Article

Regularity of steady periodic capillary water waves with constant vorticity

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We prove a regularity result for steady periodic travelling capillary waves of small amplitude at the free surface of water in a flow with constant vorticity over a flat bed.

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1. Introduction

The papers [19–21] make clear that nonuniform currents give rise to water flows with vorticity. There is an extensive research literature in the area of water waves with vorticity, see [2–4] for existence results, [13,22] for matters of uniqueness, [5,6,8] and [18] for symmetry results and [11,14,17] for regularity results. We would also like to mention the important numerical simulations from [15] and [20].

We consider in this paper steady periodic capillary water waves with constant vorticity under the assumption that there might be stagnation points in the fluid domain and that the free surface is not necessarily a graph. More precisely, we study the regularity of periodic traveling capillary waves at the free surface. The local existence of such waves was proved in [16]. We base our approach on method developed in [10] and which uses conformal mappings to transform a free boundary problem into a quasilinear pseudodifferential equation for a periodic function of one variable.

It is worth to point out that zero vorticity means either no underlying current (a situation corresponding to swell due to a distant storm and entering a region of still water cf. the discussion in [7]) or a uniform underlying current (cf. the discussion in [9]), while constant vorticity is the hallmark of tidal currents cf. the discussion in [12,20].

Let us now present the free-boundary problem of steady periodic traveling capillary water waves with constant vorticity γ in a flow of finite depth. The waves that we consider here are two-dimensional and propagate over water with a flat bed. We will use variables (X,Y) where X represents the direction of propagation and Y denotes the height. The water domain Ω in the XY-plane is bounded below by the impermeable flat bed

$$\mathcal{B} = \{(X,0); X \in \mathbb{R}\},$$

and above by an a priori unknown curve

$$\mathcal{S}(t) = \{u(t,s), v(t,s); s \in \mathbb{R}\}, t \geq 0 \tag{1.1}$$

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with

$$(u_s(t,s))^2 + (v_s(t,s))^2 > 0 \text{ for all } s \in \mathbb{R}, t \geq 0 \tag{1.2}$$

and

$$u(t, s+L) = u(t,s) + L,\ v(t,s+L) = v(t,s) \text{ for all } s \in \mathbb{R}, t \geq 0, \tag{1.3}$$

representing the free surface of the water, which is L-periodic in the horizontal direction. In addition to (1.1), (1.2) and (1.3) we ask that $u, v \in C^{2,\alpha}$ where by $C^{p,\alpha}$ ($p \geq 0$ an integer and $\alpha \in (0,1)$) we understand the standard space of functions whose partial derivatives up to order p are Hölder continuous with exponent α over their domain of definition. The equation of mass conservation for water waves gives rise to the existence of a stream function $\psi(X,Y)$ which is L-periodic in X throughout Ω and moreover $(\psi_Y, -\psi_X)$ gives the velocity field. Choosing a parametrization such that u and v are independent of t in the moving frame we have the following governing equations and boundary conditions (see [16]).

$$\begin{aligned} \Delta\psi &= -\gamma \text{ in } \Omega, \\ \psi &= -m \text{ on } \mathcal{B}, \\ \psi &= 0 \quad \text{ on } \mathcal{S}, \\ |\nabla\psi|^2 - 2\sigma\tfrac{u_s v_{ss} - u_{ss} v_s}{((u_s)^2+(v_s)^2)^{3/2}} &= Q \quad \text{ on } \mathcal{S}. \end{aligned} \tag{1.4}$$

where γ represents the constant vorticity, m denotes the relative mass flux, σ is the coefficient of surface tension (see [12]) and the constant Q is related to the hydraulic head E by the relation $Q = 2(E - P_0)$, where P_0 represents the constant atmospheric pressure.

Definition 1.1. We say that a solution (Ω, ψ) of the water wave equation (1.4) is of class $C^{2,\alpha}$ if the free surface satisfies (1.1),(1.2) and (1.3), with $u, v \in C^{2,\alpha}$ and $\psi \in C^2(\Omega) \cap C^{2,\alpha}(\overline{\Omega})$.

