of $\mathbf{R}$ *(here $H^m(E)$ denotes the space of sections with derivatives up to order m in $L^2(E)$). Assume $Q(t)$ is elliptic and π is an Agmon angle for any t, uniformly in t.*[2] *Then $\zeta_{Q(t)}(s)$ is a family of holomorphic functions in the neighbourhood of $s = 0$, which depends in C^r-fashion on t.*

Theorem 1.1 (1) allows to introduce the ζ-regularized determinant of Q

$$\det Q := \exp\left(-\frac{d}{ds}\zeta_Q(s)\right)_{s=0} \tag{1.6}$$

Theorem 1.1 (2) implies that $\det Q(t)$ is C^r in t.

Let $Q(t) : C^\infty(E) \to C^\infty(E)$ be a family of classical pseudodifferential operators of order m depending on a parameter $t \in \mathbf{R}_+$, $\mathbf{R}_+ = [0, \infty)$.

[2]Uniformity in t means that there exists $\varepsilon > 0$ such that both the angle $V_{\pi,\varepsilon}$ and the ball $B_\epsilon(0)$ do not intersect the spectrum of $Q(t)$ for any t. Note that the condition of existence of an Agmon angle uniformly in t is not algebraic: it can not be verified by looking at the symbol of an operator-valued family. An algebraic condition is that there exists a principal angle, say π, uniformly in t. It may happen, however, that, as the parameter changes, the arguments of a finite number of eigenvalues of $Q(t)$ lying in $V_{\pi,\varepsilon}$ cover all possible angles. If such a situation occurs, the zeta-function as a continuous function of t can be defined only locally (in a small neighborhood of any t_0). Nevertheless, the determinant is a C^r-function of t.

Definition.

A family $Q(t)$ is a family of operators with parameter of order m and weight $\chi > 0$ if the following two conditions are met (see [Sh]):

(a) with respect to an arbitrary coordinate chart (φ, ψ) of $(M, E \xrightarrow{p} M)$, for any compact set $K \subset R^d$ and multiindices α and β there exists a constant $C = C_{\alpha,\beta,K}$ such that

$$|\partial_x^\alpha \partial_\xi^\beta q(x, \xi, t)| \leq C_{\alpha,\beta,k}(1 + |\xi| + |t|^{1/\chi})^{m-|\beta|} \tag{1.7}$$

where $q(x, \xi, t)$ denotes the symbol of $Q(t)$.

(b) The symbol $q(x, \xi, t)$ of Q(with respect to an arbitrary chart (φ, ψ)) admits an asymptotic expansion $\sum_{j\geq 0} q_{m-j}(x, \xi, t)$ with

$$q_{m-j}(x, \lambda\xi, \lambda^\chi t) = \lambda^{m-j} q_{m-j}(x, \xi, t) \tag{1.8}$$

for $\lambda > 0$, $\xi \in \mathbf{R}^d \backslash 0$, and $t \in \mathbf{R}_+$.

The principal symbol with parameter $q_m(x, \xi, t)$ is invariantly defined as a map $T_x^* M \times \mathbf{R}_+ \to End E_x$

Definition.

A family $Q(t)$ is called elliptic with parameter [Sh] if the principal symbol $q_m(x, \xi, t)$ of $Q(t)$ is invertible for all $(x, \xi, t) \in T^*M \times \mathbf{R}_+$ with $(\xi, t) \in T_x^*M \times \mathbf{R}_+$.

In that case one can construct a parametrix $R(\mu, t)$; this is a family of elliptic classical pseudodifferential operators depending on t and μ, $t \in \mathbf{R}_+, \mu \in C\backslash \cup_{x,\xi,t}$ Spec $q_m(x, \xi, t)$, $(x, \xi, t) \in T^*M \times \mathbf{R}_+$, representing an inverse of $(\mu - Q(t))$ up to smoothing operators whose symbol with respect to a chart (φ, ψ) is determined inductively by

$$r_{-m}(x, \xi, t, \mu) = (\mu - q_m(x, \xi, t))^{-1} \tag{1.1$'$}$$

$$r_{-m-j}(x, \xi, t, \mu) = (\mu - q_m(x, \xi, t))^{-1} \sum_{k=0}^{j-1} \sum_{|\alpha|+l+k=j} \frac{1}{\alpha!} \partial_\xi^\alpha q_{m-l}(x, \xi, t) \left(\frac{1}{i}\partial_x\right)^\alpha r_{-m-k}(x, \xi, t, \mu). \tag{1.2$'$}$$

The component $r_{-m-j}(x, \xi, t, \mu)$ is positive homogeneous of degree $(-m - j)$ in $\xi, \mu^{1/m}, t^{1/\chi}$. The angle π is said to be a principal angle for the family $Q(t)$ if there exists $\varepsilon > 0$ so that for any (x, ξ, t), $(\xi, t) \in T_x^*M \times \mathbf{R}_+$ the spectrum of the principal symbol $q_m(Q)(x, \xi, t)$ does not intersect the solid angle $V_{\pi,\varepsilon}$. The angle π is said to be an Agmon angle for the family $Q(t)$ if in addition $\text{Spec}Q(t) \cap (V_{\pi,\varepsilon} \cup B_\varepsilon(0)) = \emptyset$. In [BFK] Appendix, the following result concerning the complete asymptotic expansion for $t \to \infty$ of $\log \det Q(t)$ is proven

Theorem 1.2. *(*[BFK]*) Assume that $Q(t)$ is a family of pseudodifferential operators, elliptic with parameter of order m and weight χ. Assume that π is an Agmon angle for $Q(t)$. Then the function* $\log\det Q(t)$ *admits a complete asymptotic expansion for $t\to\infty$ of the form*

$$\log\det Q(t) \sim \sum_{j=-\infty}^{d} \bar{a}_j t^{j/\chi} + \sum_{j=0}^{d} \bar{b}_j t^{j/\chi}\log t \tag{1.9}$$

where $\bar{a}_j = \int_M a_j(x)dx, \bar{b}_j = \int_M b_j(x)dx$ are defined by smooth densities $a_j(x)$ and $b_j(x)$ on M which can be computed in terms of the symbol of $Q(t)$.

In particular, with respect to a coordinate chart, $a_0(x)$ is given by

(1.10)

$$\begin{aligned} a_0(x) &= \frac{\partial}{\partial s}|_{s=0}\frac{1}{(2\pi)^d}\left(\frac{1}{2\pi i}\int_{\mathbf{R}^d} d\xi \int_\Gamma d\mu\mu^{-s} Tr \quad r_{-m-d}(x,\xi,t=1,\mu)\right) \\ &= \frac{-1}{(2\pi)^d}\int_{R^d} d\xi \int_0^\infty Tr \quad r_{-m-d}(x,\xi,t=1,-\mu)d\mu. \end{aligned}$$

Let us consider now the case $\partial M \neq \emptyset$. For the purpose of this paper we only need to consider the Dirichlet problem for an elliptic differential operator of order 2.

Introduce the operator

$$Q_D : C^\infty_D(E) \to C^\infty(E)$$

where

$$C^\infty_D(E) =: \{u \in C^\infty(E) | \quad u|_{\partial M} = 0\}.$$

The ellipticity of Q insures that Spec Q_D is discrete. Assume that π is an Agmon angle [BFK]. In [Se 2] Seeley constructs a parametrix $R_D(\mu)$ for Q_D in a similar fashion as in the case $\partial M = \emptyset$, describing inductively the asymptotic expansion of its symbol. The only difference is that to each term in the symbol expansion (1.1-1.2) a term coming from the boundary conditions is added. These terms depend only on the symbol expansion of Q and its derivatives along the boundary ∂M. Having constructed a parametrix, Seeley [Se 2] introduces complex powers of Q_D (1.5) and the generalized zeta function $\zeta_{\alpha,Q_D}(s)$ (1.6). As a special case one obtains from Seeley's results [Se 2] the following

Theorem 1.1′. *Assume Q is an elliptic differential operator of order 2 so that Q_D has π as an Agmon angle. Then the function $\zeta_{\alpha,Q_D}(s)$ admits a meromorphic continuation to the entire s-plane. It has at most simple poles and $s=0$ is its regular point. The value of $\zeta_{\alpha,Q_D}(s)$ at $s=0$ is given by $\zeta_{\alpha,Q_D}(0) = \int_M \alpha(x)I_d(x,0) + \int_{\partial M}\alpha(x)B_d(x)$ where in a coordinate chart of $(M, E\to M)$, $I_d(x)$ is defined as in* (1.5)*. In a coordinate chart of $(\partial M, E|\partial M \to \partial M)$, $B_d(x)$ is given by a formula* [Se 2] *involving at most the first d terms of the symbol expansion of Q and its derivatives up to order d.*

One defines the ζ-regularized determinant of Q_D by

$$\det Q_D := \exp\left(-\frac{d}{ds}\zeta_{Q_D}(s)\right)_{s=0} \tag{1.6′}$$

As mentioned in [BFK, A.19,p64], a result analogous to Theorem 1.2 holds:

Theorem 1.2′. *Assume that $Q(t)$ is a family of differential operators, elliptic with parameter, of order 2 and weight $\chi > 0$. Further assume that π is a principal angle for $Q(t)_D$ and Spec $Q(t)_D \cap (V_{\pi,\varepsilon} \cup B_\varepsilon(0)) = \emptyset$ for some $\varepsilon > 0$. Then the function $\log\det Q(t)_D$ admits a complete asymptotic expansion for $t \to \infty$ of the form*

$$\log\det Q(t)_D \sim \sum_{j=-\infty}^{d} (\bar{a}_j + \bar{a}_j^b) t^{j/\chi} + \sum_{j=0}^{d} (\bar{b}_j + \bar{b}_j^b) t^{j/\chi} \log t$$

where $\bar{a}_j$ and $\bar{b}_j$ are given as in Theorem 1.2. The quantities $\bar{a}_j^b$ and $\bar{b}_j^b$ are contributions from the boundary conditions and are of the form

$$\bar{a}_j^b = \int_{\partial M} a_j^b(x); \bar{b}_j^b = \int_{\partial M} b_j^b(x). \tag{1.11}$$

In a coordinate chart of $(\partial M, E|_{\partial M} \to \partial M)$, the densities $a_j^b(x), b_j^b(x)$ are given by a formula involving the terms in the symbol expansion of $Q(t)$ and its derivatives.

Next we recall a Mayer-Vietoris type formula for determinants [BFK]. We restrict ourselves to the case needed in this paper. Assume that $\partial M^d = \emptyset$ and let Γ be a smooth hypersurface in M^d. Consider an elliptic differential operator Q of order 2, $Q : C^\infty(E) \to C^\infty(E)$. Denote by M_Γ the manifold whose interior is $M \backslash \Gamma$ and whose boundary is $\partial M_\Gamma = \Gamma^+ \sqcup \Gamma^-$, and let $E_\Gamma \to M_\Gamma$ be the pull back of $E \to M$. Consider $Q_\Gamma : C^\infty(E_\Gamma) \to C^\infty(E_\Gamma)$ with Dirichlet boundary conditions, and assume that π is a principal and Agmon angle for both Q and Q_Γ. In [BFK] we have introduced the Dirichlet to Neumann operator R_{DN} associated to the vector field X along Γ which is transversal to Γ. This operator is defined as the composition

$$C^\infty(E|\Gamma) \xrightarrow{\Delta_{ia}} C^\infty(E|\Gamma^+) \oplus C^\infty(E|\Gamma^-) \xrightarrow{P_D} C^\infty(E_\Gamma) \xrightarrow{N}$$
$$C^\infty(E|\Gamma^+) \oplus C^\infty(E|\Gamma^-) \xrightarrow{\Delta_{\text{iff}}} C^\infty(E|\Gamma)$$

where $\Delta_{ia}(f) = (f, f)$ is the diagonal operator, P_D in the Poisson operator associated to Q_Γ, N is the operator induced by the vector field X and Δ_{iff} is the difference operator $\Delta_{\text{iff}}(f^+, f^-) = f^+ - f^-$. In [BFK], Proposition 3.2 and Theorem A the following result has been proved

Theorem 1.3. [BFK]

(1) R_{DN} is an invertible elliptic classical pseudodifferential operator of order 1. In a coordinate chart of $(\Gamma, E|\Gamma \to \Gamma)$ the symbol of R_{DN} has an expansion whose terms depend only on the terms of the expansion of the symbol of Q in an arbitrary small neighbourhood of Γ and their derivatives as well as on the vector field X along Γ.

