EXISTENCE OF WILTON RIPPLES FOR WATER WAVES WITH CONSTANT VORTICITY AND CAPILLARY EFFECTS

CALIN IULIAN MARTIN AND BOGDAN–VASILE MATIOC

Abstract. In this paper we study the water wave problem with capillary effects and constant vorticity when stagnation points are not excluded. When the constant vorticity is close to certain critical values we show that there exist Wilton ripples solutions of the water wave problem with two crests and two troughs per minimal period. They form smooth secondary bifurcation curves which emerge from primary bifurcation branches, that contain a laminar flow solution and consist of symmetric waves of half of the period of the Wilton ripples, at some non-laminar solution. We also prove that any Wilton ripple contains an internal critical layer provided its minimal period is sufficiently small.

1. Introduction

The situation considered in this paper is that of two-dimensional steady periodic rotational waves traveling over a flat bed when the gravity and the surface tension, or only the surface tension, are the restoring forces. The latter appears in the dynamics of water waves in many physical situations like that when the wind blows over a still fluid surface. What one observes first are two-dimensional small amplitude wave trains which are driven by capillarity [19] and, as they grow larger, turn into capillary-gravity waves. While irrotational flows are suitable for waves travelling into still water or over a uniform current, see e.g. [1, 5, 10, 31], non-uniform currents give rise to water flows with vorticity [2]. It is worth pointing out that a constant vorticity, which is a main feature of the setting considered in this paper, is the hallmark of tidal currents, see the discussion in [2, 30].

In the context of irrotational waves it was Wilton [37] who observed that the presence of both capillary and gravity forces determines sometimes a mode to interact with another one having twice its frequency, the resulting waves, called Wilton ripples, possessing two crests within a minimal period. While Reeder and Shinbrot [28] obtained a rigorous existence theory for this type of waves, a more detailed investigation of the local bifurcation picture was performed by Jones and Toland in [17, 18] in the context of irrotational deep water waves and by Jones in [16] for irrotational waves over a fluid layer of finite depth. The existence theory for water-waves with capillary and with an arbitrary vorticity distribution is recent [32, 33] and is restricted to waves without stagnation points. In this setting one can use the so called hodograph transformation, see [4], and express the problem as a quasilinear elliptic problem to which local bifurcation tools may be applied. The existence of water waves with capillary effects traveling on fluids which posses a vertical density stratification has been established only recently [14, 35] (see also [9]). We remark at this point that the many properties of water waves confined to surface tension effects, such as the regularity of the streamlines and of the wave profile [12, 13, 15, 20, 21, 25, 36]

2010 *Mathematics Subject Classification.* 76B15, 76B45, 47J15,
Key words and phrases. Secondary bifurcation; stagnation points; capillarity.

or the description of the particles trajectories within the fluid [11], were only recently investigated (see [1, 3, 7, 24, 27] for the case when surface tension is neglected).

As mentioned above, we restrict in this paper to considering water waves traveling over linearly sheared currents, so that the vorticity is constant, and we do not exclude the presence of stagnation points. In this setting the local bifurcation problem for capillary and capillary-gravity water waves has been investigated in [23, 22] by using a formulation, obtained initially for waves confined merely to gravity forces [6], which allows one to consider waves with stagnation points and overhanging profiles. The global bifurcation problem was addressed in [26], the author using the invertibility of the curvature operator to recast the governing equations as an operator equation for a compact perturbation of the identity.

Secondary bifurcation appears in connection to bifurcation problems for which primary bifurcation branches are continuous functions of a perturbation parameter. In the irrotational case [16, 17, 18] the authors find primary bifurcation branches, by using the wavespeed as the bifurcation parameter, and secondary bifurcation branches emerging from the primary curves for values of the surface tension, set as being the perturbation parameter, close to some critical values for which the eigenspace of the linearization has dimension two. The setting of waves with constant vorticity is more suitable, from the physical point of view, for considering the secondary bifurcation problem because we keep the surface tension coefficient fixed and use instead the constant vorticity as the perturbation parameter. More precisely, when the vorticity is close to some critical values for which the linearization has a two-dimensional kernel consisting of two modes, one of them twice the frequency of the other, we find, by using again the horizontal speed at the wave surface as parameter, primary bifurcation branches consisting of symmetric waves with one crest and trough per period. These primary branches depend smoothly on the constant vorticity and we show that a smooth secondary bifurcation branch, consisting of Wilton ripples with two crests per minimal period, emerges from each of the primary branches when the vorticity is sufficiently close to the critical values mentioned before. We also show that some of the secondary bifurcation branches consist only of Wilton ripples that have a critical layer in the interior of the fluid layer. We emphasize that adding vorticity to the water wave problem makes the analysis more involved and therefore we did not aim to give a complete description of the local bifurcation picture. Our results are true for capillary and capillary-gravity waves as well.

The outline of the paper is as follows. In the first part of Section 2 we present an abstract secondary bifurcation result from [29] which is then used to establish the existence of Wilton ripples for the water wave problem with capillary effects, cf. Theorem 2.7. In the last part of Section 2 we show that the Wilton ripples that we obtained possess a critical layer provided that their wavelength is sufficiently small.

2. Existence of Wilton ripples for waves with capillary and constant vorticity

Secondary bifurcations near a double eigenvalue. When considering the bifurcation problem

$$F(\lambda,\gamma,x)=0, \tag{2.1}$$

for a smooth map $F:\mathbb{R}\times\mathbb{R}\times X\to Y$, where X and Y are Banach spaces and

$$F(\lambda,\gamma,0)=0 \qquad \text{for all } (\lambda,\gamma)\in\mathbb{R}^2, \tag{2.2}$$

secondary bifurcation may occur when, for some $(\lambda_0, \gamma_0) \in \mathbb{R}^2$, the Fréchet derivative $F_x(\lambda_0, \gamma_0, 0) : X \to Y$ is a Fredholm operator of index zero with a two-dimensional kernel. This is the setting that we consider to hold throughout this paragraph. We present now a special case when secondary bifurcation occurs for the operator equation (2.1), and refer to [29] for the detailed analysis.

First of all, a primary branch of solutions has to be found. The next theorem establishes, under some additional assumptions on F, that for each γ close to γ_0, there exists of a primary branch $\mathcal{P}_\gamma$ of solutions of (2.1) which bifurcates from the set of trivial solutions $x = 0$. The result is an adaptation of the theorem on bifurcations from simple eigenvalues due to Crandall and Rabinowitz [8] to operator equations with two parameters.

Theorem 2.1 ([29, Theorem 2.1]). *Assume that there exist closed subspaces X_1 of X and Y_1 of Y such that $F : \mathbb{R}^2 \times X_1 \to Y_1$ and*

- $F_x(\lambda_0, \gamma_0, 0) : X_1 \to Y_1$ *is a Fredholm operator with Fredholm index zero and one-dimensional kernel* $\operatorname{span}\{\phi_1\}$*;*
- $F_{\lambda x}(\lambda_0, \gamma_0, 0)[\phi_1] \notin \operatorname{Im} F_x(\lambda_0, \gamma_0, 0)$.

