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# The spectrum of Schrödinger operators in $L_p(\mathbb{R}^d)$ and $C_0(\mathbb{R}^d)$

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# The Spectrum of Schrödinger Operators in $L_p(\mathbf{R}^d)$ and in $C_0(\mathbf{R}^d)^\dagger$

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## Introduction

The aim of this paper is to present results on the independence of the spectrum of Schrödinger operators in different spaces. We treat Schrödinger operators of a very general kind, namely $-\frac{1}{2}\Delta$ perturbed by certain measures $\mu$.

In Section 1 we recall what measures can be used and we review results stating the $p$-independence of the spectrum of the realizations of $-\frac{1}{2}\Delta + \mu$ in $L_p(\mathbf{R}^d)$, $1 \le p \le \infty$.

In Section 2 we show that the realizations of $-\frac{1}{2}\Delta + \mu$ in spaces of continuous functions, e.g., the bounded uniformly continuous functions or the continuous functions vanishing at infinity, again have the same spectrum, for suitable $\mu$. In fact, this is derived in a much more general context, utilizing the semigroup dual of a Banach space with respect to a strongly continuous semigroup.

In Section 3 it is shown thal Shnol's method of constructing singular sequences can also be employed in a proof of the inclusions $\sigma(H_{2,V}) \subset \sigma(H_{p,V})$ and $\sigma(H_{2,V}) \subset \sigma(H_{C_0,V})$, for suitable potentials $V$. This establishes the connection between the spectrum in $L_p$ and $C_0$ and the existence of polynomially bounded generalized eigenfunctions.

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## 1. Review of $L_p$-results.

In order to state the results we have to recall some notations. Let

$$M_0 := \{\mu : \mathcal{B} \to [0,\infty];\ \mu\ \sigma\text{-additive},\ \mu(B) = 0 \quad \text{for all sets } B \in \mathcal{B} \text{ with capacity zero}\},$$

where $\mathcal{B}$ denotes the $\sigma$-algebra of Borel subsets of $\mathbf{R}^d$.

For the definition of the extended Kato class $\hat{S}_K \subset M_0$ of measures and of the constant $c(\mu)$ defined for $\mu \in \hat{S}_K$ we refer to [StV]. We recall that for $\mu_+ \in M_0$, $\mu_- \in \hat{S}_K$ with $c(\mu_-) < 1$ a closed form in $L_2(\mathbf{R}^d)$ is defined by

$$(\mathbf{h} - \mu_- + \mu_+)[u,v] := \frac{1}{2}\int \nabla u \cdot \overline{\nabla v} dx - \int u^{\sim}\, \overline{v^{\sim}}\, d\mu_- + \int u^{\sim}\, \overline{v^{\sim}}\, d\mu_+,$$

with domain

$$D(\mathbf{h} - \mu_- + \mu_+) = \{u \in W_2^1(\mathbf{R}^d);\ \int |u^{\sim}|^2\, d\mu_+ < \infty\}$$

($u^{\sim}$ denoting a quasi-continuous version of $u$). The closure of $D(\mathbf{h} - \mu_- + \mu_+)$ in $L_2(\mathbf{R}^d)$ is of the form $L_2(Y)$, for a suitable set $Y \in \mathcal{B}$. The operator $H_\mu = H_{\mu_+ - \mu_-}$ is the self-adjoint operator in $L_2(Y)$ associated with $\mathbf{h} - \mu_- + \mu_+$. It is shown in [StV; Corollary 4.2] that the semi-group $\left(e^{-tH_\mu};\ t \geq 0\right)$ on $L_2(Y)$ acts also as a strongly continuous semigroup $U_{p,\mu}(.)$ on $L_p(Y)$, for all $p \in [1,\infty)$; the generators of these semigroups will be denoted by $-H_{p,\mu}$. Also, $H_{\infty,\mu} := H_{1,\mu}^*$. The corresponding unperturbed operators (for $\mu = 0$) will be denoted by $H_p$.

**1.1. Theorem.** *With the notations introduced so far, we have*

$$\sigma(H_{p,\mu}) = \sigma(H_{2,\mu})$$

*for all* $p \in [1,\infty]$.

We are going to give an outline of the proof of this result. In order to do so we first collect several facts which are needed in the proof.