(2) Denote by $x = (x', w)$ coordinates in a collar neighbourhood of Γ such that x' are coordinates for Γ, and let X the vector field given by $\partial/\partial w$. The principal symbol $\sigma(R_{DN}^{-1})(x', \xi')$ of R_{DN}^{-1} can be computed in these coordinates [BFK, (4.6)] *in terms of the principal symbol $\sigma(Q^{-1})(x', w', \xi', \eta)$ of Q^{-1},*

$$\sigma(R_{DN}^{-1})(x', \xi') = \frac{1}{2\pi} \int_{\mathbf{R}} \sigma(Q^{-1})(x', 0, \xi', \eta) d\eta.$$

(3) In the case when π is a principal angle for R_{DN} one has

$$\det(Q_D) = \bar{c}\det(Q_\Gamma)\det R_{DN}$$

where

$$\bar{c} = \exp\left\{\int_\Gamma c(x)\right\},$$

and the density $c(x)$, when expressed in a coordinate chart of $(\Gamma, E|\Gamma \to \Gamma)$, depends only on the first d terms of the symbol expansion of Q and their derivatives in an arbitrary small neighbourhood of Γ as well as on the vector field X.

(4) Assume that instead of a single operator Q,there is a family $Q(t) : C^\infty(E) \to C^\infty(E)$ of differential operators of order 2 with parameter t of weight χ so that $Q(t)$ is elliptic and invertible for each t. Introduce, as above $Q(t)_\Gamma$, $R_{DN}(t)$ and assume that $SpecQ(t) \cap (V_{\pi,\varepsilon} \cup B_\varepsilon) = \emptyset$, $Spec\ Q(t)_\Gamma \cap (V_{\pi,\varepsilon} \cup B_\varepsilon) = \emptyset$ for some $\varepsilon > 0$ and that π is a principal angle for $Q(t)$ and $Q(t)_\Gamma$. Then $R_{DN}(t)$ is an invertible family of pseudodifferential operators with parameter [BFK,3.13] *of order 1 and weight χ.*

Now we review the analysis of the Witten complex developed by Helffer and Sjöstrand [HS].

As we have seen in the introduction, the system (M, ρ, h, g, μ) provides a cochain complex $(C^q(M, \rho, \mathcal{T}, \mu), \delta)$ with $\mathcal{T} = (h, g)$ for which we have introduced an orthonormal basis $E_{q;i,r}$ $(1 \le i \le m_q; 1 \le r \le l)$. Write $\delta_q : C^q(M, \rho, \mathcal{T}) \to C^{q+1}(M, \rho, \mathcal{T})$ in this basis

$$\delta_q(E_{q;ir}) = \sum_{\substack{1\le i'\le m_{q+1}\\ 1\le r'\le l}} \gamma_{q;ir,i'r'} E_{q+1;i'r'}. \tag{1.12}$$

In [BZ], Bismut-Zhang have verified that the analysis of $\Delta_q(t)$, done in the case where $V = \mathbf{R}$ and $\mathcal{E}$ is the trivial line bundle by Helffer-Sjöstrand, can be carried out in the case where $\dim V \ge 1$, ρ is an arbitrary representation of $\pi_1(M)$, and μ is an arbitrary Hermitian structure for $\mathcal{E} \to M$. We begin by reviewing the t-asymptotics of an orthonormal basis $a_{q;ir}(t)$ $(1 \le i \le m_q; 1 \le r \le l)$ of $\Lambda^q(M; \mathcal{E})_{sm}$ as constructed by Helffer-Sjöstrand.

Denote by U_{qj} a small connected neighbourhood of a critical point $x_{q;j}$ and by $\hat{U}_{qj}$ the component of $p^{-1}(U_{qj})$ containing $\hat{x}_{q;j}$. According to hypothesis (H), which is supposed to hold throughout the remaining of this section, we can find coordinates $x = (x_1, \dots, x_d)$ in $\hat{U}_{q;j}$ so that $\hat{h}$ is given by

$$\hat{h}(x) = q - (x_1^2 + \dots x_q^2)/2 + (x_{q+1}^2 + \dots x_d^2)/2,$$

the Riemannian metric g given by $g_{ij} = \delta_{ij}$, and the Hermitian structure μ is given on $\hat{U}_{qj} \times V$ by $\mu_{ij} = \delta_{ij}$. The forms $a_{q;jr}$ are concentrated in U_{qj} as the following estimates show; they are versions of much more refined estimates due to [HS].

Theorem 1.4. *(*[HS, Proposition 1.7] [BZ, Theorem 8.15]*) There exist $\eta > 0$ and $C > 0$ such that, for t sufficiently large, $1 \le r \le l$,*

$$\sup_{x\in M\setminus U_{qj}} ||a_{q;jr}(x,t)|| \le Ce^{-\eta t}.$$

Similar estimates hold for the derivatives of $a_{q;jr}(x,t)$.

Recall that $W^-_{q;j}$ denotes the unstable manifold of the critical point $\hat{x}_{q;j}$ with respect to grad $_g\hat{h}$. By choosing U_{qj} sufficiently small we may assume that $\hat{U}_{qj} \cap W^-_{q;j'} = \emptyset$ for $j \neq j'$. When expressed in the coordinates introduced above, $a_{q;jr}$ admits the following expansion in t on $\hat{U}_{qj} \cap W^-_{q;j}$:

Theorem 1.5. *(*[HS, Theorem 2.5] [BZ, Theorem 8.27]*) On* $\hat{U}_{qj} \cap W^-_{q;j}$

$$a_{q;jr}(t) = (t/\pi)^{d/4} e^{-t|x|^2/2}(1 + O(\frac{1}{t}))dx_1 \wedge \cdots \wedge dx_q \bar{e}_{q;jr} \tag{1.13}$$

with $\bar{e}_{q;jr}$ *defined as in introduction.*

Moreover we need

Theorem 1.6. *(*[HS, Theorem 3.1, Proposition 3.3], [BZ, Theorem 8.30]*) The coefficients* $\eta_{q;ir,i'r'}(t)$ *in the representation*

$$d_q(t)a_{q;ir}(t) = \sum_{\substack{1\leq i'\leq m_{q+1}\\ 1\leq r'\leq l}} \eta_{q;ir,i'r'} a_{q+1;i'r'}(t) \tag{1.14}$$

satisfy, for t sufficiently large, the following estimate

$$\eta_{q;ir,i'r'}(t) = \left(\gamma_{q;ir,i'r'}(t/\pi)^{1/2} + O(1)\right) e^{-t}$$

where the $\gamma_{q;ir,i'r'}$ *are defined in* (1.12).

We conclude this auxiliary section with the following application of the above results.

First, recall that by de Rham's theory the linear isomorphism (see Introduction) $r_q : Ker\Delta_q \rightarrow Ker\underline{\Delta}_q$are induced by the linear maps $\sigma_q : \mathcal{Z}^q(M;\mathcal{E}) \rightarrow Z^q$ where $\mathcal{Z}^q(M;\mathcal{E})$ denotes the space of closed q forms $\mathcal{Z}^q(M;\mathcal{E}) \subset \Lambda^q(M;\mathcal{E})$, and Z^q is the space of cocycles $Z^q \subset C^q$. To define $\sigma_q(a)$ for $a \in \mathcal{Z}^q(M;\mathcal{E})$, take the pull back of a on $\hat{M}$, view it as a $\pi_1(M)$-invariant form in $\Lambda^q(\hat{M},V)$, and define $\sigma_q(a) \in C^q$ by the formula

$$\sigma_q(a)(\hat{x}_{q;j}) = \int_{W^-_{q;j}} a.$$

Here we use that, in view of hypothesis (H), c.f [L], the unstable manifolds $W^-_{q;j}$ *'s* provide a cell decomposition of $\hat{M}$.

Corollary 1.7.

$$\sigma_q(e^{ht}a_{q;jr}(t)) = (t/\pi)^{(d-2q)/4} e^{tq}(E_{q;jr} + O(\frac{1}{t}))$$

Proof.

It is to show that for any $\hat{x}_{q;j'}$

$$\int_{W^-_{q;j'}} a_{q;jr}(t)e^{ht} = (t/\pi)^{(d-2q)/4}e^{tq}(\delta_{jj'}\bar{e}_{q;jr} + O\left(\frac{1}{t}\right)).$$

First, note that, due to Theorem 1.4 and to the choice of $U_{qj'}$, it suffices to consider the case $j' = j$. Moreover, it suffices to estimate

$$\int_{W^-_{q;j}\cap U_{qj}} a_{q;jr}(t)e^{ht}.$$

Note that on $W^-_{q,j} \cap U_{qj}$, the function e^{ht} is at the form

$$e^{ht} = e^{qt}e^{-t(\sum_1^q x_k^2)/2}.$$

By Theorem 1.5, we conclude that

$$\begin{aligned}\int_{W^-_{q;j}\cap U_{qj}} a_{q;jr}(t)e^{ht} &= (t/\pi)^{d/4}e^{qt}\int_{W^-_{q;j}\cap U_{qj}} e^{t\sum_1^q x_k^2}(1+O(1/t))dx_1\wedge\dots dx_q.\bar{e}_{q;jr}\\ &= e^{qt}(t/\pi)^{d/4}(t/\pi)^{-q/2}(\bar{e}_{q;jr}+O(1/t)).\end{aligned}$$

2. Asymptotic expansion of Witten's deformation of the analytic torsion

In this section we prove Theorem A. Throughout we assume the (M, ρ, h, g, μ) satisfies (H) as defined in the introduction.

We begin by deriving an alternative formula for the analytic torsion (cf[Ch]). The space of q-forms $\Lambda^q(M;\mathcal{E})$ can be decomposed

$$\Lambda^q(M;\mathcal{E}) = \Lambda_t^{+,q}(M;\mathcal{E}) \oplus \Lambda_t^{-,q}(M;\mathcal{E}) \oplus \mathcal{H}_t^q$$

where

$$\begin{aligned}\Lambda_t^{+,q}(M;\mathcal{E}) &:= d_{q-1}(t)\Lambda^{q-1}(M;\mathcal{E}); \Lambda_t^{-,q}(M;\mathcal{E}) := d_q(t)^*\Lambda^{q+1}(M;\mathcal{E});\\ \mathcal{H}_t^q &:= \{\omega \in \Lambda^q(M;\mathcal{E}) : \Delta_q(t)\omega = 0\}\end{aligned}$$

Note that the spaces $\Lambda_t^{\pm,q}(M;\mathcal{E})$ are invariant with respect to the Laplacians $\Delta_q(t)$. Therefore the zeta function $\zeta_q(t,s)$ corresponding to $\Delta_q(t)$ can be written as a sum

$$\zeta_q(t,s) = \zeta_q^+(t,s) + \zeta_q^-(t,s)$$

where $\zeta_q^\pm(t,s)$ is the zeta function of the operator $\Delta_q(t)$, restricted to $\Lambda_t^{\pm,q}(M;\mathcal{E})$. The operator $d_q(t)$ maps the space $\Lambda_t^{-,q}(M;\mathcal{E})$ isomorphically onto $\Lambda_t^{+,q+1}(M;\mathcal{E})$, and intertwines $\Delta_q(t)$ and $\Delta_{q+1}(t)$. This implies that

$$\zeta_q^-(t,s) = \zeta_{q+1}^+(t,s)$$

which is used to write the zeta function $\zeta(t,s)$, defined by

$$\zeta(t,s) := \sum_{q=0}^{d} (-1)^q q \zeta_q(t,s),$$

in the following way:

$$\begin{aligned}\zeta(t,s) &= \sum_{q=0}^{d} (-1)^q q (\zeta_q^+(t,s) + \zeta_q^-(t,s)) \\ &= \sum_{q=0}^{d} (-1)^q q \zeta_q^+(t,s) - \sum_{q=0}^{d} (-1)^q (q-1) \zeta_q^+(t,s) \\ &= \sum_{q=0}^{d} (-1)^q \zeta_q^+(t,s) = -\sum_{q=0}^{d} (-1)^q \zeta_q^-(t,s).\end{aligned}$$