In order to deal with (1.4) we need a reformulation of it as a quasilinear equation in a fixed domain for a periodic function of one variable. We need first a few notations.
By $C^{p,\alpha}_{\text{loc}}$ we will denote the set of functions of class $C^{p,\alpha}$ over any compact subset of their domain of definition. By $C^{p,\alpha}_{2\pi}$ we denote the space of functions of one real variable which are 2π periodic and of class $C^{p,\alpha}_{\text{loc}}$ in $\mathbb{R}$. By $C^{p,\alpha}_{2\pi,o}$ we denote the functions that are in $C^{p,\alpha}_{2\pi}$ and have zero mean over one period. For any $d > 0$ let

$$\mathcal{R}_d = \{(x,y) \in \mathbb{R}^2 : -d < y < 0\}.$$

For any $w \in C^{p,\alpha}_{2\pi}$ let $W \in C^{p,\alpha}(\overline{\mathcal{R}}_d)$ be the unique solution of

$$\begin{aligned} \Delta W &= 0 \text{ in } \mathcal{R}_d, \\ W(x,-d) &= 0, x \in \mathbb{R}, \\ W(x,0) &= w(x), x \in \mathbb{R}. \end{aligned} \tag{1.5}$$

The function $(x,y) \to W(x,y)$ is 2π-periodic in x throughout $\mathcal{R}_d$. For $p \in \mathbb{Z}, p \geq 1$ and $\alpha \in (0,1)$ we define the *periodic Dirichlet-Neumann operator for a strip* $\mathcal{G}_d$ by

$$\mathcal{G}_d(w)(x) = W_y(x,0),\ x \in \mathbb{R}.$$

We have that $\mathcal{G}_d : C^{p,\alpha}_{2\pi} \to C^{p-1,\alpha}_{2\pi}$ is a bounded linear operator.
If the function w takes the constant value c then

$$\mathcal{G}_d(c) = \frac{c}{d}. \tag{1.6}$$

Let Z be the unique (up to a constant) harmonic function in $\mathcal{R}_d$, such that $Z + iW$ is holomorphic in $\mathcal{R}_d$. If $w \in C^{p,\alpha}_{2\pi,o}$ it follows from the discussion in Section 2 of [10] that the function $(x, y) \to Z(x, y)$ is 2π-periodic in x throughout $\mathcal{R}_d$. We specify the constant in the definition of Z by asking that $x \to Z(x, 0)$ has zero mean over one period. We define $\mathcal{C}_d(w)$ by

$$\mathcal{C}_d(w)(x) = Z(x, 0),\ x \in \mathbb{R}.$$

The obtained mapping $\mathcal{C}_d : C^{p,\alpha}_{2\pi,o} \to C^{p,\alpha}_{2\pi,o}$ is a bounded linear operator and is called the *periodic Hilbert transform for a strip*. If $w \in C^{p,\alpha}_{2\pi,o}$ for $p \geq 1$ we have

$$\mathcal{G}_d(w) = (\mathcal{C}_d(w))' = \mathcal{C}_d(w'). \tag{1.7}$$

It also follows (see [10]) that for $p \geq 1$,

$$\mathcal{G}_d(w) = \frac{[w]}{d} + \mathcal{C}_d(w'), \tag{1.8}$$

where $[w]$ denotes the average of w over one period.

Definition 1.2.

- We say that $\Omega \subset \mathbb{R}^2$ is an *L-periodic strip like domain* if it is contained in the upper half (X, Y)-plane and if its boundary consists of the real axis $\mathcal{B}$ and a parametric curve $\mathcal{S}$ defined by (1.1) and which satisfies (1.2) and (1.3).
- For any such domain, the *conformal mean depth* is defined to be the unique positive number h such that there exists an onto conformal mapping $\tilde{U} + i\tilde{V} : \mathcal{R}_h \to \Omega$ which admits an extension between the closures of these domains, with onto mappings

  $$\{(x, 0) : x \in \mathbb{R}\} \to \mathcal{S},$$

  and

  $$\{(x, -h) : x \in \mathbb{R}\} \to \mathcal{B},$$

  and such that

$$\begin{matrix} \tilde{U}(x + L, y) = \tilde{U}(x, y) + L, \\ \tilde{V}(x + L, y) = \tilde{V}(x, y), \end{matrix} (x, y) \in \mathcal{R}_h \tag{1.9}$$

The existence and uniqueness of such an h was proved in Appendix A of the paper [10]. We are now able to formulate the equivalence of (1.4) with a quasilinear equation for a periodic function of one variable in a fixed domain.

Theorem 1.1. *If* (Ω,ψ) *of class* $C^{2,\alpha}$ *is a solution of* (1.4) *then there exists a positive number* h *and a function* $v\in C^{2,\alpha}_{2\pi}$ *such that*

$$\begin{array}{l}\left\{\frac{m}{kh}+\gamma\left(\mathcal{G}_{kh}(\frac{v^2}{2})-v\mathcal{G}_{kh}(v)\right)\right\}^2=\left(Q+2\sigma\frac{\mathcal{G}_{kh}(v)v''-\mathcal{G}_{kh}(v')v'}{(v'^2+\mathcal{G}_{kh}(v)^2)^{3/2}}\right)\left(v'^2+\mathcal{G}_{kh}(v)^2\right)\\ [v]=h\\ v(x)>0 \text{ for all } x\in\mathbb{R},\\ \text{the mapping } x\to\left(\frac{x}{k}+\mathcal{C}_{kh}(v-h)(x),v(x)\right) \text{ is injective on } \mathbb{R},\\ v'(x)^2+\mathcal{G}_{kh}(v)(x)^2\neq 0 \text{ for all } x\in\mathbb{R},\end{array} \tag{1.10}$$