**1.2. Remark.** (a) Let $\varepsilon > 0$. There exist constants $C$, $\omega$ such that

$$\|e^{\xi\cdot x}\, e^{-tH_{p,\mu}}\, e^{-\xi\cdot x}\|_{p,q} \leq C\, t^{-\gamma} e^{\omega t}$$

for all $t > 0$, $1 \leq p \leq q \leq \infty$, $\xi \in \mathbf{R}^d$ with $|\xi| \leq \varepsilon$, where $\gamma = \frac{d}{2}(\frac{1}{p} - \frac{1}{q})$. (Here $\|\cdot\|_{p,q}$) denotes the norm in $L(L_p, L_q)$.)

The proof of this fact consists in two steps. In both of these steps it is essential that there exists $a > 1$ such that $a\mu$ is also in the class considered above (in particular, $c(a\mu) < 1$).

(i) One shows the inequality for $\xi = 0$, using Stein interpolation; cf. [StV; Theorem 5.1 (b)].

(ii) From the fact that the desired statement is true for the unperturbed heat semigroup ($\mu = 0$) one concludes it for the perturbed semigroup, again using Stein interpolation; cf. [ScV; Remark 3.4 (b), (c)].

(b) Let $\epsilon > 0, \omega$ be as in (a). Then there exists $C$ such that

$$\left\| e^{\xi\cdot x}(H_\mu - w)^{-1}e^{-\xi\cdot x} \right\|_{p,q} \leq C\left(\frac{1}{1-\gamma} + \frac{1}{-w-\omega}\right)$$

for all $w \in \mathbf{R}$ with $w < -\omega,\ p \leq q$ with $\gamma = \frac{d}{2}(\frac{1}{p} - \frac{1}{q}) < 1,\ |\xi| \leq \epsilon$. Further, $(-\infty, -\omega) \subset \rho(H_{p,\mu})$ for all $p \in [1, \infty]$, and

$$(H_{p,\mu} - w)^{-1} = (H_\mu - w)^{-1}$$

on $L_p(Y) \cap L_2(Y)$, for $w < -\omega$.

The proof consists in integrating the inequality in (a) after multiplying by $e^{wt}$; cf. [HV; Proposition 3.7], [ScV; Remark 3.4 (d)].

**1.3. Lemma.** ([ScV; Corollary 3.3]) *Let $1 \leq p \leq q \leq \infty,\ 0 < \epsilon' < \epsilon''$. Then there exists $C \geq 0$ such that for each linear operator*

$$A : L_{\infty,c}(\mathbf{R}^d) \to L_{\infty,\mathrm{loc}}(\mathbf{R}^d)$$

*($L_{\infty,c}$ denoting $L_\infty$-functions with compact support) satisfying*

$$\|e^{\xi\cdot x}Ae^{-\xi\cdot x}\|_{p,q} \leq 1 \quad \textit{for all} \quad \xi \in \mathbf{R}^d \quad \text{with} \quad |\xi| \leq \epsilon''$$

*one has*

$$\left\| e^{\xi\cdot x}Ae^{-\xi\cdot x} \right\|_{r,r} \leq C$$

*for $p \leq r \leq q, \quad |\xi| \leq \epsilon'$.*

The inclusion $\rho(H_{p,\mu}) \subset \rho(H_{2,\mu})$ in Theorem 1.1 is obtained as in [HV; section 2], using Remark 1.2 (a) for $\xi = 0$.

**Sketch of the proof of the inclusion** $\rho(H_{2,\mu}) \subset \rho(H_{p,\mu})$ (compare [ScV]). It is sufficient to prove the assertion for all $p \in [1, 2]$. According to Remark 1.2 (b) we find $w\ (< -\omega),\ C$ such that

$$\|e^{\xi\cdot x}(H_\mu - w)^{-1}e^{-\xi\cdot x}\|_{p,q} \leq C$$

whenever $1 \leq p \leq q \leq 2,\ \frac{d}{2}(\frac{1}{p} - \frac{1}{q}) \leq \frac{1}{2},\ |\xi| \leq 1$.