Then, there exist positive constants ε and δ and smooth functions

$$(\overline{\lambda}, \overline{z}) : D_\delta = \{(\gamma, a) \in \mathbb{R}^2 \,:\, |\gamma - \gamma_0| < \delta, |a| < \delta\} \to \mathbb{R} \times Z_1,$$

with Z_1 denoting the complement of $\operatorname{span}\{\phi_1\}$ *in X_1 such that:*

(i) *For each $\gamma \in (\gamma_0 - \delta, \gamma_0 + \delta)$, the curve $\mathcal{P}_\gamma := \{(\overline{\lambda}(\gamma, a), a(\phi_1 + \overline{z}(\gamma, a))) \,:\, |a| < \delta\}$ is a primary branch of solutions of* (2.1)*;*

(ii) $(\overline{\lambda}, \overline{z})(\gamma_0, 0) = (\lambda_0, 0)$*;*

(iii) *If $(\lambda, \gamma, x) \in \mathbb{R}^2 \times X_1$ is a solution of* (2.1) *and $|\lambda - \lambda_0| < \varepsilon$, $\|x\| < \varepsilon$, $|\gamma - \gamma_0| < \delta$, then either $x = 0$ or $(\lambda, x) \in \mathcal{P}_\gamma$.*

In order to establish the existence of a secondary branch $\mathcal{S}_\gamma$ which bifurcates from the primary branch $\mathcal{P}_\gamma$ one has to make more structural assumptions on the bifurcation problem (2.1):

$$\text{There exist closed spaces } X_2, Y_2 \text{ such that } X = X_1 \oplus X_2 \text{ and } Y = Y_1 \oplus Y_2; \tag{2.3}$$

$$\operatorname{Ker} F_x(\lambda_0, \gamma_0, 0) \cap X_i = \operatorname{span}\{\phi_i\} \text{ for } i = 1, 2; \tag{2.4}$$

$$\text{There exist } 0 \neq \psi_i \in Y \text{ with } \operatorname{span}\{\psi_i\} \oplus \big(Y_i \cap \operatorname{Im} F_x(\lambda_0, \gamma_0, 0)\big) = Y_i \text{ for } i = 1, 2; \tag{2.5}$$

$$F_x(\lambda_0, \gamma_0, 0) X_2 \subset Y_2. \tag{2.6}$$

Moreover, a more general transversality condition than that in Theorem 2.1 is required

$$F_{\lambda x}(\lambda_0, \gamma_0, 0)[\phi_i] \notin \operatorname{Im} F_x(\lambda_0, \gamma_0, 0) \text{ for } i = 1, 2. \tag{2.7}$$

Letting $\operatorname{span}\{\phi_1, \phi_2\} \oplus Z = X$ and defining $P : X \to \operatorname{span}\{\phi_1, \phi_2\}$ to be the projection onto $\operatorname{span}\{\phi_1, \phi_2\}$, the assumption that $F_x(\lambda_0, \gamma_0, 0) : X \to Y$ is a Fredholm operator of index zero with a two-dimensional kernel and the implicit function theorem ensure the existence of a smooth mapping

$$\widehat{z} : \{(\lambda, \gamma, v) \,:\, \|(\lambda, \gamma, v) - (\lambda_0, \gamma_0, 0)\|_{\mathbb{R}^2 \times \operatorname{span}\{\phi_1, \phi_2\}} < \vartheta\} \to Z, \tag{2.8}$$

where $\vartheta > 0$ is small, which has the property that $(I - P)F(\lambda, \gamma, v + \widehat{z}(\lambda, \gamma, v)) = 0$ for all (λ, γ, v) in the definition domain of $\widehat{z}$, and moreover $\widehat{z}(\lambda_0, \gamma_0, 0) = 0$. By the Lyapunov-Schmidt reduction method the original problem is recast into a finite dimensional system

$$\big\langle F(\lambda, \gamma, \alpha\phi_1 + \beta\phi_2 + \widehat{z}(\lambda, \gamma, \alpha\phi_1 + \beta\phi_2)) \big| \psi_i' \big\rangle = 0, \qquad i = 1, 2, \tag{2.9}$$

whereby α, β are small real parameters and $\psi'_i \in Y'$, $i = 1, 2$, are the linear functionals which satisfy $\langle \psi_i | \psi'_j \rangle = \delta_{ij}$ and $\psi'_i = 0$ on $\operatorname{Im} F_x(\lambda_0, \gamma_0, 0)$. Hereby $\langle \cdot | \cdot \rangle$ denotes the duality pairing on $Y \times Y'$. Observe that any non-constant curve $s \mapsto (\alpha(s), \beta(s))$ with the property that the equations (2.9) are satisfied at each pair $(\alpha(s), \beta(s))$ corresponds to a branch of solutions of (2.1). The next theorem guarantees that if $\gamma \neq \gamma_0$ is closed to γ_0, then a secondary bifurcation branch $\mathcal{S}_\gamma$ will emerge from the primary branch $\mathcal{P}_\gamma$ away from the trivial branch of solutions $x = 0$.

Theorem 2.2 ([29, Theorem 3.3]). *Suppose that F satisfies the assumptions of Theorem 2.1, the structural assumptions (2.3)-(2.6), the transversality condition (2.7), and the non-degeneracy conditions*

$$\begin{vmatrix} \langle F_{\lambda x}(\lambda_0, \gamma_0, 0)[\phi_1] | \psi'_1 \rangle & \langle F_{\gamma x}(\lambda_0, \gamma_0, 0)[\phi_1] | \psi'_1 \rangle \\ \langle F_{\lambda x}(\lambda_0, \gamma_0, 0)[\phi_2] | \psi'_2 \rangle & \langle F_{\gamma x}(\lambda_0, \gamma_0, 0)[\phi_2] | \psi'_2 \rangle \end{vmatrix} \neq 0; \tag{2.10}$$

and

$$\begin{vmatrix} \langle F_{\lambda x}(\lambda_0, \gamma_0, 0)[\phi_1] | \psi'_1 \rangle & \langle F_{xx}(\lambda_0, \gamma_0, 0)[\phi_1, \phi_1] | \psi'_1 \rangle \\ \langle F_{\lambda x}(\lambda_0, \gamma_0, 0)[\phi_2] | \psi'_2 \rangle & \langle F_{xx}(\lambda_0, \gamma_0, 0)[\phi_1, \phi_2] | \psi'_2 \rangle \end{vmatrix} \neq 0. \tag{2.11}$$

Then, there exists an interval $\mathcal{I}$ containing zero and smooth functions $\Lambda, \Xi : \mathcal{I} \times \mathcal{I} \to \mathbb{R}$ such that $\Xi(0,0) \neq 0$, and for each $\gamma = \gamma_0 + \theta$ with $0 \neq \theta \in \mathcal{I}$, the curve

$$\mathcal{S}_\gamma := \{(\widetilde{\lambda}(\gamma - \gamma_0, s), \gamma, \widetilde{x}(\gamma - \gamma_0, s)) \, : \, s \in \mathcal{I}\}$$

with

$$\begin{aligned} \widetilde{\lambda}(\theta, s) &:= \lambda_0 + \theta \Lambda(\theta, s) \\ \widetilde{x}(\theta, s) &:= \theta \Xi(\theta, s)\phi_1 + \theta s \phi_2 + \widehat{z}(\lambda_0 + \theta\Lambda(\theta, s), \gamma_0 + \theta, \theta\Xi(\theta, s)\phi_1 + \theta s \phi_2) \end{aligned}$$

is a secondary branch of solutions intersecting $\mathcal{P}_\gamma$ at $(\widetilde{\lambda}(\gamma - \gamma_0, 0), \gamma, \widetilde{x}(\gamma - \gamma_0, 0))$, whereby $\widetilde{x}(\gamma - \gamma_0, 0) \neq 0$.