Thus,

$$\begin{aligned}\log T(h,t) &= \frac{1}{2}\frac{d}{ds}\zeta(t,s)_{s=0} \\ &= \frac{1}{2}\sum_{q=0}^{d} (-1)^q \log\det \Delta_q^-(t) \\ &= \frac{1}{2}\sum_{q=0}^{d} (-1)^{(q+1)} \log\det \Delta_q^+(t)\end{aligned}$$

where $\Delta_q^{\pm}(t)$ is the restriction of $\Delta_q(t)$ to $\Lambda_t^{\pm,q}(M;\mathcal{E})$. Note that

$$\Delta_q^+(t) = d_{q-1}(t) d_{q-1}(t)^*; \Delta_q^-(t) = d_q(t)^* d_q(t).$$

Let

$$W_q^{\pm}(t) := \log\det \Delta_q^{\pm}(t).$$

Our first goal is to compute the variation $\dot{W}_q^{\pm}(t)$ of $W_q^{\pm}(t)$ with respect to t $\left(\cdot = \frac{d}{dt}\right)$ by using the following well known variational formula for determinants [RS]: Let $Q(t)$ be elliptic pseudodifferential operator of order m with π a principal angle and Spec $Q(t) \cap (V_{\pi,\varepsilon} \cup B_\varepsilon(0)) = \emptyset$ for some $\varepsilon > 0$. Further assume that $Q(t)$ is continuously differentiable when considered as a function with values in the space of linear operators $H^m(M,\mathcal{E}) \to L^2(M,\mathcal{E})$, where $H^m(M,\mathcal{E})$ denotes the Sobolev space of L^2-sections with derivatives up to order m in $L^2(E)$. Then $Tr\dot{Q}(t)Q(t)^{-s-1}$ is holomorphic in s for $Res > d/m$ and has a meromorphic extension to the whole complex s-plane with the point $s=0$ being either a regular point or a simple pole. Denoting by $F.p._{s=0}\dot{Q}(t)Q(t)^{-s-1}$ the 0'th order term in the Laurent expansion of $Tr\dot{Q}(t)Q(t)^{-s-1}$ at $s=0$, the variational formula for $\log\det Q(t)$ takes the form

$$\frac{d}{dt}\log\det Q(t) = F.p._{s=0} Tr\dot{Q}(t)Q(t)^{-s-1}. \tag{2.1}$$

To compute $\dot{W}_q^{\pm}(t)$ consider the operator $\Delta_q^+(t)$ which equals the restriction of

$$\begin{aligned}d_{q-1}(t)d^*_{q-1}(t) = e^{-th}d_{q-1}(t)e^{2th}d^*_{q-1}(t)e^{-th}\\ = e^{th}(d_{q-1}(t)d^*_{q-1}(t) + 2tdh\wedge d^*_{q-1}(t))e^{-th}\end{aligned}$$

to the space

$$\Lambda_t^{+,q} = d_{q-1}(t)\Lambda^{q-1} = e^{-th}d_{q-1}t\Lambda^{q-1} = e^{-th}\Lambda^{+,q}.$$

Note that the operator

$$e^{th}d_{q-1}(t)d^*_{q-1}(t)e^{-th} = e^{2th}(d_{q-1}d^*_{q-1} + 2tdh\wedge d^*_{q-1})e^{-2th},$$

when restricted to $\Lambda_t^{+,q}$, is isospectral to $\Delta_q^+(t)$ and therefore, with [A,B] denoting the commutator of two operators A and B we obtain

$$\begin{aligned}\dot{W}_q^+(t) &= F.p._{s=0}Tr\frac{d}{dt}(e^{2th}(d_{q-1}d^*_{q-1} + 2tdh\wedge d^*_{q-1})e^{-2th})\\ &\cdot(e^{2th}(d_{q-1}d^*_{q-1} + 2tdh\wedge d^*_{q-1})e^{-2th})^{-s-1}\\ &= F.p._{s=0}Tr\{2[h, e^{2th}d_{q-1}d^*_{q-1} + 2tdh\wedge d^*_{q-1})e^{-2th})]\\ &+ 2e^{2th}dh\wedge d^*_{q-1}e^{-2th}\}(e^{2th}(d_{q-1}d^*_{q-1} + 2tdh\wedge d^*_{q-1})e^{-2th})^{-s-1}\\ &= 2F.p._{s=0}Tre^{2th}dh\wedge d^*_{q-1}e^{-2th}(e^{2th}(d_{q-1}d^*_{q-1} + 2tdh\wedge d^*_{q-1})e^{-2th})\end{aligned}$$

where we used that $Tr[A,B]B^{-s-1} = 0$. Note that

$$(e^{2th}(d_{q-1}d^*_{q-1} + 2tdh\wedge d^*_{q-1})e^{-2th})^{-s-1} = (e^{th}\Delta_q^+(t)e^{-th})^{-s-1}$$

and therefore

$$\begin{aligned}\dot{W}_q^+(t) &= 2F.p._{s=0}Tre^{th}dh\wedge d^*_{q-1}(t)e^{-th}(e^{th}(d_{q-1}d^*_{q-1} + 2tdh\wedge d^*_{q-1}(t))e^{-th})^{-s-1}\\ &= 2F.p._{s=0}Trdh\wedge d^*_{q-1}(t)(d_{q-1}(t)d^*_{q-1}(t))^{-s-1}\\ &= 2F.p._{s=0}Trdh\wedge d_{q-1}(t)^{-1}(d_{q-1}(t)d_{q-1}(t)^*)^{-s}\end{aligned}$$

where the operator $d_{q-1}(t)^{-1}$ is defined on $\Lambda_t^{+,q}$,

$$d_{q-1}(t)^{-1} : \Lambda_t^{+,q} \to \Lambda_t^{-,q-1}.$$

Substituting

$$d_{q-1}(t)hd_{q-1}(t)^{-1} = h + dh\wedge d_{q-1}(t)^{-1}$$

leads to

$$\dot{W}_q^+(t) = 2F.p._{s=0}Trd_{q-1}(t)hd_{q-1}(t)^{-1}(\Delta_q^+(t))^{-s} - 2F.p._{s=0}Trh(\Delta_q^+(t)^{-s}.$$

As the operator $d_{q-1}(t)$ intertwines $\Delta_q^+(t)$ and $\Delta_{q-1}^+(t)$ one concludes that

$$Trd_{q-1}(t)hd_{q-1}(t)^{-1}\Delta_q^+(t)^{-s} = Trh\Delta_{q-1}^-(t)^{-s},$$

and obtains

(2.2) $$\dot W_q^+(t) = 2F.p._{s=0} Trh\Delta_{q-1}^-(t)^{-s} - 2F.p._{s=0} Trh\Delta_q^+(t)^{-s}.$$

From (2.2) we derive the following variational formula for $\log T(h,t)$:

$$\begin{aligned}\frac{d}{dt}\log T(h,t) &= \sum_{q=0}^{d}(-1)^{q+1}F.p._{s=0}Trh\Delta_{q-1}^-(t)^{-s} \\ &- \sum_{q=0}^{d}(-1)^{q+1}F.p._{s=0}Trh\Delta_q^+(t)^{-s} \\ &= \sum_{q=0}^{d}(-1)^{q}F.p._{s=0}Trh\Delta_q(t)^{-s}\end{aligned}$$

where the operator $\Delta_q(t)^{-s}$ is defined as the $(-s)th$ power of $\Delta_q(t)$ on $\Lambda_t^{+,q}\oplus\Lambda_t^{-,q}$ and 0 on $\mathcal{H}_t^q$. Denote by $P_q(t)$ the orthogonal projector onto $\mathcal{H}_t^q$, the space of t-harmonic q-forms. Note that

$$\lim_{\varepsilon\to 0} F.p._{s=0}Trh(\Delta_q(t)+\varepsilon)^{-s} = F.p._{s=0}Trh(\Delta_q(t))^{-s} + TrP_q(t)hP_q(t).$$

We point out that according to Theorem 1.1, $F.p._{s=0}Trh(\Delta_q(t)+\varepsilon)^{-s}$ can be computed in local charts and is identically 0 in the case M is of odd dimension. We summarize the above considerations in the following

Lemma 2.1.

$$\begin{aligned}(i)\frac{d}{dt}\log T(h,t) &= \sum_{q=0}^{d}(-1)^q \lim_{\varepsilon\to 0}F.p._{s=0}Trh(\Delta_q(t)+\varepsilon)^{-s} \\ &+ \sum_{q=0}^{d}(-1)^{q+1}TrP_q(t)hP_q(t).\end{aligned}$$

(ii)If M is of odd dimension,

$$\frac{d}{dt}\log T(h,t) = \sum_{q=0}^{d}(-1)^{q+1}TrP_q(t)hP_q(t).$$

Next we want to express the terms $TrP_q(t)hP_q(t)$ in a more explicit way. Let $w_1,\dots,w_\beta$ be a basis in the space $\mathcal{H}^q(M;\mathcal{E})$ of harmonic q-forms where $\beta=\beta_q=\dim\mathcal{H}^q$ is the q'th Betti number. Introduce the $\beta\times\beta$ matrix $K_q(t)=(\mathcal{H}_{ij}(t))$ with

(2.3) $$\mathcal{H}_{ij}(t) = < P_q(t)e^{-th}w_i, e^{-th}w_j > = < P_q(t)e^{-th}w_i, P_q(t)e^{-th}w_j > .$$

We claim that the matrix $K_q(t)$ is nonsingular for all values of t. To see it, note that the forms $w_j(t)=e^{-tq}w_j$ are t-closed, i.e. are elements in $\Lambda_t^{+,q}\oplus\mathcal{H}_t^q$. If $K_q(t)$ were singular a nontrivial linear combination

$$\sum_1^\beta a_jw_j(t) = e^{-th}(\sum_1^\beta a_jw_j)$$

would belong to the space $\Lambda_t^{+,q}$ and, therefore, $\sum_1^\beta a_jw_j$ would belong to $\Lambda^{+,q}$ which contradicts the choice of the $w_j's$. Next we show the following

Lemma 2.2.

$$TrP_q(t)hP_q(t) = -1/2\frac{d}{dt}\log|\det K_q(t)| \tag{2.4}$$

Proof. Denote $P_q(t)w_j(t)$ by $\eta_j(t)$ and let $\eta_j^*(t)$ be the basis which is biorthogonal to $\eta_j(t)$, i.e.

$$< \eta_i^*(t), \eta_j(t) >= \delta_{ij}. \tag{2.5}$$

Expressing the forms $\eta_j^*(t)$ with respect to $\eta_i(t)(1 \le i \le \beta)$ one obtains

$$\eta_j^*(t) = \sum_{i=1}^{\beta} a_{ji}(t)\eta_i(t)$$

where, by (2.2), $A(t) := (a_{ji}(t))$ is given by $K_q(t)^{-1}$. This leads to

$$TrP_q(t)hP_q(t) = \sum_{j=1}^{\beta} < h\eta_j, \eta_j^* >= \sum_{1\le i,j\le\beta} a_{ji} < h\eta_j, \eta_i >= TrK_q^{-1}\Sigma \tag{2.6}$$

where $\Sigma = \Sigma(t) = (\sigma_{ij}(t))$ is the matrix with $\sigma_{ij}(t) =< h\eta_i(t), \eta_j(t) >$. Formula (2.4) follows, once we show that

$$\Sigma(t) = -\frac{1}{2}\dot{K}_q(t). \tag{2.7}$$

To verify (2.7) note that

$$\eta_i(t) = w_i(t) + e^{-th}\theta_i(t) = e^{-th}w_i + e^{-th}\theta_i(t) \in \mathcal{H}_t^q \oplus \Lambda_t^{+,q}$$

where $\theta_i(t) \in \Lambda^{+,q}$. Therefore $-\dot{\eta}_i(t) = h\eta_i(t) + e^{-th}\dot{\theta}_i(t)$ and $< \dot{\eta}_i(t), \eta_j(t) >= - < h\eta_i(t), \eta_j(t) >$. Hence we obtain

$$\dot{\mathcal{H}}_{ij}(t) =< \dot{\eta}_i(t), \eta_j(t) > + < \eta_i(t), \dot{\eta}_j(t) >= -2 < h\eta_i(t), \eta_j(t) > .$$

which is (2.7). $\square$

The right hand side of (2.4) has a simple geometric interpretation. Denote by $C_q(t)$ the canonical isomorphism $C_q(t) : \mathcal{H}_t^q \to \mathcal{H}^q$ which maps a t-harmonic form η to the unique harmonic form w cohomologous to $e^{th}\eta$. Note that $C_q(t)\eta_j(t) = w_j$ and, therefore we conclude that

$$\log V_q(t) = \frac{1}{2}\log|\det K_q(t)| \tag{2.8}$$

where $V_q(t)$ denotes the volume of the parallepiped in $\mathcal{H}_t^q$ spanned by the forms $\eta_j(t)$. Note that $V_q(0) = 1$. Introduce the positive numbers $C(t) > 0$ by setting

$$\log C(t) := \sum_{q=0}^{d}(-1)^q \log V_q(t). \tag{2.9}$$

Integrate Lemma 2.1 (i) with respect to t we obtain, combined with Lemma 2.2

$$\log T(h,t) - \log T = \log C(t) + \sum_{q=0}^{d}(-1)^q \lim_{\varepsilon\to 0}\int_0^t F.p._{s=0}Trh(\Delta_q(t)+\varepsilon)^{-s}. \tag{2.10}$$

To prove Theorem A we provide an asymptotic expansion for each of the terms on the right hand side of (2.10).