Let $K \subset \rho(H_{2,\mu})$ be compact, $\overset{\circ}{K}$ connected, $K = \overline{\overset{\circ}{K}},\ w \in \overset{\circ}{K}$. Then there exist $\epsilon \in (0, 1]$ and a constant $C'$ such that $K \subset \rho(e^{\xi\cdot x}H_{2,\mu}e^{-\xi\cdot x})$ for $|\xi| \leq \epsilon$, and

$$\begin{aligned} \|e^{\xi\cdot x}(H_{2,\mu} - z)^{-1}e^{-\xi\cdot x}\| &= \|(e^{\xi\cdot x}H_{2,\mu}e^{-\xi\cdot x} - z)^{-1}\| \\ &\leq C' \qquad (|\xi| \leq \epsilon,\ z \in K). \end{aligned}$$

This follows from perturbation theory and analytic continuation. (Note that the equality

$$e^{\xi\cdot x}(H_{2,\mu}-z)^{-1}e^{-\xi\cdot x} = \left(e^{\xi\cdot x}H_{2,\mu}e^{-\xi\cdot x}-z\right)^{-1}$$

on $L_2(Y)\cap L_{2,c}(\mathbf{R}^d)$, whose validity for $z=w$ is obtained by Laplace transform, has to be extended to $K$ by analytic continuation. The absence of this argument in [HV] was pointed out to the authors by W. Arendt.)

Using the resolvent equation

$$(H_{2,\mu}-z)^{-1} = (I+(z-w)(H_{2,\mu}-z)^{-1})(H_{2,\mu}-w)^{-1}$$

together with Lemma 1.3 one concludes the existence of $C''$ such that

$$\|e^{\xi\cdot x}(H_{2,\mu}-z)^{-1}e^{-\xi\cdot x}\|_{p,q} \le C''$$

for $z\in K,\ 1\le p\le 2$ with $\frac{d}{2}(\frac{1}{p}-\frac{1}{q})\le\frac{1}{2},\ |\xi|\le\frac{\epsilon}{2}$.

Iterating this argument one obtains the last inequality for all $p\in[1,2]$ and small $|\xi|$. Using this estimate for $\xi=0$ and the fact that

$$(H_{2,\mu}-w)^{-1} = (H_{p,\mu}-w)^{-1} \quad \text{on } L_p\cap L_2(Y)$$

one obtains $K\subset\rho(H_{p,\mu})$. ∎

**1.4. Remarks.** (a) A slightly different situation has been treated in [ScV]. In this paper the perturbation $\mu$ is the sum of a form small distributional part $\mu_0$ (cf. [HS]) and $\mu_+\in M_0$. This implies that the semigroup $(e^{-tH_\mu};\ t\ge 0)$ acts as a strongly continuous semigroup on $L_p(Y)$ for $p_0\le p\le p_0'$ where $p_0\in[1,2)$ depends on the form bound of $\mu_0$ (cf. [BS]). It is then shown that $\sigma(H_{p,\mu})=\sigma(H_{2,\mu})$ for all $p\in(p_0,p_0')$.

(b) The $p$-independence of the $L_p$-spectrum of elliptic operators on certain Riemannian manifolds was shown in [Stu]. In a similar context the $p$-independence for $1<p<\infty$ was shown in [Sh; Proposition 2.6].

(c) The $p$-independence of spectra has been shown in [Al] for perturbations of certain translation invariant operators.

(d) If $U(\cdot)$ is a strongly continuous semigroup on $L_2(\Omega)$ (where $\Omega\subset\mathbf{R}^d$) satisfying a Gaussian estimate, then it was shown in [Ar] that the spectra of the generators of the corresponding semigroups on $L_p(\Omega)$ are $p$-independent.

## 2. The spectrum of $-\frac{1}{2}\Delta+\mu$ in spaces of continuous functions

We want to show that under suitable hypotheses the spectrum of $-\frac{1}{2}\Delta+\mu$ in

$$C_0(\mathbf{R}^d) = \{f\in C(\mathbf{R}^d);\ f(x)\to 0\ (|x|\to\infty)\}$$

(or in other spaces of bounded continuous functions) is the same as the $L_p$-spectrum.

It turns out that the main point which is specific about this situation is the question whether $(e^{-tH_\mu};\ t \geq 0)$ acts as a strongly continuous semigroup on $C_0(\mathbf{R}^d)$. The fact that then coincidence of spectra can be concluded will follow from very general considerations presented next.