The bifurcation analysis for the water wave problem. We prove now that the abstract setting presented above can be used to show that secondary bifurcation is a particular feature of the water wave problem when capillary effects are taken into account and when the flow beneath the wave has constant vorticity. To this end we shall use the new formulation for the water wave problem which has been derived in [6] in the context of gravity water waves with constant vorticity and adapted later in [23, 22] to a more general setting which includes also capillary forces.

More precisely, in the regime when surface tension is not negligible, two-dimensional $2\pi-$periodic water waves traveling at constant speed over a flat horizontal bottom are described, when the vorticity of the flow is assumed constant, by some of the solutions of the equation

$$\begin{aligned} &\left(\lambda + \gamma\left(\frac{[w^2]}{2h} - w + \mathcal{C}_h(ww') - w\mathcal{C}_h(w')\right)\right)^2 \\ &\quad = \left(\lambda^2 + \mu + 2\sigma\frac{w'' + w''\mathcal{C}_h(w') - w'\mathcal{C}_h(w'')}{\left(w'^2 + (1 + \mathcal{C}_h(w'))^2\right)^{3/2}} - 2gw\right)\left(w'^2 + \left(1 + \mathcal{C}_h(w')\right)^2\right), \end{aligned} \tag{2.12}$$

where h is a positive constant called conformal mean depth, γ is the (constant) vorticity of the flow, g is the gravity constant, σ is the surface tension coefficient. Moreover, μ and λ are two real parameters, $[w]$ denotes the average of w over one period and $\mathcal{C}_h$ is the Fourier multiplier

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} a_n e^{inx} \mapsto \sum_{n\in\mathbb{Z}\setminus\{0\}} -i\coth(nh)a_n e^{inx}. \tag{2.13}$$

It is shown in [23, 22] (see also [6]) that if $w\in C^{2+\alpha}(\mathbb{R})$ is a $2\pi-$periodic solution of the equation (2.12) which satisfies additionally the following conditions

$$[w]=0, \tag{2.14}$$

$$w>-h, \tag{2.15}$$

$$\mathbb{R}\ni x\mapsto (x+\mathcal{C}_h(w)(x), w(x)+h) \quad \text{is an one-to-one regular curve,} \tag{2.16}$$

then the curve defined by (2.16) is the $2\pi-$periodic profile of a capillary-gravity (or only capillary when $g=0$) wave traveling above the flat bed located at $y=0$. We enhance that the formulation (2.12) describes also waves with overhanging profiles, when the free surface is no longer a graph, but all the solutions we find herein do not possess this property. Let us notice that the constant function $w=0$ is a solution of (2.12) if and only if

$$\mu=0. \tag{2.17}$$

Thus, for each choice of the parameters $(\lambda,\gamma)\in\mathbb{R}^2$, the pair $(\mu,w)=(0,0)$ is a solution of the problem (2.12) which satisfies the conditions (2.14)-(2.16). These are the laminar flow solutions of the capillary or capillary-gravity water wave problem, that is waves with a flat surface and with parallel streamlines. The constant λ can be identified as the speed at the surface of these waves as the stream function associated with these solutions is given by

$$\psi(x,y) = -\frac{\gamma}{2}y^2+(\lambda+\gamma h)y-\left(\lambda h+\frac{\gamma h^2}{2}\right) \qquad 0\le y\le h, \tag{2.18}$$

the fluid occupying the strip $[0\le y\le h]$ (see [6, 23] for more details).

We shall use here the parameter λ as a bifurcation parameter and γ as a perturbation parameter to prove that secondary bifurcation occurs on some of the bifurcation branches found in [23, 22] provided that the constant vorticity is close to some critical values. To do so we need to introduce a functional analytic setting which allows us to rewrite the problem as a bifurcation equation and to use the abstract results presented in the Theorems 2.1-2.2. Because closed subspaces of Banach spaces do not possess in general a closed complement, in view of the structure restrictions imposed in the Theorems 2.1-2.2 we choose a Hilbert space setting. More precisely, we define

$$X := \left\{(\mu,w) := \big(\mu, \sum_{n=1}^{\infty} a_n\cos(nx)\big) \,:\, \mu, a_n\in\mathbb{R} \text{ and } \sum_{n=1}^{\infty} a_n^2 n^6<\infty\right\},$$

$$Y := \left\{\sum_{n=0}^{\infty} a_n\cos(nx) \,:\; a_n\in\mathbb{R} \text{ and } \sum_{n=1}^{\infty} a_n^2 n^2<\infty\right\},$$

which are identified as subspaces of $H^3(\mathbb{S})$ and $H^1(\mathbb{S})$, respectively ($\mathbb{S}$ being the unit circle $\mathbb{R}/(2\pi\mathbb{Z})$). Therefore, they are endowed with the usual Sobolev norms. We define now

$\mathcal{O} := \{(\mu, w) \in X \,:\, w'^2 + (1 + \mathcal{C}_h(w'))^2 > 0\}$, which is a zero neighborhood of X, and the operator $F : \mathbb{R}^2 \times \mathcal{O} \subset \mathbb{R}^2 \times X \to Y$ by the relation

$$\begin{aligned} F(\lambda, \gamma, (\mu, w)) :=& \left(\lambda + \gamma\left(\frac{[w^2]}{2h} - w + \mathcal{C}_h(ww') - w\mathcal{C}_h(w')\right)\right)^2 \\ &- 2\sigma \frac{w'' + w''\mathcal{C}_h(w') - w'\mathcal{C}_h(w'')}{\left(w'^2 + (1 + \mathcal{C}_h(w'))^2\right)^{1/2}} \\ &- \left(\lambda^2 + \mu - 2gw\right)\left(w'^2 + \left(1 + \mathcal{C}_h(w')\right)^2\right). \end{aligned} \tag{2.19}$$

With this notation we have reduced the original problem to solving the equation

$$F(\lambda, \gamma, (\mu, w)) = 0. \tag{2.20}$$

We observe that the operator F is well-defined, it is real-analytic in all its variables and, as noticed before, $F(\lambda, \gamma, (0, 0)) = 0$ for all $(\lambda, \gamma) \in \mathbb{R}^2$. Our choice for X as subspace of $H^3(\mathbb{S})$ is to guarantee that the solutions (μ, w) that we find are of class $C^{2+\alpha}$ and solve thus the original water wave problem.

We are concerned now to verify the assumptions made in the first part of the paper. As a first result we therefore state.