where $k=\frac{2\pi}{L}$. *Moreover*

$$\mathcal{S}=\left\{\left(a+\frac{x}{k}+\mathcal{C}_{kh}(v-h)(x),v(x)\right):x\in\mathbb{R}\right\}, \tag{1.11}$$

for some constant $a\in\mathbb{R}$, *whose presence in the formula* (1.11) *is due to the invariance of problem* (1.4) *to horizontal translations. Conversely, let* $h>0$ *and* $v\in C^{2,\alpha}_{2\pi}$ *be such that* (1.10) *holds. Assume also that* $\mathcal{S}$ *is defined by* (1.11), *let* Ω *be the domain whose boundary consists of* $\mathcal{S}$ *and of the real axis* $\mathcal{B}$ *and let* $a\in\mathbb{R}$ *be arbitrary. Then there exists a function* ψ *in* Ω *such that* (Ω,ψ) *is a solution of* (1.4) *of class* $C^{2,\alpha}$.

For the proof of Theorem 1.1 we refer the reader to [16] for the case of capillary water waves, and to [10] for the case of gravity water waves. Concerning the existence of solutions to the problem (1.10) we refer the reader to [16] (Theorem 3.3). There we proved the following theorem.

Theorem 1.2. *For any* $h>0$, $k>0$, $\gamma\in\mathbb{R}$ *and* $m\in\mathbb{R}$ *satisfying* $k^3\geq\frac{\gamma^2}{\sigma}$, $kh\geq\frac{1}{2}$ *there exists laminar flows with a flat free surface in water of depth* h, *of constant vorticity* γ *and relative mass flux* m. *The laminar flows of flux*

$$m_\pm=\frac{\gamma h^2}{2}-\frac{\gamma h\tanh(kh)}{2k}\pm h\sqrt{\frac{\gamma^2\tanh^2(kh)}{4k^2}+k\sigma\tanh(kh)}$$

are exactely those with horizontal speeds at the flat free surface equal to

$$\lambda_\pm=-\frac{\gamma\tanh(kh)}{2k}\pm\sqrt{\frac{\gamma^2\tanh^2(kh)}{4k^2}+k\sigma\tanh(kh)}.$$

The flows of mass flux given by $m_\pm$ *trigger the appearance of periodic steady waves of small amplitude, with period* $\frac{2\pi}{k}$ *and conformal mean depth* h, *which have a smooth profile with one crest and one trough per period, monotone between consecutive crests and troughs and symmetric about any crest line.*

2. Regularity

We are now able to present the regularity result for solutions of the problem (1.10).

Theorem 2.1. *Let* $h>0$, $\alpha\in(0,1)$ *and* $v\in C^{2,\alpha}_{2\pi}$ *be a solution of* (1.10). *Then* $v\in C^{\infty}_{2\pi}$.

Proof. From (1.6) and (1.8) we find that

$$\mathcal{G}_{kh}(v)v'' - \mathcal{G}_{kh}(v')v' = \left(\frac{1}{k} + \mathcal{C}_{kh}(v')\right) v'' - v'\mathcal{C}_{kh}(v'') \tag{2.1}$$

From (1.8) and the second equation of (1.10) we have that

$$\mathcal{G}_{kh}(\frac{v^2}{2}) - v\mathcal{G}_{kh}(v) = \frac{[v^2]}{2kh} + \mathcal{C}_{kh}(vv') - \frac{v}{k} - v\mathcal{C}_{kh}(v') = \frac{[v^2]}{2kh} - \frac{v}{k} - \mathcal{Q}_{kh}(v), \tag{2.2}$$

where $\mathcal{Q}_{kh}(v) = v\mathcal{C}_{kh}(v') - \mathcal{C}_{kh}(vv')$. From Lemma 3.3 we have that $\mathcal{Q}_{kh}(v) \in C^{2,\alpha/3}_{2\pi}$ since $v \in C^{2,\alpha}_{2\pi}$. The latter fact together with the formulas (2.1), (2.2), (1.10) and using $v'^2 + \mathcal{G}_{kh}(v)^2 \in C^{1,\alpha/3}_{2\pi}$ yield

$$\left(\frac{1}{k} + \mathcal{C}_{kh}(v')\right) v'' - v'\mathcal{C}_{kh}(v'') \in C^{1,\alpha/3}_{2\pi} \tag{2.3}$$