Let $X$ be a Banach space, $(U(t); t \geq 0)$ a strongly continuous semigroup on $X$, and $T$ its generator. The semigroup dual of $X$ is then defined by

$$X^\odot := \{x^* \in X^*;\ T(t)^* x^* \to x^*\ (t \to 0)\};$$

see, e.g., [HP; Chap. XIV], [BB; Sec. 1.4] (where $X^\odot$ is denoted by $X_0^*$), [Ne]. (We use the adjoint space $X^*$ of continuous conjugate linear functionals on $X$ in order to stay consistent with duality in $L_2$.)

**2.1. Theorem.** *Let $Y \subset X^\odot$ be a closed subspace which is invariant under $U^*(t) \quad (t \geq 0)$. Denote by $U_Y(\cdot)$ the part of the semigroup $U^*(\cdot)$ in $Y$, and by $T_Y$ the generator of $U_Y(\cdot)$.*

*(a) Then $T_Y$ is the part of $T^*$ in $Y$,*

$$\begin{aligned} D(T_Y) &= \{x^* \in Y \cap D(T^*);\ T^* x^* \in Y\}, \\ T_Y &= T^* | D(T_Y). \end{aligned}$$

*(b) $\rho_\infty(T) \subset \rho_\infty(T_Y)$, and $(\lambda - T_Y)^{-1}$ is the part of $((\lambda - T)^{-1})^*$ in $Y$, for $\lambda \in \rho_\infty(T)$. (Here $\rho_\infty(T)$ denotes the component of $\rho(T)$ containing a right half plane; and similarly for $T_Y$.)*

*(c) If additionally $Y$ is equi-norming for $X$, i.e., the norm*

$$\|x\|_Y := \sup\{|< x^*, x >|;\ x^* \in Y,\ \|x^*\| \leq 1\} \quad (x \in X)$$

*is equivalent to the original norm in $X$, then*

$$\rho_\infty(T) = \rho_\infty(T_Y).$$

**Proof.** (a) This is known for $Y = X^\odot$, and the proof carries over to our case (cf. [BB; p. 51], [Ne; Theorem 1.3.3]).

(b) For $\lambda \in \mathbf{C}$ with $\mathrm{Re}\lambda$ larger than the type of $U(\cdot)$, the resolvents of $T$ and $T_Y$ are given by the Laplace transform of $U(\cdot)$ and $U_Y(\cdot)$, respectively, and therefore

$$< x^*, (\lambda - T)^{-1} x > \quad = \quad < (\lambda - T_Y)^{-1} x^*, x >$$

for all $x \in X,\ x^* \in Y$. Therefore $(\lambda - T_Y)^{-1}$ is the part of $((\lambda - T)^{-1})^*$ in $Y$. This implies that $((\lambda - T)^{-1})^*$ maps $Y$ to $Y$ for all $\lambda \in \rho_\infty(T)$. By uniqueness we obtain the claimed assertions.

(c) The equivalence of $\|\cdot\|$ and $\|\cdot\|_Y$ implies that there exists a constant $c$ such that

$$\|(\lambda - T)^{-1}\| \le c\|(\lambda - T_Y)^{-1}\| \quad \text{for all} \quad \lambda \in \rho_\infty(T).$$

This implies $\partial(\rho_\infty(T)) \subset \sigma(T_Y)$, and therefore $\rho_\infty(T) = \rho_\infty(T_Y)$. ∎

**2.2. Remark.** The assymptions made in the previous theorem are satisfied, in particular, for $Y = X^\odot$. For this case, however, one has $\rho(T^\odot) = \rho(T)$; cf. [Ne; Theorem 1.4.2].

**2.3. Corollary.** *Assume that $\mu$ satisfies the hypotheses of Theorem 1.1. Let $Y$ be a closed subspace of $L_\infty$ which is equi-norming for $L_1$, invariant under $(e^{-tH_{1,\mu}})^*$ $(t \ge 0)$ and such that*

$$\|(e^{-tH_{1,\mu}})^* f - f\|_\infty \to 0 \quad (t \to 0)$$

*for all $f \in Y$. Denote by $-H_{Y,\mu}$ the generator of the strongly continuous semigroup on $Y$ induced by $((e^{-tH_{1,\mu}})^*;\ t \ge 0)$. Then*

$$\sigma(H_{Y,\mu}) = \sigma(H_{2,\mu}).$$

**2.4. Remarks.** (a) The semigroup dual of $L_1(\mathbf{R}^d)$ for the unperturbed Schrödinger semigroup $(e^{-tH_1};\ t \ge 0)$ is

$$C_{b,u}(\mathbf{R}^d) = \{f \in C(\mathbf{R}^d);\ f \text{ bounded and uniformly continuous}\}.$$