Lemma 2.3.
(i) For d odd,

$$\lim_{\varepsilon \to 0} \int_0^t F.p._{s=0} Trh(\Delta_q(t) + \varepsilon)^{-s} = 0 \quad (0 \leq q \leq d)$$

(ii) For d even

$$\lim_{\varepsilon \to 0} \int_0^t F.p._{s=0} Trh(\Delta_q(t) + \varepsilon)^{-s}$$

is a polynormal in t of degree at most $d+1$, $\sum_{j=o}^{d+1} p_{q;j} t^j$ where the coefficients $p_{q;j}$ can be computed in local charts (see Theorem 1.1). Moreover

$$p_{q;0} = 0 \quad (0 \leq q \leq d). \tag{2.11}$$

Proof. Note that (i) is already contained in Lemma 2.1. Concerning (ii), recall from Theorem 1.1. that for $\varepsilon > 0$, $F.p._{s=0} Trh(\Delta_q(t) + \varepsilon)^{-s}$ is given by $\int_M h(x) I_d(x, t, \varepsilon) dx$ where, in a local chart, $I_d(x, t, \varepsilon)$ can be computed by a formula of the type (1.7), which is obtained from the d-th term in the expansion of the symbol of the resolvent of $\Delta_q(t) + \varepsilon$. The symbol of $(\Delta_q(t) + \varepsilon)$, in a chart is given by

$$a_2(x, \xi) + t^2 ||\nabla h||^2 Id + a_1(x, \xi) + tL_q + \varepsilon$$

where L_q is a certain multiplication operator acting on $\Lambda^q(T^*M) \otimes V$. The symbol expansion of the resolvent, which we want to consider, is constructed inductively as follows

$$r_{-2}(x, \xi, \lambda, t, \varepsilon) = (\lambda - a_2(x, \xi))^{-1}$$

and for $j \geq 1$

$$\begin{aligned} r_{-2-j}(x, \xi, \lambda, t, \varepsilon) = (a_2(x, \xi) - \lambda)^{-1} & \sum_{\substack{1 \leq |\alpha| \leq 2 \\ l + |\alpha| = j}} \frac{1}{\alpha!} \\ & \partial_\xi^\alpha a_2(x, \xi) \left(\frac{1}{i} \partial_x \right)^\alpha r_{-2-l}(x, \xi, \lambda, t, \varepsilon) \\ & + (a_2((x, \xi) - \lambda)^{-1}) \sum_{\substack{0 \leq |\alpha| \leq 1 \\ l + |\alpha| = j}} \partial_\xi^\alpha a_1(x, \xi) \left(\frac{1}{i} \partial_x \right)^\alpha r_{-2-l}(x, \xi, \lambda, t, \varepsilon) \\ & + (a_2(x, \xi) - \lambda)^{-1} \sum_{2 + l = j} (t^2 ||\nabla h||^2 + tL_q + \varepsilon) r_{-2-l}(x, \xi, \lambda, t, \varepsilon). \end{aligned}$$

This shows that

$$r_{-2-d}(x, \xi, \lambda, t,) := \lim_{\varepsilon \to 0} r_{-2-d}(x, \xi, \lambda, t, \varepsilon)$$

is a polynormal in t of degree at most d. Integrating in t we conclude that (ii) holds. $\square$

Recall that we have introduced in section O the volumes $V_q = |\det r_q^{-1}|$ and we denote by β_q the q-th Betti number, $\beta_q = \dim \mathcal{H}^q$.

Lemma 2.4.
(i)

$$\log\det C_q(t) = \log V_q + q\beta_q t + \beta_q \frac{d-2q}{4}\log(t/\pi) + 0(1/\sqrt{t}).$$

(ii)

$$\log C(t) = -\sum_{q=0}^{d}(-1)^q \log V_q + \left(\sum_{q=0}^{d}(-1)^{q+1}\frac{d-2q}{4}\beta_q\right)\log t/\pi$$
$$+ (\sum_{q=0}^{d}(-1)^{q+1}q\beta_q)t + O(1/\sqrt{t}).$$

Proof. Summing with respect to q, statement (ii) follows from (i), together with $\log V_q(t) = -\log\det C_q(t)$. To compute $\det C_q(t)$ we express $C_q(t)$ with respect to orthomormal bases of $\mathcal{H}_q^t$ and $\mathcal{H}_q$. We decompose $C_q(t) = r_q^{-1}\circ C_q'(t)$. Where $C_q'(t) : \mathcal{H}_t^q \to Ker\underline{\Delta}_q$. To describe $C_q'(t)$ we choose a fixed orthomormal basis of $Ker\underline{\Delta}_q$, $\mathcal{H}_{qj}(1 \le j \le \beta_q)$ and express it with respect to the orthomormal basis $E_{q;ir}$ of C^q constructed in the introduction. One has

$$\mathcal{H}_{qj} = \sum_{i,r} b_{jir}E_{q;ir}(1 \le j \le \beta_q). \tag{2.12}$$

In section 1 we introduced an orthonormal basis of q-forms $a_{q;ir}(t)$ in $\Lambda^q(M,\mathcal{E})_{sm}$ constructed by Helffer-Sjöstrand. Define

$$w'_{qj}(t) = \sum_{i,r} b_{jir}a_{q;ir}(t) \tag{2.13}$$

where the coefficients $b_{jir}$are the same as in (2.12). To prove that $w'_{qj}(t)$ is close to a t-harmonic q-form we write $w'_{qj}(t) = P_q(t)(w'_{qj}(t)) + w''_{qj}(t)$ and estimate $w''_{qj}(t)$ by

$$w''_{qj}(t) = \Delta_q(t)^{-1}(\Delta_q(t)w'_{qj}(t)) = O(1/\sqrt{t})$$

where we used that, according to Theorem 1.6, the operator $\Delta_q(t)$ when expressed in the orthonormal basis $a_{q;jr}$ is given by

$$\left(\eta^{(q)}(t)\right)^T\eta^{(q)}(t) + \eta^{(q-1)}(t)\left(\eta^{(q-1)}(t)\right)^T$$
$$= \frac{t}{\pi}e^{-2t}\left\{(\gamma^{(q)})^T\gamma^{(q)} + \gamma^{(q-1)}(\gamma^{(q-1)})^T + O\left(\frac{1}{t^{1/2}}\right)\right\}.$$

Therefore, we can find an orthonormal basis $w_{qj}(t)$ of $\mathcal{H}_t^q$ such that

$$w_{qj}(t) - w'_{qj}(t) = O(1/\sqrt{t}).$$

Note that $C'_q(t)(w_{qj}(t)) = \pi_q(\sigma_q(t))(e^{th}w_{qj}(t))$ where the map $\sigma_q(t)$ has been introduced at the end of section 1 and π_q denotes the orthogonal projection from the space $\mathcal{Z}^q$ of closed q-cocycles into $Ker\underline{\Delta}_q$. Using that $\pi_q(\mathcal{H}_{qj}) = \mathcal{H}_{qj}$, we obtain

$$C'_q(t)w'_{qj}(t) = (t/\pi)^{(d-2q)/4}e^{tq}(\mathcal{H}_{qj} + O(1/\sqrt{t})) \tag{2.14}$$

and, by estimating $\sigma_q(t)(e^{th}w_{qj}(t) - e^{th}w'_{qj}(t))$ in a straightforward way,

$$C'_q(t)(w_{qj}(t) - w'_{qj}(t)) = t^{-q/2}e^{tq}O(1/\sqrt{t}). \tag{2.15}$$

From (2.14) and (2.15) we conclude that, for t sufficiently large

$$\det C'_q(t) = (t/\pi)^{\beta_q(d-2q)/4}e^{q\beta_q t}(1 + O(1/\sqrt{t})$$

or

$$\log\det C'_q(t) = \beta_q(d-2q)/4\log(t/\pi) + q\beta_q t + O(1/\sqrt{t}).$$

This fact together with $\log\det C_q(t) = \log\det C'_q(t) + \log|\det r_q^{-1}|$ and together with $\log V_q = \log|\det r_q^{-1}|$, as defined in the introduction, yields statement (i). □

Lemma 2.5. *For t sufficiently large,*

$$\log T_{sm} = \log\tau_{comb}(\mathcal{T}) + \frac{1}{2}\sum_{q=0}^{d}(-1)^q q(m_q l - \beta_q)(2t - \log t/\pi) + O(1/\sqrt{t}).$$

Proof. Recall that

$$\log T_{sm}(h,t) = \frac{1}{2}\sum_{q=0}^{d}(-1)^{q+1}q\log\det\Delta_q(t)|_{\Lambda_t^q(M;\mathcal{E})_{sm}}$$

and that $Ker\Delta_q$ as well as $Ker\underline{\Delta}_q$ have both dimensions β_q. Using Theorem 1.7 we conclude that (with $\det'$ denoting the product of nonzero eigenvalues)

$$\begin{aligned}
\log\det\Delta_q(t)|_{\Lambda_t^q(M;\mathcal{E})_{sm}} &= \log\det{}'((\eta^{(q)})^T\eta^{(q)} + \eta^{(q-1)}(\eta^{(q-1)})^T) \\
&= \log(\sqrt{t/\pi}e^{-t})^{2(m_q l-\beta_q)}\det{}'((\gamma^{(q)})^T\gamma^{(q)} + \gamma^{(q-1)}(\gamma^{(q-1)})^T + O(1/\sqrt{t})) \\
&= -2(m_q l - \beta_q)t + 2(m_q l - \beta_q)\frac{1}{2}\log(t/\pi) + \log\det{}'\underline{\Delta}_q + O(1/\sqrt{t})
\end{aligned}$$

where the matrices η^q and $\gamma^{(q)}$ have been defined in (1.14) and (1.12). □

The proof of Theorem A now follows easily. First note that $\log T_{la}(h,t) = \log T(h,t) - \log T_{sm}(h,t)$. Therefore the asymptotic expansion of $\log T_{la}(h,t)$ is obtained from the expansions of $\log T(h,t)$ and $\log T_{sm}(h,t)$. The asymptotic expansion (1.4) for $\log T(h,t)$ follows from (2.10) together with Lemma 2.3 and Lemma 2.4 where we use the definition of $\log\tau_{met}(\mathcal{T}) = \sum_0^d(-1)^q\log V_q$. The expansion (1.5) is contained in Lemma 2.5.

3. Comparison theorem for Witten's deformation of the analytic torsion

In this section we prove the comparison result stated in Theorem B. Throughout this section we assume that all systems involved satisfy hypothesis (H). Let (M, ρ, h, g, μ) be such a system.

Let $x_{q;j} \in Cr_q(h)$ be a critical point of h of index q and U_{qj} an open neighbourhood of $x_{q;j}$.

Definition.