Now from Lemma (3.3) with $f = -v' \in C^{1,\alpha}_{2\pi}$ and $g = \mathcal{C}_{kh}(v'') \in C^{0,\alpha}_{2\pi}$ it follows that

$$-v'\mathcal{C}_{kh}(\mathcal{C}_{kh}(v'')) - \mathcal{C}_{kh}(-v'\mathcal{C}_{kh}(v'')) \in C^{1,\alpha/3}_{2\pi}, \tag{2.4}$$

and taking into account that $\mathcal{C}_{kh}^{-1} = -\mathcal{C}_{kh}$ (see Lemma (3.2)) we obtain from above that

$$v'v'' - \mathcal{C}_{kh}(-v'\mathcal{C}_{kh}(v'')) \in C^{1,\alpha/3}_{2\pi}. \tag{2.5}$$

By applying $\mathcal{C}_{kh}$ to (2.3) and using (2.5) we get

$$\frac{1}{k}\mathcal{C}_{kh}(v'') + C_{kh}(v''\mathcal{C}_{kh}(v')) + v'v'' \in C^{1,\alpha/3}_{2\pi} \tag{2.6}$$

Setting $f = \mathcal{C}_{kh}(v') \in C^{1,\alpha}_{2\pi}$ and $g = v'' \in C^{0,\alpha}_{2\pi}$ we get by applying Lemma (3.3)

$$\mathcal{C}_{kh}(v')\mathcal{C}_{kh}(v'') - \mathcal{C}_{kh}(v''\mathcal{C}_{kh}(v')) \in C^{1,\alpha/3}_{2\pi} \tag{2.7}$$

Adding up (2.6) and (2.7) yields

$$\left(\frac{1}{k} + \mathcal{C}_{kh}(v')\right) \mathcal{C}_{kh}(v'') + v'v'' \in C^{1,\alpha/3}_{2\pi} \tag{2.8}$$

We now multiply (2.3) by $\frac{1}{k} + \mathcal{C}_{kh}(v') \in C^{1,\alpha}_{2\pi}$ and (2.8) by $v' \in C^{1,\alpha}_{2\pi}$ and by adding up the resulting expressions we obtain

$$\left(\left(\frac{1}{k} + \mathcal{C}_{kh}(v')\right)^2 + v'^2\right) v'' \in C^{1,\alpha/3}_{2\pi} \tag{2.9}$$

Since the expression in the bracket on the left-hand side of (2.9) is strictly positive and belongs to $C^{1,\alpha}_{2\pi}$ we obtain that $v'' \in C^{1,\alpha/3}_{2\pi}$. Therefore $v \in C^{3,\alpha/3}_{2\pi}$. An interation of this method shows that $v \in C^{\infty}_{2\pi}$.

3. Appendix

This section contains a more precise description of the operator $\mathcal{C}_d$ obtained in [10]. Denote by $L^2_{2\pi}$ the space of 2π-periodic locally square integrable functions of one real variable. By $L^2_{2\pi,\mathrm{o}}$ we denote the subspace of $L^2_{2\pi}$ whose elements have zero mean over one period.

Lemma 3.1. *If*

$$w = \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx),$$

is the Fourier series expansion of $w \in L^2_{2\pi}$ *then*

$$\mathcal{C}_d(w) = \sum_{n=1}^{\infty} a_n \coth(nd)\sin(nx) - \sum_{n=1}^{\infty} b_n \coth(nd)\cos(nx) \tag{3.1}$$

Lemma 3.2. *For any* $d > 0$, $p \geq 0$ *integer and* $\alpha \in (0,1)$, $\mathcal{C}_d : C^{p,\alpha}_{2\pi,o} \to C^{p,\alpha}_{2\pi,o}$ *is a bounded linear operator. Moreover* $\mathcal{C}_d$ *is a bijection from* $L^2_{2\pi,o}$ *onto itself, and* $\mathcal{C}_d^{-1} = -\mathcal{C}_d : C^{p,\alpha}_{2\pi,o} \to C^{p,\alpha}_{2\pi,o}$ *is also a bounded linear operator.*

Lemma 3.3. *Let* $p \geq 1$ *be an integer,* $\alpha \in (0,1)$ *and* $d > 0$. *If* $f \in C^{p,\alpha}_{2\pi}$ *and* $g \in C^{p-1,\alpha}_{2\pi}$ *then*

$$f\mathcal{C}_d(g) - \mathcal{C}_d(fg) \in C^{p,\delta}_{2\pi} \text{ for all } \delta \in (0,\alpha).$$

Proof. The proof follows the line of the proofs of Lemma 3.2 and of Lemma B1 from the paper [10]. □

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