The generator is then the part of $-H_\infty$ in $C_{b,u}$,

$$D(H_{C_{b,u}}) = \{f \in C_{b,u}(\mathbf{R}^d);\ H_{C_{b,u}} f = -\frac{1}{2}\Delta f \in C_{b,u}\}.$$

For $V \in C_{b,u}(\mathbf{R}^d)$, the multiplication operator by $V$ is a bounded operator in $C_{b,u}(\mathbf{R}^d)$, and therefore Theorem 2.3 is applicable to $H + V$ with $Y = C_{b,u}(\mathbf{R}^d)$.

(b) The space $C_0(\mathbf{R}^d)$ is invariant under the unperturbed Schrödinger semigroup, and

$$D(H_{C_0}) = \{f \in C_0(\mathbf{R}^d);\ H_{C_0} f = -\frac{1}{2}\Delta f \in C_0\}.$$

For bounded $V \in C(\mathbf{R}^d)$ the multiplication by $V$ is a bounded operator on $C_0(\mathbf{R}^d)$. Therefore Theorem 2.3 is applicable to $H + V$ with $Y = C_0(\mathbf{R}^d)$.

(c) For $V = V_+ - V_-$, $V_\pm \ge 0$, $V_- \in K_d$, $V_+ \in K_{d,\text{loc}}$ it is shown in [S; Theorem B.3.1] that $e^{-tH_V}$ maps $L_\infty$-functions to continuous functions, for $t > 0$. As a consequence,

$$Y := L_1(\mathbf{R}^d)^\odot$$

consists of continuous functions, in this case.

## 3. An application of Shnol's method.

In order to establish a connection with the PDE-world, we will now discuss an alternative proof of the inclusions

$$\sigma(H_{p,V}) \supset \sigma(H_{2,V}), \qquad \sigma(H_{C_0,V}) \supset \sigma(H_{2,V}). \tag{3.1}$$

To this end, we will produce rather explicit "Weyl sequences" in $L_p$ and also in $C_0$ which are obtained by applying suitably chosen cut-offs to generalized eigenfunctions associated with the expansion theorem for $H_{2,V}$ ([B], [S], [PStW]); this requires some mild modifications of Shnol's method (cf. [Shn], [S; Section C.4], and [HSt]). Therefore, we learn that properties of the Schrödinger operator in Hilbert space $L_2$ fully determine the spectra in $L_p$ and even in $C_0$: while estimates for the resolvent kernel $(H_{2,V} - z)^{-1}(x, y)$ give the inclusion $\varrho(H_{p,V}) \supset \varrho(H_{2,V})$, the converse inclusion will now be a consequence of the eigenfunction expansion theorem for $H_{2,V}$. Related ideas are also discussed in [Sh].

It should be stressed, however, that the approach proposed here requires more restrictive assumptions on the potential $V$, as compared with the "duality and interpolation"-proof described in Section 2. In the following, we will restrict the discussion to the case $V \in L_\infty(\mathbf{R}^d)$ where it is easy to obtain $L_p$-bounds for the gradient of a generalized eigenfunction.

We first collect a few facts (where we always assume that $V$ is bounded):

(1) For $1 \le p \le \infty$, we have ([HV1])

$$D(H_{p,V}) = D(H_p) = \{u \in L_p;\ \Delta u \in L_p\}. \tag{3.2}$$

If, more strongly, $V$ is bounded and continuous, then (cf. Section 2)

$$D(H_{C_0,V}) = D(H_{C_0}) = \{u \in C_0;\ \Delta u \in C_0\}. \tag{3.3}$$

(2) From the generalized eigenfunction expansion theorem for $H_{2,V}$ ([B], [S], [PStW]), we can draw the following conclusion: for any $\mu \in \sigma(H_{2,V})$ and any $\varepsilon > 0$, there exists a $\lambda \in (\mu - \varepsilon, \mu + \varepsilon)$ and a (non-trivial) distributional solution $u$ of the PDE

$$-\frac{1}{2}\Delta u + Vu = \lambda u, \tag{3.4}$$

satisfying a polynomial growth bound

$$|u(x)| \le c_1(1 + |x|)^K, \tag{3.5}$$

with some constants $c_1 > 0$ and $K \in \mathbf{N}$. For $V$ bounded, it is also known that $u$ is (equivalent to) a continuous function (cf., e.g., [S]).