Lemma 2.3. *Given $(\lambda, \gamma) \in \mathbb{R}^2$, the Fréchet derivative $F_{(\mu,w)}(\lambda, \gamma, 0) : X \to Y$ is a Fourier multiplier. More precisely, we have*

$$F_{(\mu,w)}(\lambda, \gamma, 0)\big[\big(\nu, \sum_{n=1}^{\infty} a_n \cos(nx)\big)\big] = -\nu + \sum_{n=1}^{\infty} \mathfrak{m}_n(\lambda, \gamma) a_n \cos(nx) \tag{2.21}$$

whereby

$$\mathfrak{m}_n(\lambda, \gamma) = -2\left(\frac{n}{\tanh(nh)}\lambda^2 + \gamma\lambda - (g + \sigma n^2)\right). \tag{2.22}$$

Proof. The proof follows by using the partial differentiation formula

$$F_{(\mu,w)}(\lambda, \gamma, (\mu, w))[(\nu, v)] = \nu F_\mu(\lambda, \gamma, (\mu, w)) + F_w(\lambda, \gamma, (\mu, w))[v] \tag{2.23}$$

together with the expressions for the partial derivatives

$$\begin{aligned} F_\mu(\lambda, \gamma, (\mu, w)) =& -\left(w'^2 + \left(1 + \mathcal{C}_h(w')\right)^2\right) \qquad (2.24) \\ F_w(\lambda, \gamma, (\mu, w))[v] =& 2\gamma\left(\lambda + \gamma\left(\frac{[w^2]}{2h} - w + \mathcal{C}_h(ww') - w\mathcal{C}_h(w')\right)\right) \\ &\times \left(\frac{[wv]}{h} - v + \mathcal{C}_h(vw') + \mathcal{C}_h(wv') - v\mathcal{C}_h(w') - w\mathcal{C}_h(v')\right) \\ &- 2\sigma \frac{v'' + v''\mathcal{C}_h(w') + w''\mathcal{C}_h(v') - w'\mathcal{C}_h(v'') - v'\mathcal{C}_h(w'')}{\left(w'^2 + (1 + \mathcal{C}_h(w'))^2\right)^{1/2}} \end{aligned}$$

$$\begin{aligned}&+2\sigma\frac{w''+w''\mathcal{C}_h(w')-w'\mathcal{C}_h(w'')}{\left(w'^2+(1+\mathcal{C}_h(w'))^2\right)^{3/2}}\left(w'v'+(1+\mathcal{C}_h(w'))\mathcal{C}_h(v')\right)\\&-2\left(\lambda^2+\mu-2gw\right)\left(w'v'+(1+\mathcal{C}_h(w'))\mathcal{C}_h(v')\right)\\&+2gv\left(w'^2+\left(1+\mathcal{C}_h(w')\right)^2\right).\end{aligned}\tag{2.25}$$

Setting $(\mu, w) = 0$ in (2.23)-(2.25) and using the definition (2.13) we obtain the desired assertion. □

In order to identify primary branches of bifurcations, we need to find the parameters (λ, γ) for which zero is an eigenvalue of $F_{(\mu,w)}(\lambda, \gamma, 0) : X \to Y$ and whether this operator is Fredholm of index zero. Of particular interest are in this setting the values of (λ, γ) for which zero is double eigenvalue of $F_{(\mu,w)}(\lambda, \gamma, 0) : X \to Y$.

Proposition 2.4. *The operator $F_{(\mu,w)}(\lambda, \gamma, 0) : X \to Y$ is a Fredholm operator of index zero for any choice of the parameters $(\lambda, \gamma) \in \mathbb{R}^2$. More precisely, defining the constants*

$$\lambda_n^\pm := -\frac{\gamma}{2}T_n \pm \sqrt{\frac{\gamma^2}{4}T_n^2 + (g+\sigma n^2)T_n} \qquad \text{for } n \in \mathbb{N}\setminus\{0\}, \tag{2.26}$$

$$\gamma_{n,m} := \frac{\left((g+\sigma n^2)T_n - (g+\sigma m^2)T_m\right)^2}{\sigma T_n T_m (T_n - T_m)(m^2 - n^2)} \qquad \text{for } n \neq m \in \mathbb{N}\setminus\{0\}, \tag{2.27}$$

whereby $T_n := n^{-1}\tanh(nh)$ for $n \in \mathbb{N}\setminus\{0\}$, we have that $\gamma_{n,m} > 0, \lambda_n^+ > 0$, $\lambda_n^- < 0$, and:

(i) *If $\lambda \notin \{\lambda_n^\pm : n \in \mathbb{N}\setminus\{0\}\}$, then $F_{(\mu,w)}(\lambda, \gamma, 0)$ is an invertible operator;*

(ii) *If $\lambda = \lambda_n^+$ or $\lambda = \lambda_n^-$ and $\gamma^2 \notin \{\gamma_{n,m} : m \neq n\}$, then $\mathfrak{m}_n(\lambda, \gamma) = 0$ is an eigenvalue of $F_{(\mu,w)}(\lambda, \gamma, 0)$ and the corresponding eigenspace is one-dimensional;*

(iii) *If $\gamma^2 = \gamma_{n,m}$ for some $m \neq n$, then, depending on the sign of γ, either $\lambda_n^+ = \lambda_m^+$ or $\lambda_n^- = \lambda_m^-$. Moreover, when $\lambda_n^+ = \lambda_m^+$ (resp. when $\lambda_n^- = \lambda_m^-$), the operator $F_{(\mu,w)}(\lambda_n^+, \gamma, 0)$ (resp. $F_{(\mu,w)}(\lambda_n^-, \gamma, 0)$) has a two-dimensional kernel, while $F_{(\mu,w)}(\lambda_k^-, \gamma, 0)$ (resp. $F_{(\mu,w)}(\lambda_k^+, \gamma, 0)$), $k \in \{n, m\}$, has a one-dimensional kernel.*

Proof. We note that the sign of the constants defined by (2.26) follows directly from their definition, while the positivity of $\gamma_{n,m}$ is implied by the property of the sequence $(T_n)_n$ of being decreasing.

Since $F_{(\mu,w)}(\lambda, \gamma, 0)$ is a Fourier multiplier with symbol given by (2.22), it follows readily from the definition of the norms on X and Y, that the spectrum of $F_{(\mu,w)}(\lambda, \gamma, 0)$ consists only of the eigenvalues $\{-1\} \cup \{\mathfrak{m}_n(\lambda, \gamma) : n \in \mathbb{N}\setminus\{0\}\}$. Moreover, because $\mathfrak{m}_n(\lambda, \gamma) \to_{n\to\infty} \infty$ the eigenspace corresponding to each eigenvalue is finite dimensional. These facts together with the uniqueness of the Fourier series representation of functions in Y ensures that

$$\operatorname{Ker} F_{(\mu,w)}(\lambda, \gamma, 0) \oplus \operatorname{Im} F_{(\mu,w)}(\lambda, \gamma, 0) = Y,$$

which shows that $F_{(\mu,w)}(\lambda, \gamma, 0)$ is indeed a Fredholm operator of index zero.

In order to prove (i), let us note that zero is an eigenvalue of $F_{(\mu,w)}(\lambda, \gamma, 0)$ if and only if $\mathfrak{m}_n(\lambda, \gamma) = 0$ for some $n \geq 1$. Solving the latter quadratic equation, we conclude that $\mathfrak{m}_n(\lambda, \gamma) = 0$ exactly when $\lambda = \lambda_n^+$ or $\lambda = \lambda_n^-$, whereby $\lambda_n^\pm$ are given by (2.26). This proves (i).