U_{qj} is said to be a H-neighbourhood if there is a ball $B_{2\alpha} := \{x \in \mathbf{R}^d : |x| < 2\alpha\}$ and diffeomorphisms $\varphi : B_{2\alpha} \to U_{qj}$ and $\psi : B_{2\alpha} \times V \to \mathcal{E}/_{U_{qj}}$ with the following properties:

(i) $\varphi(0) = x_{q;j}$;

(ii) When expressed in the coordinates of φ, h is of the form $h(x) = q - (x_1^2 + \cdots + x_q^2)/2 + (x_{q+1}^2 + \cdots + x_d^2)/2$;

(iii) the pull back $\varphi^*(g)$ of the Riemannian metric g is the Euclidean metric;

(iv) ψ is a trivialization of $\mathcal{E}|_{U_{qj}}$, and the pull back $\psi^*(\mu)$ of the Hermitian structure μ is given at any point in $B_{2\alpha}$ by the scalar product $\psi^*(\mu_{x_{q;j}})$.

For later use we define $U'_{qj} := \varphi(B_\alpha)$.

Definition.

U_{qj} $(0 \le q \le d, 1 \le j \le \#Cr_q(h))$ is said to be a system of H-neighbourhoods if

(i) U_{qj} are H-neighbourhoods

(ii) U_{qj} are pairwise disjoint

Given a system of H-neighbourhoods U_{qj} introduce the manifolds

$$M_I := M \setminus \cup_{q,j} U'_{qj}; \quad M_{II} := \cup_{q,j} \overline{U'_{qj}},$$

where U'_{qj} is defined as in the above definition. Both manifolds M_I and M_{II} have the same boundary, given by a disjoint union of spheres of dimension $d-1$.

Fix $\varepsilon > 0$ and consider the auxiliary operator $\Delta_q(t) + \varepsilon$. Its symbol with respect to arbitrary coordinates (φ, ψ) of $(M, \mathcal{E} \to M)$ is of the form

$$a_2(x, \xi) + t^2 ||\nabla h||^2 + a_1(x, \xi) + tL(x) + \varepsilon \tag{3.1}$$

where $a_i : B_{2\alpha} \times \mathbf{R}^d \to End(\Lambda^q(\mathbf{R}^d) \otimes V)$ $(i = 1, 2)$ are homogeneous of degree i in ξ, where $||\nabla h||^2 : B_{2\alpha} \to \mathbf{R}$ is given by

$$||\nabla h||^2 = \sum_{1 \le i,j \le d} g^{ij} \frac{\partial h}{\partial x_i} \frac{\partial h}{\partial x_j}$$

and where $L : B_{2\alpha} \to End(\Lambda^q(\mathbf{R}^d))$ is the operator $L = \mathcal{L}_{\nabla h} + \mathcal{L}^*_{\nabla h}$ of order 0 with $\mathcal{L}_{\nabla h}$ denoting the Lie-derivative of q-forms along the vector field

$$\nabla h = \sum_{i,j} g^{ij} \frac{\partial h}{\partial x_i} \frac{\partial}{\partial x_j}.$$

The operator $\mathcal{L}^*_{\nabla h}$ is the adjoint of $\mathcal{L}_{\nabla h}$ with respect to the metric g and is given by

$$\mathcal{L}^*_{\nabla h} = -(-1)^{q(d+q)} J_{d-q} \mathcal{L}_{\nabla h} J_q \tag{3.2}$$

where $J_q : \Lambda^q(B_{2\alpha}) \to \Lambda^{d-q}(B_{2\alpha})$ is the Hodge operator associated to the metric $\varphi^* g$. Recall that we have denoted by $Cr(h)$ the set of all critical points of h. Set $M^* := M \backslash Cr(h)$. For an arbitrary chart (φ, ψ) of $(M^*, \mathcal{E}|_{M^*} \to M^*)$, define, as discussed in section 1 for operators elliptic with parameter, the symbol expansion $\sum_{j \geq 0} r_{-2-j}(h, \varepsilon, x, \xi, t, \mu)$ of the resolvent $(\mu - \Delta_q(t) - \varepsilon)^{-1}$ inductively as follows:

$$r_{-2}(h, \varepsilon, x, \xi, t, \mu) = (\mu - a_2(x, \xi) - t^2 ||\nabla h||^2)^{-1}$$

and, for $j \geq 1$,

$$\begin{aligned} r_{-2-j} = &-(\mu - a_2 - t^2 ||\nabla h||^2)^{-1} \sum_{\substack{1 \leq |\alpha| \leq 2 \\ l+|\alpha|=j}} \frac{1}{\alpha!} \partial_\xi^\alpha a_2 \left(\frac{1}{i}\partial_x\right)^\alpha r_{-2-l} \\ &- (\mu - a_2 - t^2 ||\nabla h||^2)^{-1} \sum_{\substack{0 \leq |\alpha| \leq 1 \\ l+|\alpha|=j}} \partial_\xi^\alpha (a_1 + tL) \left(\frac{1}{i}\partial_x\right)^\alpha r_{-2-l} \\ &- (\mu - a_2 - t^2 ||\nabla h||^2)^{-1} \varepsilon r_{-j}. \end{aligned} \tag{3.3}$$

Note that r_{-2-j} have the following homogeneity property: for $\lambda \in \mathbf{R}_+$

$$r_{-2-j}(h, \varepsilon, x, \lambda\xi, \lambda t, \lambda^{1/2}\mu) = \lambda^{-2-j} r_{-2-j}(h, \varepsilon, x, \xi, t, h). \tag{3.4}$$

For later use, we introduce the densities $a_0(h, \varepsilon, x)$ on M^* with values in $\mathbf{R}$, defined with respect to the chart (φ, ψ) and arbitrary ε as

$$\begin{aligned} a_0(h, \varepsilon, x) = &\frac{\partial}{\partial s}/_{s=0} \left(\frac{1}{2\pi}\right)^d \frac{1}{2\pi i} \int_{\mathbf{R}^d} d\xi \\ &\int_\Gamma d\mu \mu^{-s} Tr r_{-2-d}(h, \varepsilon, x, \xi, t = 1, \mu) \\ = &\frac{-1}{(2\pi)^d} \int_{\mathbf{R}^d} d\xi \int_0^\infty d\mu Tr r_{-2-d}(h, \varepsilon, x, t = 1, -\mu). \end{aligned} \tag{3.5}$$

Proposition 3.1. *Assume that (M^d, ρ, h, g, μ) and $(\tilde{M}^d, \tilde{\rho}, \tilde{h}, \tilde{g}, \tilde{\mu}))$ both satisfy (H) so that $Cr_q(h) = Cr_q(\tilde{h})$ $(0 \leq q \leq d)$ and $l = \tilde{l} = \dim V$. Then for any $\varepsilon > 0$*

(i) $\log\det(\Delta_q(h,t)+\varepsilon) - \log\det(\Delta_q(\tilde{h},t)+\varepsilon)$ has a complete asymptotic expansion for $t \to \infty$ whose free term is denoted by $\overline{a}_0 := \overline{a}_0(h, \tilde{h}, \varepsilon)$

(ii) The coefficient $\overline{a}_0$ can be represented in the form

$$\overline{a}_0 = \int_{M_I} a_0(h, \varepsilon, x) - \int_{\tilde{M}_I} a_0(\tilde{h}, \varepsilon, \tilde{x}) \tag{3.6}$$

where $a_0(h,\varepsilon,x)$ *and* $a_0(\tilde h,\varepsilon,\tilde x)$ *are the densities introduced in* (3.5) *for arbitrary* ε.

(iii) In the case, $\dim M = d$ *is odd*

$$\overline{a}_0(h,\tilde h,\varepsilon) + \overline{a}_0(d-h, d-\tilde h,\varepsilon) = 0 \quad (\text{ all } \varepsilon > 0). \tag{3.7}$$

Proof. The proof is based on a Mayer-Vietoris type formula (Theorem 1.3). Note that $\Delta_q(h,t)+\varepsilon$ is a family of invertible, selfadjoint elliptic operators with parameter of order 2 and weight 1 for any $\varepsilon > 0$. The same is true for the operators $(\Delta_q^I(h,t)+\varepsilon)_D$ and $(\Delta_q^{II}(h,t)+\varepsilon)_D$ obtained by restricting $\Delta_q(h,t)+\varepsilon$ to M_I and M_{II} respectively, and by imposing Dirichlet boundary conditions. Therefore we can apply Theorem 1.3. Denote by $R_{DN}(h,t,\varepsilon)$ the Dirichlet to Neumann operator defined in section 1 where the vector field X is chosen to be the unit normal vector field along ∂M_I. We conclude from Theorem 1.3 (iv) that $R_{DN}(h,t,\varepsilon)$ is an invertible pseudodifferential operator with parameter of order 1 and weight 2 and from Theorem 1.3 (ii) we conclude that $R_{DN}(h,t,\varepsilon)$ is elliptic with parameter. According to Theorem 1.2, $\log\det R_{DN}(h,t,\varepsilon)$ has a complete asymptotic expansion for $t\to\infty$. Inspecting the principal symbol of $(\Delta_q^I(h,t)+\varepsilon)_D$ one observes that $(\Delta_q^I(h,t)+\varepsilon)_D$ is a family of invertible, selfadjoint differential operators with parameter of order 2 and weight 1 which is elliptic with parameter. From Theorem 1.2' we therefore conclude that $\log\det(\Delta_q^I(h,t)+\varepsilon)_D$ admits a complete asymptotic expansion as $t\to\infty$. Finally $(\Delta_q^{II}(h,t)+\varepsilon)_D$ is a family of invertible selfadjoint operators with parameter of order 2 and weight 1, which is however not elliptic with parameter.

Of course the same considerations can be made for the system $(\tilde M,\tilde\rho,\tilde h,\tilde g,\tilde\mu)$ to conclude that $\log\det R_{DN}(\tilde h,t,\varepsilon)$ and $\log\det\Delta_q^I(\tilde h,t)+\varepsilon)_D$ have both complete asymptotic expansions for $t\to\infty$. Applying the Mayer-Vietoris type formula (Theorem 1.3 (iii)) for $\log\det(\Delta_q(h,t)+\varepsilon)$ and $\log\det(\Delta_q(\tilde h,t)+\varepsilon)$ we obtain for the difference

$$\begin{aligned}
&\log\det(\Delta_q(h,t)+\varepsilon) - \log\det(\Delta_q(\tilde h,t)+\varepsilon)\\
&\quad= \log\det(\Delta_q^I(h,t)+\varepsilon)_D - \log\det(\Delta_q^I(\tilde h,t)+\varepsilon)_D\\
&\quad+ \log\det(\Delta_q^{II}(h,t)+\varepsilon)_D) - \log\det(\Delta_q^{II}(\tilde h,t)+\varepsilon)_D\\
&\quad+ \log\det R_{DN}(h,t,\varepsilon) - \log\det R_{DN}(\tilde h,t,\varepsilon)\\
&\quad+ \log\bar C(h,t,\varepsilon) - \log\bar C(\tilde h,t,\varepsilon).
\end{aligned} \tag{3.8}$$

Note that M_{II} and $\tilde M_{II}$ are isometric and $\mathcal{E}_{|M_{II}}$ as well as $\tilde{\mathcal{E}}_{|\tilde M_{II}}$ are trivial. Consequently

$$\log\det(\Delta_q^{II}(h,t)+\varepsilon)_D = \log\det(\Delta_q^{II}(\tilde h,t)+\varepsilon)_D.$$

Due to our definition of H-coordinates the isometry between M_{II} and $\tilde M_{II}$ extends to neighbourhoods of M_{II} and $\tilde M_{II}$. As a consequence we conclude from Theorem 1.2 and Theorem 1.3 (iii) that $\bar C(h,t,\varepsilon) = \bar C(\tilde h,t,\varepsilon)$ and that $\log\det R_{DN}(h,t,\varepsilon)$ and $\log\det R_{DN}(\tilde h,t,\varepsilon)$ have identical asymptotic expansions.