(3) To control the cut-off errors, we need an $L_p$-bound on $\nabla u$, for $u$ satisfying (3.4), (3.5). Note that there is no $L_p$-analogue of the $L_2$-gradient bound given in

[S; Lemma C.2.1]. Here we proceed as in [HV1], using an argument of L. Schwartz, to obtain the following lemma.

**3.1. Lemma.** *Let $p \in [1,\infty]$, and suppose that $\Omega \subset \Omega'$ are open sets in $\mathbf{R}^d$ with the property that* $\mathrm{dist}(\Omega, \partial\Omega') \geq 1$. *Then there exists a constant $C = C(p)$, which is independent of both $\Omega$ and $\Omega'$, such that*

$$\|\nabla u\|_{L_p(\Omega)} \leq C \left( \|u\|_{L_p(\Omega')} + \|\Delta u\|_{L_p(\Omega')} \right), \tag{3.6}$$

*for all $u \in L_p(\Omega')$ with the property that $\Delta u \in L_p(\Omega')$.*

**Proof.** We proceed as in [HV1]: letting $T$ denote the usual fundamental solution for $-\Delta$, and picking some $\chi \in C_c^\infty(\mathbf{R}^d)$ with support in the unit ball and $\chi(x) = 1$ for $|x| \leq 1/2$, we have

$$\nabla u = (\nabla(\chi T)) * \Delta u - \nabla\zeta * u, \tag{3.7}$$

(where $\zeta = (\Delta\chi)T + 2\nabla\chi \cdot \nabla T \in C_c^\infty(\mathbf{R}^d)$), and the required estimate follows from Young's inequality ([RS]). Furthermore, it is clear from eq. (3.7) that $\nabla u$ is continuous, provided $u$ and $\Delta u$ are continuous functions. ∎

Now let $u$ be a (continuous) generalized eigenfunction of $H_2$ and $\varphi \in C_c^\infty(\mathbf{R}^d)$. Then it follows from Lemma 3.1 and $\Delta(\varphi u) = \varphi\Delta u + 2\nabla\varphi\nabla u + (\Delta\varphi)u$ that $\varphi u$ will belong to the domain of $H_p$, for $1 \leq p \leq \infty$. Similarly, if $V$ is bounded and continuous, then $\varphi u$ will belong to the domain of $H_{C_0,V}$.

(4) Central to Shnol's method is the observation that the growth bound (3.5) implies that the $L_2$-norm of $u$, considered on a suitable sequence of balls, will not grow too rapidly (cf. [S]). While the exposition given in [S; Section C.4] can directly be carried over to the $L_p$-case for $1 \leq p < \infty$, it has to be modified for $p = \infty$ and, similarly, also for the space $C_0$. We therefore change the scenario used in [S] and consider

$$\mathcal{E}_n = \{x \in \mathbf{R}^d;\ |x| < 2^n\}, \qquad \mathcal{F}_n = \mathcal{E}_{n+1} \setminus \mathcal{E}_n \qquad (n \in \mathbf{N}). \tag{3.8}$$

We then have the following lemma.

**3.2. Lemma.** *Let $1 \leq p \leq \infty$, and let $u$ be as in (3.5). Let $a > 2$ and set $c_2 = c_2(p) = a^{K+\frac{d}{p}}$. Then there exists a sequence $(n_j)_{j\in\mathbf{N}} \subset \mathbf{N}$, $n_j \to \infty$, such that*

$$\left\| u|_{\mathcal{F}_{n_j}} \right\|_p \leq c_2 \left\| u|_{\mathcal{E}_{n_j}} \right\|_p \qquad (j \in \mathbf{N}). \tag{3.9}$$

**Proof.** If the statement of the lemma were not true, there would exist some $n_0$ such that

$$\|u|_{\mathcal{F}_n}\|_p \geq c_2\ \|u|_{\mathcal{E}_n}\|_p > 0 \qquad (n \geq n_0), \tag{3.10}$$

so that

$$\|u|_{\mathcal{E}_n}\|_p \geq \left\|u|_{\mathcal{F}_{n-1}}\right\|_p \geq c_2 \left\|u|_{\mathcal{E}_{n-1}}\right\|_p \quad (n > n_0). \tag{3.11}$$

This leads to

$$\|u|_{\mathcal{E}_n}\|_p \geq c_2^{n-n_0} \|u|_{\mathcal{E}_{n_0}}\|_p \quad (n \geq n_0), \tag{3.12}$$

in contradiction with the polynomial growth bound of $u$. ∎

With these preparations, it is now easy to prove the inclusions stated in eq. (3.1).