For (ii), let us observe that if $\lambda_n^+ = \lambda_m^+$ or $\lambda_n^- = \lambda_m^-$ for some $n \neq m$, we obtain by using algebraic manipulations that $\gamma^2 = \gamma_{n,m}$. Thus, if $\gamma^2 \notin \{\gamma_{n,m} : m \neq n\}$, then

$\lambda_n^+ \neq \lambda_m^+$ and $\lambda_n^- \neq \lambda_m^-$ when $n \neq m$, and since $\lambda_n^+\lambda_n^- < 0$ we conclude that the kernel of $F_{(\mu,w)}(\lambda,\gamma,0)$ is one-dimensional for any $\lambda \in \{\lambda_n^+, \lambda_n^-\}$.

Finally, the previous manipulations show that if $\gamma^2 = \gamma_{n,m}$ then either $\lambda_n^+ = \lambda_m^+$ or $\lambda_n^- = \lambda_m^-$. More precisely, we have that if $\lambda_n^+ = \lambda_m^+$ when $\gamma = \overline{\gamma} \in \{\pm\sqrt{\gamma_{n,m}}\}$, then $\lambda_n^- \neq \lambda_m^-$, and, when $\gamma = -\overline{\gamma}$, then $\lambda_n^- = \lambda_m^-$ and $\lambda_n^+ \neq \lambda_m^+$. So, we are left to show that there cannot be a third integer $p \notin \{n,m\}$ such that $\lambda_n^+ = \lambda_m^+ = \lambda_p^+$ or $\lambda_n^- = \lambda_m^- = \lambda_p^-$. This and more is shown in the next lemma. □

Lemma 2.5. *We define the constants*

$$\Theta^\pm := \frac{gh^2}{3} + \frac{\gamma^2 h^3}{6} \mp \frac{\gamma h^2}{6}\sqrt{\gamma^2 h^2 + 4gh}. \tag{2.28}$$

and the functions

$$\lambda^\pm(x) := -\frac{\gamma}{2}\frac{\tanh(hx)}{x} \pm \sqrt{\frac{\gamma^2}{4}\frac{\tanh^2(hx)}{x^2} + (g+\sigma x^2)\frac{\tanh(hx)}{x}} \qquad \textit{for } x \geq 0.$$

If $\sigma \geq \Theta^+$ (resp. $\sigma \geq \Theta^-$) then the function λ^+ (resp. $-\lambda^-$) is strictly increasing on $(0,\infty)$. On the other hand, if $\sigma < \Theta^+$ (resp. $\sigma < \Theta^-$) , then the function λ^+ (resp. $-\lambda^-$) has a unique local extremum in $(0,\infty)$, namely a minimum.

Proof. For the proof we refer to the Lemmas 3 and 4 in [22]. □

The local bifurcation problem for the equation (2.20) was studied in the papers [23, 22] in the case when the kernel of the derivative $F_{(\mu,w)}(\lambda,\gamma,0)$ is one-dimensional. The author uses a Hölder space setting and constructs local bifurcation curves consisting only of solutions of (2.20). We show in this paper that if γ^2 is sufficiently close to one of the constants

$$\gamma_N := \gamma_{N,2N} \tag{2.29}$$

whereby $N \in \mathbb{N}\setminus\{0\}$, then at least one secondary bifurcation branch emerges from some of these primary branches at a point which does not belong to the set of laminar flow solutions. The advantage of working in a Hilbert space context is that we can decompose X and Y as orthogonal sums $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$, whereby we set

$$X_1 := \left\{(\mu,w) \in X \,:\, w = \sum_{n=1}^{\infty} a_n\cos(2Nnx)\right\},$$

$$X_2 := \left\{(0,w) \in X \,:\, w = \sum_{\frac{n}{2N}\notin\mathbb{N}} a_n\cos(nx)\right\},$$

$$Y_1 := \left\{v \in Y \,:\, v = \sum_{n=0}^{\infty} a_n\cos(2Nnx)\right\}, \quad Y_2 := \left\{v \in Y \,:\, v = \sum_{\frac{n}{2N}\notin\mathbb{N}} a_n\cos(nx)\right\}.$$

Let us motivate these decompositions: if γ_0 is chosen such that $\gamma_0^2 = \gamma_N$ for some $N \in \mathbb{N}$, then we know from Proposition 2.4 that either $\lambda_N^+ = \lambda_{2N}^+$ or $\lambda_N^- = \lambda_{2N}^-$, and the operator $F_{(\mu,w)}(\lambda_0,\gamma_0,0)$ has a two-dimensional kernel if we define

$$\lambda_0 := \begin{cases} \lambda_{2N}^+, & \text{if } \lambda_N^+ = \lambda_{2N}^+ \text{ when } \gamma = \gamma_0; \\ \lambda_{2N}^-, & \text{if } \lambda_N^- = \lambda_{2N}^- \text{ when } \gamma = \gamma_0. \end{cases} \tag{2.30}$$

Note that λ_0 depends on the parameter γ and $\operatorname{Ker} F_{(\mu,w)}(\lambda_0,\gamma_0,0) = \operatorname{span}\{\phi_1,\phi_2\}$, whereby $\phi_1 := \cos(2Nx) = (0,\cos(2Nx)) \in X_1$ and $\phi_2 = \cos(Nx) = (0,\cos(Nx)) \in X_2$. In order to apply the Theorem 2.1 to our setting we define $\widetilde{\mathcal{O}} := \mathcal{O} \cap X_1$ and observe that since all the functions $w \in \widetilde{\mathcal{O}}$ have the same period π/N, the function $F(\lambda,\gamma,(\mu,w))$ has also period π/N, cf. (2.13) and (2.19). Particularly, the restriction $F : \mathbb{R}^2 \times \widetilde{\mathcal{O}} \to Y_1$ is a well-defined smooth mapping, and, because $\phi_2 \in X_2$, the operator $F_{(\mu,w)} : X_1 \to Y_1$ is a Fredholm operator of index zero. Moreover, in virtue of (2.21) and (2.22) we have

$$\begin{aligned} F_{\lambda(\mu,w)}(\lambda_0,\gamma_0,0)[\phi_1] &= -2\left(\frac{2}{T_{2N}}\lambda_0+\gamma_0\right)\cos(2Nx) \\ &= -2\operatorname{sign}(\lambda_0)\sqrt{\gamma_0^2+\frac{4(g+4\sigma N^2)}{T_{2N}}}\cos(2Nx), \end{aligned} \tag{2.31}$$

and we infer from (2.21) that $F_{\lambda(\mu,w)}(\lambda_0,\gamma_0,0)[\phi_1] \notin \operatorname{Im} F_{(\mu,w)}(\lambda_0,\gamma_0,0)$. Since all the assumptions needed to apply Theorem 2.1 to our equation (2.20) are satisfied, we obtain the following result establishing the existence of local bifurcation branches consisting only of solutions of the (2.20) describing capillary-gravity (or only capillary when $g = 0$) water waves.

Theorem 2.6 (Existence of primary bifurcation branches). *There exists a smooth function*

$$(\overline{\lambda},(\overline{\mu},\overline{w})) : \{(\gamma,a) \,:\, |\gamma-\gamma_0|<\delta, |a|<\delta\} \to \mathbb{R}\times X_1,$$

with $\delta>0$, which intersects the set of laminar solutions at $(\overline{\lambda},(\overline{\mu},\overline{w}))(\gamma,0) := (\lambda_0,0)$ and has the property that for each $|\gamma-\gamma_0|<\delta$ the curve $\mathcal{P}_\gamma := \{(\overline{\lambda},(\overline{\mu},\overline{w}))(\gamma,a) \,:\, |a|<\delta\}$ is a primary branch of solutions of (2.20) with $\overline{w}(\gamma,a)$ having minimal period π/N when $a \neq 0$.