Therefore we have proved that

$$\log\det(\Delta_q(h,t)+\varepsilon) - \log\det(\Delta_q(\tilde h,t)+\varepsilon)$$

has a complete asymptotic expansion as $t \to \infty$ which is identical with the complete asymptotic expansion for $\log\det(\Delta_q^I(h,t)+\varepsilon)_D - \log\det(\Delta_q^I(\tilde{h},t)+\varepsilon)_D$. According to Theorem 1.2' the free term in the asymptotic expansions of both $\log\det(\Delta_q^I(h,t)+\varepsilon)_D$ and $\log\det(\Delta_q^I(\tilde{h},t)+\varepsilon)_D$ consists of a boundary contribution and a contribution from the interior. Recall that ∂M_I and $\partial\tilde{M}_I$ are isometric and that in collar neighbourhoods of ∂M_I and of $\partial\tilde{M}_I$ the symbols of $(\Delta_q^I(h,t)+\varepsilon)_D$ and $(\Delta_q^I(\tilde{h},t)+\varepsilon)_D$ are identical when expressed in (H)-coordinates. Therefore the boundary contributions are the same and the free term in the asymptotic expansion of $\log\det(\Delta_q^I(h,t)+\varepsilon) - \log\det(\Delta_q^I(\tilde{h},t)+\varepsilon)$ is given by

$$\overline{a}_0 = \int_{M_I} a_0(h,\varepsilon,x) - \int_{\tilde{M}_I} a_0(\tilde{h},\varepsilon,\tilde{x}) \tag{3.9}$$

where the densities $a_0(h,\varepsilon,x)$ and $a_0(\tilde{h},\varepsilon,\tilde{x})$ are given by (3.5).

Noting that $a_0(h,\varepsilon,x)$ and $a_0(\tilde{h},\varepsilon,\tilde{x})$ are identical on $M_{II}\backslash Cr(h) \cong \tilde{M}_{II}\backslash Cr(\tilde{h})$ statement (ii) follows. Towards (iii), note that if M is of odd dimension, the quantity $r_{-d-2}(h,\varepsilon,x,\xi,t,\mu)$ defining $a_0(h,\varepsilon,x)$ satisfies according to (3.3) and (3.4)

$$r_{-d-2}(d-h,\varepsilon,x,\xi,t,\mu) = r_{-d-2}(h,\varepsilon,x,\xi,-t,\mu) \tag{3.10}$$

and

$$r_{-d-2}(h,\varepsilon,x,-\xi,-t,\mu) = -r_{-d-2}(h,\varepsilon,x,\xi,t,\mu). \tag{3.11}$$

Therefore $r_{-d-2}(h,\varepsilon,x,\xi,t,\mu) + r_{-d-2}(-h,\varepsilon,x,\xi,t,\mu)$ is an odd function of ξ. Integrating over $|\xi| = 1$ we conclude that $a_0(h,\varepsilon,x) + a_0(d-h,\varepsilon,x) = 0$. $\square$

Introduce the following perturbed version of $\log T(h,t)$ for any $\varepsilon > 0$

$$A(h,t,\varepsilon) := \frac{1}{2}\sum_{q=0}^{d}(-1)^{q+1} q \log\det(\Delta_q(h,t)+\varepsilon). \tag{3.12}$$

Note that $A(h,t,\varepsilon)$ can be written as a sum

$$A(h,t,\varepsilon) = A_{sm}(h,t,\varepsilon) + A_{la}(h,t,\varepsilon)$$

where A_{sm} is defined similarly as $\log T_{sm}(h,t)$,

$$A_{sm}(h,t,\varepsilon) := \frac{1}{2}\sum_{q=0}^{d}(-1)^{q+1} q \log\det(\Delta_q^{sm}(h,t)+\varepsilon)$$

with

$$\Delta_q^{sm}(h,t) := \Delta_q(h,t)|_{\Lambda^q(M;\mathcal{E})_{sm}}$$

and $A_{la}(t,h,\varepsilon)$ is given by $A(h,t,\varepsilon) - A_{sm}(h,t,\varepsilon)$. Note that the eigenvalues of the operator $\Delta_q^{sm}(h,t)$ tend to 0 as $t \to \infty$ and therefore by Theorem 1.5

$$\log\det(\Delta_q^{sm}(h,t)+\varepsilon) = m_q \log\varepsilon + O\left(\frac{1}{\varepsilon} t e^{-2t}\right)$$

for $t \to \infty$. This shows that $A_{sm}(h,t,\varepsilon) - A_{sm}(\tilde{h},t,\varepsilon)$ is exponentially small as $t \to \infty$ and hence, for any fixed $\varepsilon > 0$, it has a trivial complete asymptotic expansion for $t \to \infty$. In view of (3.13) and Proposition 3.1 we conclude that for any $\varepsilon > 0$ $A(h,t,\varepsilon) - A(\tilde{h},t,\varepsilon)$ and $A_{la}(h,t,\varepsilon) - A_{la}(\tilde{h},t,\varepsilon)$ have both complete asymptotic expansions for $t \to \infty$ and, moreover, these expansions are identical. In particular we conclude that the free terms of the two expansions are identical

$$FT(A_{la}(h,t,\varepsilon) - A_{la}(\tilde{h},t,\varepsilon)) = FT(A(h,t,\varepsilon) - A(\tilde{h},t,\varepsilon)).$$

Use Proposition 3.1 (ii) and the fact that the densities $a_0(h,\varepsilon,x)$ and $a_0(\tilde{h},\varepsilon,x)$ ((3.5)) are continuous in ε to obtain

Lemma 3.2.
(i) For any $\varepsilon > 0$, $A_{la}(h,t,\varepsilon) - A_{la}(\tilde{h},t,\varepsilon)$ has a complete asymptotic expansion for $t \to \infty$ and it is identical to the one for $A(h,t,\varepsilon) - A(\tilde{h},t,\varepsilon)$.
(ii) The limit

$$\lim_{\varepsilon \to 0} FT(A_{la}(h,t,\varepsilon) - A_{la}(\tilde{h},t,\varepsilon))$$

exists and is given by

$$\begin{aligned} &\lim_{\varepsilon \to 0} FT(A_{la}(h,t,\varepsilon) - A_{la}(\tilde{h},t,\varepsilon)) \\ &\quad = \int_{M_I} a_0(h,\varepsilon = 0,x) - \int_{\tilde{M}_I} a_0(\tilde{h},\varepsilon = 0,\tilde{x}). \end{aligned} \tag{3.14}$$

We have to investigate the left hand side of (3.14) further. For this purpose we need the following estimate for the counting function $N_q(t,\lambda)$ of Spec $\Delta_q(t)$,

$$N_q(t,\lambda) := \#\{k \in \mathbf{N} : \lambda_k^q(t) \le \lambda\}$$

Lemma 3.3. *There exists a constant $C > 0$ independent of t and λ such that, for t sufficiently large*

$$N_q(t,\lambda) \le C\lambda^d.$$

Proof. First note that for t sufficiently large and M_{II} given as in (3.1), $\Delta_q(t) \ge \Delta_q(0) = \Delta_q$ on M_{II}. By Weyl's law, we conclude

$$N_q^{II}(t,\lambda) \le N_q^{II}(0,\lambda) \le C_1 \lambda^{d/2}$$

where $N_q^{II}(t,\lambda)$ is the counting function for the spectrum of the operator $\Delta_q(t)$ restricted to M_{II}, when considered with Neumann boundary conditions (Neumann spectrum). Recall that $M_I = \cup_{q,j} U_{qj}$. On each of the discs U_{qj}, $\Delta_q(t)$, when expressed in (H)-coordinates, is the direct sum of shifted harmonic oscillators of the form

$$H_t := -\frac{d^2}{dx^2} + t^2x^2 + tc$$

with $-\alpha < x < \alpha$ (α as in (3.1)). Following [CFKS, p.218] introduce the scaling operator S_t defined by

$$S_t f(x); = t^{1/2} f(tx).$$

Then $S_{t^{1/2}}.tK.S_{t^{1/2}}^{-1} = H_t$ where

$$K := -\frac{d^2}{dx^2} + x^2 + c.$$

Therefore the Neumann spectrum of H_t on the interval $-\alpha < x < \alpha$ is the same as the Neumann spectrum of tK when considered on the interval $-\sqrt{t}\alpha < x < \sqrt{t}\alpha$. Denote by $N_{tK;\sqrt{t}}(\lambda)$ the counting function of the Neumann spectrum of tK on the interval $-\sqrt{t}\alpha < x < \sqrt{t}\alpha$ and by $N^D_{tK;\sqrt{t}}(\lambda)$ the counting function of the Dirichlet spectrum of tK on the interval $-\sqrt{t}\alpha < x < \sqrt{t}\alpha$. Note that for all $t \geq 0$ and λ sufficiently large

$$N_{tK;\sqrt{t}}(\lambda) \leq N^D_{tK;\sqrt{t}}(\lambda) + 1 \leq 2N^D_{tK;\sqrt{t}}(\lambda/t).$$

Comparing the Dirichlet problem for K on $-\sqrt{t}\alpha \leq x \leq \sqrt{t}\alpha$ with the one on the whole real line we conclude that $N^D_{K;\sqrt{t}}(\lambda/t) \leq C_2\lambda/t \leq C_2\lambda$ for $t \geq 1$ with a constant $C_2 > 0$ independent of λ and t. Hence we have shown that the counting function $N^I_q(t,\lambda)$ of the Neumann spectrum of the operator $\Delta_q(t)$ on M_I can be estimated by

$$N^I_q(t,\lambda) \leq C_3\lambda^d$$

for a constant $C_3 > 0$ independent of t and λ. The subadditive property of the Neumann counting function [CH] implies that

$$N_q(t,\lambda+0) \leq N^I_q(t,\lambda+0) + N^{II}_q(t,\lambda+0) \leq C\lambda^d$$

for some constant $C > 0$ independent of t and λ and for t sufficiently large. □

Let us introduce the following version of the trace of the heat kernel

$$\theta_q(t,\mu) := \sum_{k \geq m_q l+1} e^{-\mu\lambda^q_k(t)} \tag{3.15}$$

where $m_q = \#Cr_q(h)$ and $(\lambda^q_k(t))_{k\geq 1}$ denote the eigenvalues of $\Delta_q(h,t)$. From Lemma 3.3 we obtain

Corollary 3.4.

(i) There exists a constant $C > 0$ independent of t and μ such that, for t sufficiently large

$$\theta_q(t,\mu) \leq C\mu^{-d}. \tag{3.16}$$

(ii) There exists a constant $C > 0$ independent of t and μ such that, for t sufficiently large, and $\mu \geq 1/\sqrt{t}$

$$\theta_q(t,\mu) \leq Ce^{-\beta t\mu}. \tag{3.17}$$

Proof.

(i) Recall that for $k \geq m_q l + 1$ there exists a constant $C_1 > 0$ such that

$$\lambda_k^q(t) > C_1 t.$$

Therefore

$$\theta_q(t, \mu) = \int_{C_1 t}^{\infty} e^{-\mu\lambda} dN_q(t, \lambda).$$

Integrating by parts we obtain

$$\theta_q(t, \mu) = \mu \int_{C_1 t}^{\infty} e^{-\mu\lambda} N_q(t, \lambda) d\lambda. \tag{3.18}$$

By Lemma 3.3, one then concludes

$$\theta_q(t, \mu) \leq \frac{C}{\mu^d} \int_{C_1 t\mu}^{\infty} e^{-\lambda} \lambda^d d\lambda \leq \tilde{C}/\mu^d.$$

(ii) From (3.18) and Lemma 3.3 we obtain

$$\theta_q(t, \mu) \leq C\mu e^{-C_1 t\mu/2} \int_{C_1 t}^{\infty} e^{-\mu\lambda/2} \lambda^d d\lambda \leq \frac{\tilde{C}}{\mu^d} e^{-C_1 t\mu/2}.$$

By choosing $\beta < C_1/2$ and $C > 0$ sufficiently large we obtain (ii). $\square$

Recall from Theorem A that $logT_{la}(h, t)$ has an asymptotic expansion for $t \to \infty$.