**Proposition 3.3.** *Let $V \in L_\infty(\mathbf{R}^d)$. Then $\sigma(H_{p,V}) \supset \sigma(H_{2,V})$, for all $p \in [1,\infty]$. If, moreover, $V$ is (bounded and) continuous, then $\sigma(H_{C_0,V}) \supset \sigma(H_{2,V})$.*

**Proof.** We first choose a function $\varphi \in C_c^\infty(-2,2)$ with the property that $\varphi(x) = 1$, for $|x| \leq 4/3$, and $\varphi(x) = 0$, for $|x| \geq 5/3$, and we define

$$\varphi_n(x) = \varphi(2^{-n}|x|)\,, \qquad x \in \mathbf{R}^d.$$

Then $\mathcal{G}_n := \operatorname{supp}(\nabla\varphi_n) \subset \mathcal{F}_n$ and $\operatorname{dist}(\mathcal{G}_n, \partial\mathcal{F}_n) \geq 1$, for $n \geq 2$. Furthermore, we have $\|\nabla\varphi_n\|_\infty \leq c_3 2^{-n}$ and $\|\Delta\varphi_n\|_\infty \leq c_4 2^{-2n}$.

Now let $\mu \in \sigma(H_{2,V})$ be given, and let $\varepsilon > 0$. By what was said in point (2), there exists some $\lambda \in (\mu-\varepsilon, \mu+\varepsilon)$ and a (non-trivial) generalized eigenfunction $u$ of $H_{2,V}$ that satisfies (3.4), (3.5). For given $p \in [1,\infty]$, we will prove that there exists a sequence $(n_j) \subset \mathbf{N}$ so that

$$\left\|(H_{p,V} - \lambda)(\varphi_{n_j} u)\right\|_p / \left\|\varphi_{n_j} u\right\|_p \to 0, \quad j \to \infty. \tag{3.13}$$

Therefore, $H_{p,V} - \lambda$ does not have a bounded inverse, whence $\lambda \in \sigma(H_{p,V})$. Taking $\varepsilon \to 0$ then gives $\mu \in \sigma(H_{p,V})$.

Applying Lemma 3.2 to $u$, we find a constant $c_2$ and a sequence $(n_j)$ such that (3.9) holds. As $\varphi_{n_j} u \in \mathcal{D}(H_{p,V})$ and $(H_{p,V} - \lambda)(\varphi_{n_j} u) = -(\nabla\varphi_{n_j})\nabla u - \frac{1}{2}(\Delta\varphi_{n_j})u$, we have

$$\begin{aligned}\left\|(H_{p,V} - \lambda)(\varphi_{n_j} u)\right\|_p &\leq \left\|\nabla\varphi_{n_j}\right\|_\infty \left\|\nabla u|_{\mathcal{G}_{n_j}}\right\|_p + \left\|\Delta\varphi_{n_j}\right\|_\infty \left\|u|_{\mathcal{G}_{n_j}}\right\|_p \\ &\leq c_5\, 2^{-n_j} \left( \left\|u|_{\mathcal{F}_{n_j}}\right\|_p + \left\|\Delta u|_{\mathcal{F}_{n_j}}\right\|_p \right),\end{aligned}$$

by Lemma 3.1. From $V \in L_\infty$ and $\frac{1}{2}\Delta u = (V - \lambda)u$ we now conclude that

$$\left\|(H_{p,V} - \lambda)(\varphi_{n_j} u)\right\|_p \leq c_6 2^{-n_j} \left\|u|_{\mathcal{F}_{n_j}}\right\|_p \leq c_7\, 2^{-n_j} \left\|u|_{\mathcal{E}_{n_j}}\right\|_p \leq c_8 2^{-n_j} \left\|\varphi_{n_j} u\right\|_p\,,$$

and the result follows.

The proof in the case of the space $C_0$ is essentially identical with the $p = \infty$ proof and omitted. ∎

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