Since for every positive integer n we have $\gamma_{n,m} \to_{m\to\infty} \infty$, we infer from Proposition 2.4 that $\mathfrak{m}_{2N}(\lambda_0,\gamma) = 0$ is a simple eigenvalue of $F_{(\mu,w)}(\lambda_0,\gamma,0) : X \to Y$ when $\gamma \neq \gamma_0$ is sufficiently closed to γ_0. Thus, the primary bifurcation branches $\mathcal{P}_\gamma$ coincide with those found in the papers [23, 22] by using the bifurcation theorem for simple eigenvalues of Crandall and Rabinowitz, cf. [8]. Particularly, we know that $\mathcal{P}_\gamma$ consists of symmetric solutions of (2.20) that have exactly one crest and trough per period. The main result of this paper is the following theorem, showing next that if $\gamma \neq \gamma_0$ is close to γ_0, then a secondary bifurcation curve $\mathcal{S}_\gamma$ emerges from $\mathcal{P}_\gamma \setminus \{(\lambda_0,\gamma,0)\}$. The curve $\mathcal{S}_\gamma$ consists of capillary-gravity (or only capillary when $g = 0$) water waves.

Theorem 2.7 (Existence of secondary bifurcation branches). *Let $\delta > 0$ be the constant found in Theorem 2.6. Then, there exists $\varepsilon \in (0,\delta)$ and for each γ with $0 < |\gamma-\gamma_0| < \varepsilon$ a smooth local curve $\mathcal{S}_\gamma$ that intersects $\mathcal{P}_\gamma$ and consists only of solutions $(\lambda,(\mu,w))$ of problem (2.20) with the property that w is a function of minimal period $2\pi/N$.*

Moreover, the Wilton ripple corresponding to an arbitrary point on $\mathcal{S}_\gamma$ has exactly two crest and troughs per period.

Proof. We first show that the assumptions of Theorem 2.2 are all satisfied. In order to check the structural assumptions (2.3)-(2.6), we observe that in our setting we have $\psi_1 = \phi_1 = \cos(2Nx)$ and $\psi_2 = \phi_2 = \cos(Nx)$. Since $\phi_i \in X_i$ and $\psi_i \in Y_i$ for $i = 1,2$, the representation (2.21) together with the Proposition 2.4 ensures that (2.3)-(2.6) are verified. Invoking (2.31) and using the following relation

$$\begin{aligned} F_{\lambda(\mu,w)}(\lambda_0,\gamma_0,0)[\phi_2] =& -2\left(\frac{2}{T_N}\lambda_0+\gamma_0\right)\cos(Nx)\\ =& -2\,\mathrm{sign}(\lambda_0)\sqrt{\gamma_0^2+\frac{4(g+\sigma N^2)}{T_N}}\psi_2\notin \mathrm{Im}\, F_{(\mu,w)}(\lambda_0,\gamma_0,0) \end{aligned} \tag{2.32}$$

we see that the transversality condition (2.7) is also satisfied.

We are left to check the non-degeneracy conditions (2.10) and (2.11). To this end, we obtain from (2.21) that

$$F_{\gamma(\mu,w)}(\lambda_0,\gamma_0,0)\big[(\nu,v)\big] = -2\lambda_0\sum_{n=1}^\infty a_n\cos(nx) \qquad \forall\,(\nu,v)=(\nu,\sum_{n=1}^\infty a_n\cos(nx))\in X,$$

and, together with the first equalities in (2.31) and (2.32), we see that (2.10) is equivalent to showing that $T_N\neq T_{2N}$. This relation is guaranteed by the property of $(T_N)_N$ of being a decreasing sequence.

To deal with (2.11), we differentiate (2.24) and (2.25) once more and find that

$$\begin{aligned} F_{(\mu,w)^2}(\lambda_0,\gamma_0,0)[\phi_i,\phi_j] =& 2\gamma_0^2\phi_i\phi_j+2\sigma(\phi_i'\mathcal{C}_h(\phi_j'')+\phi_j'\mathcal{C}_h(\phi_i''))\\ &+2(2g-\gamma_0\lambda_0)\left(\phi_i\mathcal{C}_h(\phi_j')+\phi_j\mathcal{C}_h(\phi_j')\right)\\ &-2\lambda_0^2\left(\phi_i'\phi_j'+\mathcal{C}_h(\phi_i')\mathcal{C}_h(\phi_j')\right)\\ &+2\gamma_0\lambda_0\left(\frac{[\phi_i\phi_j]}{h}+\mathcal{C}_h((\phi_i\phi_j)')\right) \end{aligned} \tag{2.33}$$

for $1\leq i,j\leq 2$. Since, by (2.13), for all positive integers n we have that

$$\begin{aligned} \mathcal{C}_h(\cos(nx)) &= \coth(nh)\sin(nx),\\ \mathcal{C}_h(\sin(nx)) &= -\coth(nh)\cos(nx) \end{aligned}$$

we compute

$$\big\langle F_{(\mu,w)^2}(\lambda_0,\gamma_0,0)[\phi_1,\phi_1]\big|\psi_1'\big\rangle=0, \qquad \big\langle F_{(\mu,w)^2}(\lambda_0,\gamma_0,0)[\phi_1,\phi_2]\big|\psi_2'\big\rangle=A,$$

whereby

$$A:=\gamma_0^2+2(g+\sigma N^2)\left(\frac{1}{T_N}+\frac{1}{T_{2N}}\right)-\frac{\lambda_0\gamma_0}{T_{2N}}-\lambda_0^2\left(2N^2+\frac{1}{T_NT_{2N}}\right). \tag{2.34}$$

Recalling (2.31), the condition (2.11) is equivalent to showing that the constant A defined by (2.34) is not zero. To see this, we infer from $\mathfrak{m}_N(\lambda_0,\gamma_0)=0$ that

$$\begin{aligned} A =& \gamma_0^2+2(g+\sigma N^2)\left(\frac{1}{T_N}+\frac{1}{T_{2N}}\right)-\frac{\lambda_0\gamma_0}{T_{2N}}-\lambda_0^2\left(2N^2+\frac{1}{T_NT_{2N}}\right)\\ =& \gamma_0^2+(g+\sigma N^2)\left(\frac{2}{T_N}+\frac{1}{T_{2N}}\right)-2N^2\lambda_0^2-\frac{1}{T_{2N}}\mathfrak{m}_N(\lambda_0,\gamma_0)\\ =& \gamma_0^2+(g+\sigma N^2)\left(\frac{2}{T_N}+\frac{1}{T_{2N}}\right)-2N^2\lambda_0^2, \end{aligned}$$

and expressing λ_0^2 from $\mathfrak{m}_N(\lambda_0,\gamma_0)=0$ and using (2.26) we further obtain

$$\begin{aligned} A =&\gamma_0^2 + (g+\sigma N^2)\left(\frac{2}{T_N}+\frac{1}{T_{2N}}-2N^2T_N\right)\\ &-2N^2T_N\gamma_0\left(-\frac{\gamma_0}{2}T_N+\operatorname{sign}(\lambda_0)\sqrt{\frac{\gamma_0^2}{4}T_N^2+(g+\sigma N^2)T_N}\right). \end{aligned}$$

Since $1>N^2T_N^2$, it suffices to prove that

$$\left((1+N^2T_N^2)\gamma_0^2+(g+\sigma N^2)\frac{1}{T_{2N}}\right)^2 > N^4T_N^2\gamma_0^2\left(\gamma_0^2T_N^2+4(g+\sigma N^2)T_N\right).$$

However, the latter inequality is straightforward, so that we end by inferring that $A>0$.