Proposition 3.5.

$$\lim_{\varepsilon \to 0} FT(A_{la}(h, t, \varepsilon) - A_{la}(\tilde{h}, t, \varepsilon)) = FT(\log T_{la}(h, t)) - FT(\log T_{la}(\tilde{h}, t)).$$

Proof. We verify that the function, defined for $\varepsilon > 0$ and t sufficiently large by

$$H(t, \varepsilon) := A_{la}(h, t, \varepsilon) - A_{la}(\tilde{h}, t, \varepsilon) + \log T_{la}(h, t, \varepsilon) - \log T_{la}(\tilde{h}, t, \varepsilon)$$

is in the form

$$H(t, \varepsilon) = \sum_{k=1}^{d} \varepsilon^k f_k(t) + g(t, \varepsilon) \tag{3.19}$$

where $g(t, \varepsilon) = O(1/\sqrt{t})$ uniformly in ε. The statement of the proposition can be deduced from (3.19) as follows: Recall that for $\varepsilon > 0$, $H(t, \varepsilon)$ has a asymptotic expansion for $t \to \infty$. As $g(t, \varepsilon) = O(1/\sqrt{t})$ uniformly in ε we conclude that for any $\varepsilon > 0$, $\sum_{k=1}^{d} \varepsilon^k f_k(t)$ has an asymptotic expansion for $t \to \infty$. By taking d different values $0 < \varepsilon_1 < \cdots < \varepsilon_d$ for ε and using that the Vandermonde determinant

$$\det \begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_1^d \\ \vdots & & \vdots \\ \varepsilon_d & \dots & \varepsilon_d^d \end{pmatrix} \neq 0$$

we conclude that for any $1 \leq k \leq d$, $f_k(t)$ has an asymptotic expansion for $t \to \infty$ and that for any $\varepsilon > 0$

$$FT(H(t,\varepsilon)) = \sum_{k=1}^{d} \varepsilon^k FT(f_k(t)).$$

Hence $\lim_{\varepsilon \to 0} FT(H(t,\varepsilon))$ exists and $\lim_{\varepsilon \to 0} FT(H(t,\varepsilon)) = 0$. It remains to prove (3.19). For this purpose we introduce the zeta function $\zeta_{q,la}$ corresponding to the large eigenvalues of $\Delta_q(t) + \varepsilon$,

$$\zeta_{q,la}(t,\varepsilon,s) = \sum_{j \geqq m_q l+1} (\lambda_j^q(t) + \varepsilon)^{-s}.$$

Using the heat kernel representation, $\zeta_{q,la}(t,\varepsilon,s)$ can be written as

$$\zeta_{q,la}(t,\varepsilon,s) = \frac{1}{\Gamma(s)} \int_o^\infty \mu^{s-1} \theta_q(t,\mu) e^{-\varepsilon\mu} d\mu \tag{3.20}$$

with $\theta_q(t,\mu)$ given by (3.15). The integral in (3.20) can be splitted into two parts

$$\zeta_{q,la}^I(t,\varepsilon,s) = \frac{1}{\Gamma(s)} \int_{1/\sqrt{t}}^\infty \mu^{s-1} \theta_q(t,\mu) e^{-\varepsilon\mu} d\mu \tag{3.21}$$

and

$$\zeta_{q,la}^{II}(t,\varepsilon,s) = \frac{1}{\Gamma(s)} \int_0^{1/\sqrt{t}} \mu^{s-1} \theta_q(t,\mu) e^{-\varepsilon\mu} d\mu. \tag{3.22}$$

First let us consider

$$\zeta_{q,la}^I(t,\varepsilon,s) - \zeta_{q,la}^I(t,\varepsilon=0,s) = \frac{1}{\Gamma(s)} \int_{1/\sqrt{t}}^\infty \mu^s \theta_q(t,\mu) \frac{e^{-\varepsilon\mu} - 1}{\mu} d\mu. \tag{3.23}$$

Note that

$$\zeta_{q,la}^I(t,\varepsilon,s) - \zeta_{q,la}^I(t,\varepsilon=0,s)$$

is by Corollary 3.5 (ii) an entire function of s. Therefore, with

$$\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\right)_{s=0} = 1$$

and $1 - e^{-\varepsilon\mu} \leq \varepsilon\mu$,

$$\begin{aligned}
&|\frac{d}{ds}(\zeta_{q,la}^I(t,\varepsilon,s) - \zeta_{q,la}^I(t,\varepsilon=0,s))_{s=0}| \\
&= |\int_{1/\sqrt{t}}^\infty \theta_q(t,\mu) \frac{e^{\varepsilon\mu - 1}}{\mu} d\mu| \\
&\leq \varepsilon C \int_{1/\sqrt{t}}^\infty e^{-\beta t\mu} d\mu = \frac{\varepsilon C}{\beta t} e^{-\beta\sqrt{t}}
\end{aligned}$$

where we have used Corollary 3.4. Concerning the term

$$\frac{d}{ds}(\zeta^{II}_{q,la}(t,\varepsilon,s)-\zeta^{II}_{q,la}(t,\varepsilon=0,s))_{s=0},$$

expand $(e^{-\varepsilon\mu}-1)/\mu$

$$(e^{-\varepsilon\mu}-1)/\mu=\sum_{k=1}^{d}\frac{-1^k}{k!}\varepsilon^k\mu^{k-1}+\varepsilon^{d+1}\mu^d e(\varepsilon,\mu)$$

where the error term is given by

$$e(\varepsilon,\mu)=\left(\sum_{k=d+1}^{\infty}\frac{(-1)^k}{k!}\varepsilon^k\mu^{k-1}\right)/\varepsilon^{d+1}\mu^d.$$

Note that $\mu^d\theta_q(t,\mu)\le C$ according to Corollary 3.4. Therefore

$$\int_0^{1/\sqrt{t}}\mu^s\theta_q(t,\mu)\varepsilon^{d+1}\mu^d e(\varepsilon,\mu)d\mu$$

is an entire function of s and, for t sufficiently large

$$\begin{aligned}&|\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^{1/\sqrt{t}}\mu^s\theta_q(t,\mu)\varepsilon^{d+1}\mu^d e(\varepsilon,\mu)d\mu\right)_{s=0}|\\&\le\varepsilon^{d+1}C/\sqrt{t}\end{aligned}$$

where C is independent of t and ε, $0\le\varepsilon\le 1$. Finally, recall that $\theta_q(t,\mu)$ admits an expansion for $\mu\to 0+$ of the form

$$\theta_q(t,\mu)=\sum_{j=0}^{d}C_j(t)\mu^{(j-d)/2}+\theta'_q(t,\mu)$$

where $\theta'_q(t,\mu)$ is continuous in $\mu\ge 0$. Therefore, for $1\le k\le d$,

$$\frac{1}{\Gamma(s)}\int_0^{1/\sqrt{t}}\mu^s\theta_q(t,\mu)\frac{(-1)^k}{k!}\varepsilon^k\mu^{k-1}d\mu$$

is analytic with respect to s at $s=0$ and

$$\sum_{k=1}^{d}\frac{d}{ds}(\frac{1}{\Gamma(s)}\int_0^{1/\sqrt{t}}\mu^s\theta_q(t,\mu)\frac{(-1)^k}{k!}\varepsilon^k\mu^{k-1}d\mu)_{s=0}$$

is of the form $\sum_{k=1}^{d}\varepsilon^k f_k(t)$. This establishes (3.19). $\square$

Proof of Theorem B. From Theorem A we know that $\log T_{la}(h,t) - \log T_{la}(\tilde{h},t)$ has an asymptotic expansion for $t \to \infty$. By Proposition 3.5, the free term of the asymptotic expansion $\overline{a}_0$ is given by

$$\overline{a}_0 = \lim_{\varepsilon \to 0} FT(A_{la}(h,t,\varepsilon) - A_{la}(\tilde{h},t,\varepsilon)).$$

By Lemma 3.3 (ii) we conclude that

$$\overline{a}_0 = \int_{M_I} a_0(h,\varepsilon = 0, x) - \int_{\tilde{M}_I} a_0(\tilde{h},\varepsilon = 0,\tilde{x}).$$

Equation (0.9) is proved in Proposition 3.1 (iii). In view of the equality

$$FT(\log T(h,t) - \log T_{sm}(h,t)) = FT(\log T_{la}(h,t))$$

one can see that $\overline{a}_0$ is independent of h and $\tilde{h}$ within the class of functions h and $\tilde{h}$ which give rise to the same cochain complexes $C^*(M;\rho,\mathcal{T},\mu)$ respectively $C^*(M;\tilde{\rho},\tilde{\mathcal{T}},\tilde{\mu})$. This combined with the locality of a_0 implies that $\int_{M_I} a_0(h,\varepsilon = 0,x) = \int_{M_I} a_0(h',\varepsilon = 0,x)$; this implies (ii). □

Proof of Corollary C. Choose a bijection $\Theta : Cr(h) \to Cr(\tilde{h})$ so that $\Theta(x_{q;j})$ is a critical point $\tilde{x}_{q;j}$ of $\tilde{h}$ of index q. By assumption Θ extends to an isometry $\Theta : \cup_{q,j} U_{qj} \to \cup_{q,j} \tilde{U}_{qj}$ where (U_{qj}) and $(\tilde{U}_{qj})$ are systems of H-neighbourhoods for h respectively $\tilde{h}$. Denote by $\mathcal{T}$ respectively $\tilde{\mathcal{T}}$ the triangulation induced by (h,g) respectively $(\tilde{h},\tilde{g})$ and the dual triangulation by $\mathcal{T}_{\mathcal{D}}$ respectively $\tilde{\mathcal{T}}_{\mathcal{D}}$.

Using Theorem A for both h and $d-h$, we obtain

$$\begin{aligned} 2\log T - 2\log \tilde{T} = {} & FT(\log T(h,t) - \log T(\tilde{h},t)) \\ & + FT(\log T(d-h,t) - \log T(d-\tilde{h},t)) \\ & + \log \tau_{met}(\mathcal{T}) - \log \tau_{met}(\tilde{\mathcal{T}}) \\ & + \log \tau_{met}(\mathcal{T}_{\mathcal{D}}) - \log \tau_{met}(\tilde{\mathcal{T}}_{\mathcal{D}}). \end{aligned}$$

Decomposing $\log T(h,t) = \log T_{la}(h,t) + \log T_{sm}(h,t)$ and taking into account the asymptotics (0.7) of $\log T_{sm}(h,t)$ we conclude that

$$\begin{aligned} 2\log T - 2\log \tilde{T} = {} & \log \tau(\mathcal{T}) - \log \tau(\tilde{\mathcal{T}}) \\ & + \log \tau(\mathcal{T}_{\mathcal{D}}) - \log \tau(\tilde{\mathcal{T}}_{\mathcal{D}}) \\ & + FT(\log T_{la}(h,t) - \log T_{la}(\tilde{h},t)) \\ & + FT(\log T_{la}(d-h,t) - \log T_{la}(d-\tilde{h},t)) \end{aligned}$$

from which the Corollary follows by (0.9) and (0.10). □

4. Application

In this section we present a new and short proof of results due to, in increasing generality, Ray-Singer, Cheeger, Müller and Bismut-Zhang concerning the relation of the analytic torsion and the Reidemeister torsion. To the best of our knowledge the result as formulated can not be found in literature.

Theorem 4.1. *(Ray-Singer; Cheeger; Müller; Bismut-Zhang). Assume that* (M^d, g) *is a compact Riemannian manifold without boundary of odd dimension* d*,* ρ *is a representation of the fundamental group* $\pi_1(M)$ *on a vectors space* V *of dimension* l*,* $\rho : \pi_1(M) \to GL(V)$*,* μ *is a Hermitian structure on the flat bundle* $\mathcal{E} \to M$ *(*$\mathcal{E}$ *induced by* ρ*) and* $\mathcal{T} = (h, g')$ *is a generalized triangulation of* M *with* $\mathcal{T}_{\mathcal{D}} = (d - h, g')$ *denoting its dual. Then*

$$\log T = (\log \tau(\mathcal{T}) + \log \tau(\mathcal{T}_{\mathcal{D}}))/2.$$

For the derivation of Theorem 4.1 from Theorem A and Theorem B we need a number of well known results which we state for the convenience of the reader.They can be proved in a straight forward fashion, or found in literature as mentioned.

Lemma 4.2. *Let* (M, g) *be a Riemannian manifold. Assume that* $\mathcal{T} = (h, g')$ *is a generalized triangulation of* M*.*

Then there exists a triangulation $\mathcal{T}' = (h, g'')$ *of* M *with the following properties:*

(i) For any critical point $x_{q;j} \in Cr_q(h)$ *there exists a neighbourhood* U_{qj} *so that* g'' *and* g *coincide on* U_{qj}*.*

(ii) For any two critical points x, y*, in* $Cr(h)$ *the intersections* $W_x^+(g') \cap W_y^-(g')$ *and* $W_x^+(g'') \cap W_y^-(g'')$ *are diffeomorphic where* $W_x^+(g')$ *respectively* $W_x^+(g'')$ *denote the stable manifold associated to the critical point* x *and the gradient vector field* $\text{grad}_{g'} h$ *respectively* $\text{grad}_{g''} h$ *and where* $W_y^-(g')$ *respectively* $W_y^-(g'')$ *denote the unstable manifold associated to the critical point* y *and the gradient vector field* $\text{grad}_{g'} h$ *respectively* $\text{grad}_{g''} h$

Definition.