The assumptions of Theorem 2.2 being verified, we conclude that there exists a constant $\varepsilon>0$ and, for each $0<|\gamma-\gamma_0|<\varepsilon$, a smooth secondary branch $\mathcal{S}_\gamma$ of solutions of (2.20) bifurcating from $\mathcal{P}_\gamma\setminus\{(\lambda_0,\gamma,0)\}$. More precisely, there exists smooth functions $\Lambda,\Xi:(-\varepsilon,\varepsilon)^2\to\mathbb{R}$ with $\Xi(0,0)\neq 0$ and such that for each $\gamma=\gamma_0+\theta$ with $0\neq|\theta|<\varepsilon$, we have

$$\mathcal{S}_\gamma := \{(\widetilde{\lambda}(\gamma-\gamma_0,s),\gamma,(\widetilde{\mu},\widetilde{w})(\gamma-\gamma_0,s))\;:\;|s|<\varepsilon\}$$

whereby

$$\begin{aligned} \widetilde{\lambda}(\theta,s) &:= \lambda_0+\theta\Lambda(\theta,s)\\ (\widetilde{\mu},\widetilde{w})(\theta,s) &:= \theta\Xi(\theta,s)\phi_1+\theta s\phi_2+\widehat{z}(\lambda_0+\theta\Lambda(\theta,s),\gamma_0+\theta,\theta\Xi(\theta,s)\phi_1+\theta s\phi_2). \end{aligned}\tag{2.35}$$

Note that $\widehat{z}$ is the function from the Lyapunov-Schmidt reduction, cf. (2.8), and since $\widehat{z}$ maps into the complement Z of $\operatorname{span}\{\phi_1,\phi_2\}$ in X, the functions $\widetilde{w}(\theta,s)$ have a priori the period 2π. However, defining the subspaces

$$\widetilde{X} := \left\{(\mu,w)\in X\;:\;w=\sum_{n=1}^\infty a_n\cos(Nnx)\right\},\quad \widetilde{Y} := \left\{v\in Y\;:\;v=\sum_{n=0}^\infty a_n\cos(Nnx)\right\},$$

our previous analysis ensures that the restriction $F:\mathbb{R}^2\times(\mathcal{O}\cap\widetilde{X})\to\widetilde{Y}$ satisfies all the assumptions of Theorems 2.1 and Theorem 2.2. In particular, we obtain that the function $\widetilde{z}$ maps into the complement $\widetilde{Z}$ of $\operatorname{span}\{\phi_1,\phi_2\}$ in $\widetilde{X}$, and we conclude that indeed $\widetilde{w}(\theta,s)$ has minimal period $2\pi/N$ for all $s\neq 0$.

Finally, in order to prove that $\widetilde{w}(\theta,s)$ has exactly two troughs and crests per period, we infer from (2.35) that

$$(\widetilde{\mu},\widetilde{w})(\theta,s) := (\widetilde{\mu},\widetilde{w})(\theta,0)+((\widetilde{\mu},\widetilde{w})(\theta,s)-(\widetilde{\mu},\widetilde{w})(\theta,0))$$

whereby $(\widetilde{\mu},\widetilde{w})(\theta,0)\in X_1$ is the point on the primary branch where the secondary bifurcation occurs and has the property that the corresponding water wave solution is symmetric and has exactly one crest and trough in each interval of length π/N, cf. [6, 34]. The second term belongs to $\widetilde{X}$ and is very small when s is close to zero. Restricting, if necessary, the range for s, perturbation arguments, similar to those in [34], show that the solution corresponding to $(\widetilde{\mu},\widetilde{w})(\theta,s)$ has exactly two crests and troughs per minimal period. $\square$

Wilton ripples with internal critical layers. Recalling (2.18), we see that the laminar flows corresponding to the solutions $(\lambda, \gamma, 0)$ contain stagnation points, that is fluid particles traveling horizontally with the same speed as the wave, provided that

$$\lambda(\lambda + \gamma h) < 0. \tag{2.36}$$

In fact, if (2.36) is satisfied, then the laminar flow solution contains a streamline consisting only of stagnation points. Moreover, if $(\lambda, \gamma, 0)$ belongs to a local bifurcation branch and the linearization at this solution has a one-dimensional kernel, then the non-laminar solutions on this branch possess a critical layer inside in the form of a Kelvin cat's eye vortex with a stagnation point in the middle of the vortex, see e.g. [6, 26, 34]. Particularly, if (λ_0, γ_0) are chosen such that (2.36) is satisfied, then all the primary bifurcation branches $\mathcal{P}_\gamma$ and the secondary branches $\mathcal{S}_\gamma$ consist only of solutions with exactly one critical layer with stagnation points, provided that γ is close to γ_0.

Particularly, it is clear that γ_0 and λ_0 need to have opposite signs for (2.36) to be satisfied. Since, when $N \to \infty$, we have

$$\begin{aligned}
&\frac{\gamma_N}{N^3} \to \frac{4\sigma}{3},\\
&\frac{1}{\sqrt{N}}\left(\frac{\sqrt{\gamma_N}}{2}T_N + \sqrt{\frac{\gamma_N}{4}T_N^2 + (g + \sigma N^2)T_N}\right) \to \sqrt{3\sigma},\\
&\frac{1}{\sqrt{N}}\left(\frac{\sqrt{\gamma_N}}{2}T_{2N} + \sqrt{\frac{\gamma_N}{4}T_{2N}^2 + (g + 4\sigma N^2)T_{2N}}\right) \to \sqrt{3\sigma},
\end{aligned}$$

we conclude that if N is large, then γ_0 and λ_0 have opposite signs. Thus, the scenario (2.36) is not excluded for large N. So, assume that $\gamma_0 = -\sqrt{\gamma_N}$ (the case $\gamma_0 = \sqrt{\gamma_N}$) is analogous. Then, letting $N \to \infty$, we get

$$\lambda_0 = \frac{\sqrt{\gamma_N}}{2}T_N + \sqrt{\frac{\gamma_N}{4}T_N^2 + (g + \sigma N^2)T_N} > 0,$$

and

$$\frac{\lambda_0 + \gamma_0 h}{\sqrt{N}} = \frac{\lambda_0 - \sqrt{\gamma_N}h}{\sqrt{N}} \to -\infty,$$

which proves that the condition (2.36) is satisfied when N is sufficiently large.