Given generalized triangulations $\mathcal{T} = (h, g)$ and $\mathcal{T}' = (h', g')$, $\mathcal{T}'$ is called a subdivision of $\mathcal{T}$ if

(i) $Cr(h) \subseteq Cr(h')$

(ii) $W_x^{\pm}(h', g') = W_x^{\pm}(h, g)$ for any $x \in Cr(h)$.

The following result is implicit in [Mi2].

Lemma 4.3. *Let* $\mathcal{T} = (h, g')$ *be a generalized triangulation,* $0 \le q \le d - 1$ *an integer and* x, y *two distinct points in* $M \backslash Cr(h)$*. Then there exists a generalized triangulation* $\mathcal{T}' = (h', g'')$ *with the following properties*

(i) $Cr(h') = Cr(h) \cup \{x, y\}$*;*

(ii) $x \in Cr_q(h')$*;* $y \in Cr_{q+1}(h')$*;*

(iii) $\mathcal{T}'$ *is a subdivision of* $\mathcal{T}$*;*

(iv) $W_y^- \cap W_x^+$ *is connected.*

As a consequence one obtains

$$\tau(\mathcal{T}) = \tau(\mathcal{T}') \tag{4.1}$$

for any Riemannian manifold (M,g), any representation $\rho : \pi_1(M) \to GL(V)$ and any Hermitian structure μ for $\mathcal{E} \to M$ where $\mathcal{E}$ is the flat bundle induced by ρ.

For a Riemannian manifold (M,g), a set $F \subseteq M$ and $r > 0$, denote by $B_r(F)$ the following neighbourhood of F, $B_r(F) := \{x \in M : \text{ dist }_g(x,F) \leq r\}$.

Lemma 4.4. *Assume that (M,g) is a Riemannian manifold, $F = \{x_1, \dots, x_N\}$ a finite set of points of M and $E \to M$ a vector bundle of rank l. Then the following statements hold:*

(1) There exists a 1-parameter family g_ε, $(|\varepsilon| < 2\varepsilon_0, \varepsilon_0 > 0)$ of Riemannian metrics of class C^1 in (ε, x) with the following properties:

(i) $g_\varepsilon = g$ on $M \backslash B_{2\varepsilon}(F)$;

(ii) g_ε is flat on $B_\varepsilon(F)$;

(iii) $\lim_{\varepsilon\to 0} g_\varepsilon = g$ in C^1-topology.

(2) There exists a 1-parameter family $\mu_\varepsilon, (|\varepsilon| < 2\varepsilon_0, \varepsilon_0 > 0)$ of Hermitian structures of class C^1 in (ε, x) with the following properties:

(i) $\mu_\varepsilon = \mu$ on $M \backslash B_{2\varepsilon}(F)$;

(ii) μ_ε is a parallel on $B_{2\varepsilon}(F)$ with respect to the canonical connection induced by ρ;

(iii) $\lim_{\varepsilon\to 0} \mu_\varepsilon = \mu$ in C^1-topology.

The following result is a slight, but immediate generalization of [RS, Theorem 2.5].

Lemma 4.5. [RS] *Assume that for $j = 1, 2$ we are given Riemannian manifolds (M_j, g_j), representations $\rho_j : \pi_1(M_j) \to GL(V_j)$ and Hermitian structures $\mu_j : \mathcal{E}_j \to M_j$ where $\mathcal{E}_j$ denotes the bundle induced by ρ_j. Then*

$$\begin{aligned}\log T(M_1 \times M_2, \rho_1 \otimes \rho_2, g_1 \times g_2, \mu_1 \otimes \mu_2)\\ = \chi(M_2) \log T(M_1, \rho_1, g_1, \mu_1) + \chi(M_1) \log T(M_2, \rho_2, g_2, \mu_2)\end{aligned}$$

where $\chi(M_j)$ denotes the Euler characteristic of M_j.

We use Lemma 4.5 only to prove the following

Corollary 4.6. *Let $T^d := \mathbf{R}^d/\mathbf{L}$ be a d-dimensional torus,with $\mathbf{L}$ a lattice in $\mathbf{R}^d$, g_0 the Euclidean metric on T^d, $\rho_0 : \pi_1(T^d) \to GL(\mathbf{R}^d)$ the trivial representation and μ_0 the Hermitian structure on $\mathcal{E} \to T^d$ parallel with respect to the canonical connection of ρ_0.*

(i) If $d \geq 2$, then $\log T = 0$.

(ii) If $d = 1$, then $\log T = l \log a$ where $a > 0$ is such that $\mathbf{L} = a\mathbf{Z}$

Proof.

(i) follows from Lemma 4.5 together with the fact that the Euler characteristic $\chi(T^d)$ satisfies $\chi(T^d) = 0$ for $d \geq 1$.

(ii) For $d = 1$, T^1 is the circle $\{a/2\pi(\cos\theta, \sin\theta) : 0 \leq \theta \leq 2\pi\}$ and with $l = \dim V$,

$$-\log T = \frac{l}{2}\frac{d}{ds}|_{s=0} \sum_{k\neq,0} \left(\frac{a}{2\pi k}\right)^{2s} = \frac{d}{ds}|_{s=0}(\frac{a}{2\pi})^{2s}\zeta(2s)$$

where $\zeta(s)$ is the Riemann zeta function. Recalling that $\zeta(0) = -1/2$ and $\frac{d}{ds}\zeta(0) = -\log\sqrt{2\pi}$ we obtain $\log T = l \log a$ $\square$

The result analogous to Lemma 4.5 for the Reidemester torsion is implicit in [Mi1]:

Lemma4.7. [Mi1] *Assume that for $j = 1, 2$ we are given Riemannian manifolds (M_j, g_j), representations $\rho_j : \pi_1(M_j) \to GL(V_j)$, Hermitian structures μ_j on $\mathcal{E}_j \to M_j$ and triangulations $\mathcal{T}_j$ of (M_j, g_j). Then*

$$\log \tau(\mathcal{T}_1 \times \mathcal{T}_2) = \chi(M_2) \log \tau(\mathcal{T}_1) + \chi(M_1) \log \tau(\mathcal{T}_2).$$

Again we use Lemma 4.7 only to prove the following:

Corollary 4.8. *Let $T^d := \mathbf{R}^d/\mathbf{L}$ be a d-dimensional torus,with $\mathbf{L}$ a lattice in $\mathbf{R}^d$, g_0 the Euclidean metric on T^d, $\rho : \pi_1(T^d) \to GL(V)$ the trivial representation and μ_0 the Hermitian structure on $\mathcal{E} \to T^d$ parallel with respect to the canonical connection of ρ_0. Moreover let $h_0 : T^d \to \mathbf{R}$ be given by*

$$h_0(\theta_1, \ldots, \theta_d) := \sum_{j=1}^{d} \cos \theta_j$$

Then $\mathcal{T}_0 = (h_0, g_0)$ is a generalized triangulation with the following properties:
(i) If $d \geq 2$, then $\tau(\mathcal{T}_0) = 0$
(ii) If $d = 1$, then $\tau(\mathcal{T}_0) = l \log a$ where $a > 0$ is such that $\mathbf{L} = a\mathbf{Z}$.

Proof.
(i) follows from Lemma 4.7 together with the fact that the Euler characteristic $\chi(T^d)$ satisfies $\chi(T^d) = 0$ for $d \geq 1$.
(ii) It is easy to see that $\tau_{comb}(\mathcal{T}_0) = 1$ and $\log \tau_{met}(\mathcal{T}_0) = \log V_0 - \log V_1 = l \log a$. $\square$

The last result we need concerns the metric anomaly of the analytic torsion. In the form needed for the proof of Theorem 4.1 it is a slight generalization of a result due to Ray-Singer.

Lemma 4.9. *(*[RS]*). Let M be a manifold, $\rho : \pi_1(M) \to GL(V)$ a representation of the fundamental group of M on a vector space V and μ a Hermitian structure on the bundle $\mathcal{E} \to M$ induced by ρ. Let $g(u)$ be a 1-parameter family of class C^1 of Riemannian metrics of M. Then $\log T(M, \rho, g(u), \mu)$ is a C^1-function of u whose derivative is given by*

$$\frac{d}{du} \log T(M, \rho, g(u), \mu) = \sum_{q=0}^{d} (-1)^q \frac{d}{du} \log \mathcal{V}_q(u)$$

where $\mathcal{V}_q(u)$ is the volume defined by

$$\mathcal{V}_q(u) = \textit{vol}_{\,g(u)}(A_q(u)^{-1}(a_{q;j}), \ldots, A_q(u)^{-1}(a_{q;\beta q})).$$

Here $(a_{q;j})_{1\leq j\leq\beta_q}$ is a basis of the q'th cohomology group $H^q(M;\mathcal{E})$, chosen independently of u and $A_q(u) : Ker\Delta_q(u) \to H^q(M;\mathcal{E})$ is the canonical isomorphism provided by Hodge theory between the null space $Ker\Delta_q(u)$ of the q-Laplacian $\Delta_q(u)$ and $H^q(M;\mathcal{E})$.

Proof. Taking into account Lemma 4.1 it suffices to verify the statement for the case where the generalized triangulation $\mathcal{T} = (h, g')$ has the additional property that $g'(x) = g(x)$ for any critical point $x \in Cr(h)$.

In view of Lemma 4.4 and Theorem 1.1 we may assume in addition that in sufficiently small neighbourhoods of any of the critical points of h, $g = g'$, g is flat and μ is parallel with respect to the canonical connection induced by ρ. According to Lemma 4.9 and the definition of the Reidemeister torsion $\tau = \tau_{met}\tau_{comb}$ it suffices to verify the statement under the additional assumption that $g = g'$ on all of M. Moreover, due to (4.1) it suffices to verify the statement for a subdivision $\mathcal{T}'$ of $\mathcal{T}$ of our choice. Denote by $\beta_q(T^d)$ the q th Betti number of the d-dimensional torus $T^d = \mathbf{R}^d/\mathbf{Z}^d$. Note that for any smooth Morse function $h : M \to \mathbf{R}$ with $\#Cr(h) \geq \beta_q(T^d)$ $(0 \leq q \leq d)$ there exists a smooth Morse function $h' : T^d \to \mathbf{R}$ such that $\#Cr_q(h) = \#Cr_q(h')$ $(0 \leq q \leq d)$. Therefore, by Corollary C, we conclude that it remains to verify the statement of Theorem 4.1 for $M = T^d$, g_0 the Euclidean metric on T^d, ρ_0 the trivial representation of $\pi_1(T^d)$ on V, μ_0 the Hermitian structure on $\mathcal{E} \to T^d$, which is parallel with respect to the canonical connection induced by ρ and the generalized triangulation $\mathcal{T} = (h, g_0)$ where $h : T^d \to \mathbf{R}$ is a Morse function. In the case where ρ is a trivial and μ is the Hermitian structure, parallel with respect to the canonical connection on $\mathcal{E} \to M$ it is well known [Mi1] that the Reidemeister torsion $\tau(\mathcal{T})$ is independent of the generalized triangulation. In particular $\tau(\mathcal{T}) = \tau(\mathcal{T}_{\mathcal{D}})$. It then follows from Corollary 4.6 and Corollary 4.8, that for any $d \geq 1$,

$$T(T^d, \rho_0, g_0, \mu_0) = \tau(T^d, \rho_0, g_0, \mu_0, \mathcal{T}_0)$$

where $\mathcal{T}_0 = (h_0, g_0)$ with $h(\theta_1, \ldots, \theta_d) := \sum_{j=1}^d \cos\theta_j$ □

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D. Burghelea, Ohio State University, U.S.A.

L. Friedlander, University of Arizona, U.S.A.

T. Kappeler, Ohio State University, U.S.A.