Conclusion. In this paper we rigorously establish the existence of Wilton ripples that describe finite amplitude capillary-gravity (resp. capillary when gravity is neglected) water waves of permanent form that possess two crests per minimal period. The setting that we considered is that of flows with constant vorticity. More precisely, we show that if for a linearly sheared laminar flow the horizontal velocity at the wave surface and the constant vorticity match certain critical values, cf. (2.26) and (2.27), the linearized problem has the property that a mode interacts with another one having twice its frequency. For the full nonlinear hydrodynamical problem, the higher frequency mode corresponds to a primary curve of non-laminar water wave solutions with one crest and one trough per period bifurcating from the curve of laminar flows. Moreover, as a result of the resonant interactions of the two modes, we establish, when the constant vorticity is closed to the critical value mentioned before, the existence of a secondary bifurcation curve that emerges from the primary branch at a non-laminar solution. This local curve consists of the Wilton ripples solutions, their wavelength being twice that of the solutions on the primary branch.

Our analysis extends the result of Jones [16] to the case of waves with constant vorticity. We emphasize that by adding vorticity to the water wave problem the analysis is much more involved than in the irrotational case [16], but it confers us the advantage to keep the surface tension coefficient fixed throughout the analysis, a perspective which seems daunting in case of paper [16]. A particular feature of the Wilton ripples that we have found is that they possess, when their wavelength is small, an internal critical layer with a stagnation point inside. This property is a hallmark of the rotational setting, as irrotational waves cannot contain stagnation points beneath the wave surface.

References

[1] A. Constantin. The trajectories of particles in Stokes waves. *Invent. Math.*, 166(3):523–535, 2006.

[2] A. Constantin. *Nonlinear Water Waves with Applications to Wave-Current Interactions and Tsunamis*, volume 81 of *CBMS-NSF Conference Series in Applied Mathematics*. SIAM, Philadelphia, 2011.

[3] A. Constantin and J. Escher. Analyticity of periodic traveling free surface water waves with vorticity. *Ann. of Math.*, 173:559–568, 2011.

[4] A. Constantin and W. Strauss. Exact steady periodic water waves with vorticity. *Comm. Pure Appl. Math.*, 57(4):481–527, 2004.

[5] A. Constantin and W. Strauss. Pressure beneath a Stokes wave. *Comm. Pure Appl. Math.*, 63(4):533–557, 2010.

[6] A. Constantin and E. Varvaruca. Steady periodic water waves with constant vorticity: regularity and local bifurcation. *Arch. Ration. Mech. Anal.*, 199(1):33–67, 2011.

[7] A. Constantin and G. Villari. Particle trajectories in linear water waves. *J. Math. Fluid Mech.*, 10(1):1–18, 2008.

[8] M. G. Crandall and P. H. Rabinowitz. Bifurcation from simple eigenvalues. *J. Functional Analysis*, 8:321–340, 1971.

[9] J. Escher, A.-V. Matioc, and B.-V. Matioc. On stratified steady periodic water waves with linear density distribution and stagnation points. *J. Differential Equations*, 251(10):2932–2949, 2011.

[10] M. D. Groves. Steady water waves. *J. Nonlinear Math. Phys.*, 11(4):435–460, 2004.

[11] D. Henry. Particle trajectories in linear periodic capillary and capillary-gravity deep-water waves. *J. Nonlinear Math. Phys*, 14:1–7, 2007.

[12] D. Henry. Analyticity of the streamlines for periodic travelling free surface capillary-gravity water waves with vorticity. *SIAM J. Math. Anal*, 42(6):3103–3111, 2010.

[13] D. Henry. Analyticity of the Free Surface for Periodic Travelling Capillary-Gravity Water Waves with Vorticity. *J. Math. Fluid Mech.*, 14(2):249–254, 2012.

[14] D. Henry and B.-V. Matioc. On the existence of steady periodic capillary-gravity stratified water waves. *Ann. Scuola Norm. Sup. Pisa*, 2012. to appear.

[15] D. Henry and B.-V. Matioc. On the regularity of steady periodic stratified water waves. *Commun. Pure Appl. Anal.*, 11(4):1453–1464, 2012.

[16] M. Jones. Small amplitude capillary-gravity waves in a channel of finite depth. *Glasgow Math. J.*, 31(2):141–160, 1989.

[17] M. Jones and J. Toland. The bifurcation and secondary bifurcation of capillary-gravity waves. *Proc. Roy. Soc. London Ser. A*, 399(1817):391–417, 1985.

[18] M. Jones and J. Toland. Symmetry and the bifurcation of capillary-gravity waves. *Arch. Rational Mech. Anal.*, 96(1):29–53, 1986.

[19] W. Kinnersley. Exact large amplitude capillary waves on sheets of fluid. *J. Fluid Mech.*, 77(2):229–241, 1976.

[20] W.-X. Li and L.-J. Wang. Regularity of traveling free surface water waves with vorticity. arXiv:1112.2730v1.

[21] C. I. Martin. Regularity of steady periodic capillary water waves with constant vorticity. *J. Nonlinear Math. Phys.*, 19(supp.01):1240006, 7 p., 2012.
[22] C. I. Martin. Local bifurcation and regularity for steady periodic capillary–gravity water waves with constant vorticity. *Nonlinear Anal. Real World Appl.*, (14):131–149, 2013.
[23] C. I. Martin. Local bifurcation for steady periodic capillary water waves with constant vorticity. *J. Math. Fluid Mech.*, 15(1):155–170, 2013.
[24] A.-V. Matioc. On particle trajectories in linear water waves. *Nonlinear Anal. Real World Appl.*, 11(5):4275–4284, 2010.
[25] A.-V. Matioc. Steady internal water waves with a critical layer bounded by the wave surface. *J. Nonlinear Math. Phys.*, 19(1):1250008, 21 p., 2012.
[26] B.-V. Matioc. Global bifurcation for water waves with capillary and constant vorticity. preprint.
[27] B.-V. Matioc. Analyticity of the streamlines for periodic traveling water waves with bounded vorticity. *Int. Math. Res. Not.*, 17:3858–3871, 2011.
[28] J. Reeder and M. Shinbrot. On Wilton ripples. II. Rigorous results. *Arch. Rational Mech. Anal.*, 77(4):321–347, 1981.
[29] M. Shearer. Secondary bifurcation near a double eigenvalue. *SIAM J. Math. Anal.*, 11(2):365–389, 1980.
[30] A. F. Teles da Silva and D. H. Peregrine. Steep, steady surface waves on water of finite depth with constant vorticity. *J. Fluid Mech.*, 195:281–302, 1988.
[31] J. F. Toland. Stokes waves. *Topol. Methods Nonlinear Anal.*, 7(1):1–48, 1996.
[32] E. Wahlén. Steady periodic capillary-gravity waves with vorticity. *SIAM J. Math. Anal.*, 38(3):921–943 (electronic), 2006.
[33] E. Wahlén. Steady periodic capillary waves with vorticity. *Ark. Mat.*, 44(2):367–387, 2006.
[34] E. Wahlén. Steady water waves with a critical layer. *J. Differential Equations*, 246(6):2468–2483, 2009.
[35] S. Walsh. Steady periodic gravity waves with surface tension. preprint.
[36] L.-J. Wang. Regularity of traveling periodic stratified water waves with vorticity. 2012. arXiv:1202.3225v1.
[37] J. R. Wilton. On ripples. *Phil. Mag.*, 29:688–700, 1915.

Institut für Mathematik, Universität Wien, Nordbergstrasse 15, 1090 Wien, Austria
E-mail address: calin.martin@univie.ac.at

Institut für Mathematik, Universität Wien, Nordbergstrasse 15, 1090 Wien, Austria
E-mail address: bogdan-vasile.matioc@univie.ac.at