DISPERSION RELATIONS FOR GRAVITY WATER FLOWS WITH TWO ROTATIONAL LAYERS

CALIN IULIAN MARTIN

Abstract. We derive the dispersion relation for periodic traveling water waves propagating at the surface of water possessing a layer of constant non-zero vorticity γ_1 adjacent to the free surface above another rotational layer of vorticity γ_2 which is adjacent to the flat bed. As a by-product we give necessary and sufficient condition for local bifurcation in the frame-work of piecewise constant vorticity. Moreover, we give estimates on the wave speed at the free surface of the bifurcating laminar flows. These estimates involve only the vorticity γ_1, the mean depth of water d and the depth at which the jump in vorticity occurs.

1. Introduction

We devote this paper to the subject of wave-current interactions ([3, 24, 37]) which despite its recognized importance has seen little advancement-a circumstance generated by the complexity of the problem. The term "current" describes here a flow with a flat free surface. The prevailing feature of currents is the existence of shear in the vertical direction. The extensive studies by Peregrine [35] and Jonsson [24] document the interaction of surface gravity waves with vertically sheared currents.

Even the uniform currents (i.e. irrotational flows), which are the simplest ones, have awaited a long time until they had a firm theoretical basis that came through the extensive studies of the Stokes waves [38] and the flow beneath them concerning particle trajectories, behavior of the pressure [1, 2, 5, 10]. The substantial progress in the more complicated scenario of a non-uniform current came only relatively recently through [9] where the existence of small and large amplitude steady periodic water waves with a general (regular) vorticity distribution was proved. Paper [9] was followed by a bulk of papers treating a variety of topics like symmetry [6, 7, 30], stability [12], regularity of the free surface and of the stream lines [8, 15, 19, 20, 41] and allowing for more sophisticated features like stratifications [16, 23, 40], stagnation points and critical layers [13, 14, 26, 27, 39] or the presence of a singular (merely bounded or piece-wise constant) vorticity distribution [11, 29, 31, 32].

As far as our paper is concerned we shall deal here with non-uniform currents whose main characteristic is the presence of non-zero vorticity in the flow and, in addition, we assume that the vorticity has a discontinuous piecewise constant distribution. This situation is of practical relevance and can be observed in regions where there is a rapid change of the current strength cf. [24]. The distribution of vorticity in our setting is as follows: we consider a layer of constant non-vanishing vorticity γ_1 adjacent to the free surface above a rotational flow of vorticity γ_2. On physical grounds, this situation is justified by the fact that rotational wind generated waves possess a layer of high vorticity adjacent to the

2010 *Mathematics Subject Classification.* Primary: 35Q31, 35Q35, 76B15, 76D33; Secondary: 12D10.
Key words and phrases. Dispersion relation; discontinuous vorticity .

wave surface [33, 36], while in the near bed region there may exist currents resulting from sediment transport along the ocean bed [34].
The main topic that we address here is the dispersion relation for small-amplitude waves. This relation indicates how the relative speed of the bifurcating laminar flow at the free surface varies with respect to certain parameters like the wave-length, the mean depth of the flow, and-in the case of a piece-wise constant vorticity like the one we consider here-the position of vorticity jumps. The dispersion relation we obtain recovers the corresponding formula (23) from [4] for the case of a layer of constant non vanishing vorticity adjacent to the flat bed within an irrotational flow as well as the dispersion relation (81) from [11] in the context of a layer of constant non-zero vorticity adjacent to the free surface above fluid in irrotational flow.
To treat the above mentioned vorticity distribution we adopt the framework of weak solutions to the free boundary Euler equations and we refer the interested reader to [11] where the existence of steady two-dimensional periodic water waves of small and large amplitude in a flow with an arbitrary bounded (but discontinuous) vorticity was proven in the context of a fixed mass flux. For the context of a fixed mean-depth we refer the reader to [18]. Concerning the main topic of our paper, it was shown in [21, 22] that the dispersion relations corresponding to the fixed mean depth approach coincide with those in [11] and [4] corresponding to the fixed mass flux point of view.
We also like to mention that the dispersion relation for capillary-gravity waves for the situation of a layer of vorticity adjacent to the surface above irrotational fluid as well as for the case of an isolated layer of vorticity adjacent to the flat bed was obtained in [28]. For recent results on dispersion relations for small amplitude gravity waves with continuous non-constant vorticity we refer the reader to [25].

2. The equations of motion

This paper considers two-dimensional steady periodic water waves which travel over a rotational, incompressible and inviscid fluid propagating in the positive x-direction over the flat bed $y = -d$ (for some $d > 0$) with the free surface $y = \eta(x)$ being a small perturbation of the flat free surface $y = 0$. We assume that the only restoring force acting upon the fluid is gravity. In a reference frame moving in the same direction as the wave with wave speed $c > 0$, the equations of motion are Euler's equations

$$\left\{ \begin{array}{rcl} (\mathfrak{u}-c)\mathfrak{u}_x + \mathfrak{v}\mathfrak{u}_y & = & -P_x \\ (\mathfrak{u}-c)\mathfrak{v}_x + \mathfrak{v}\mathfrak{v}_y & = & -P_y - g, \end{array} \right. \tag{2.1}$$

together with the incompressibility condition

$$\mathfrak{u}_x + \mathfrak{v}_y = 0, \tag{2.2}$$

whereby $(\mathfrak{u}, \mathfrak{v})$ denotes the velocity field, P is the pressure and g is the gravitational constant. An assumption that we make throughout the paper is that $(\mathfrak{u}, \mathfrak{v})$, P and the surface wave profile $x \to \eta(x)$ are periodic in the variable x and for simplicity we choose the period $L = 2\pi$. The vorticity (assumed to be piecewise constant) of the flow is

$$\omega := \mathfrak{u}_y - \mathfrak{v}_x.$$

Equations (2.1) and (2.2) are supplemented by the kinematic boundary conditions

$$\left\{ \begin{array}{l} \mathfrak{v} = (\mathfrak{u}-c)\eta_x \quad \text{on} \quad y = \eta(x) \\ \mathfrak{v} = 0 \quad \text{on} \quad y = -d \end{array} \right. \tag{2.3}$$

representing essentially a necessary and sufficient condition for the flow to move along a boundary but not across/through the boundary, and the dynamic boundary condition

$$P = P_{atm} \quad \text{on} \quad y = \eta(x), \tag{2.4}$$

which decouples the motion of the air above the free surface from that of the water. Here P_{atm} denotes the constant atmospheric pressure. The details about the validity of (2.1)-(2.4) are worked out in [3]. To simplify the problem just presented we introduce the stream function ψ defined (up to a constant) by the relations

$$\psi_x = -\mathfrak{v}, \;\; \psi_y = \mathfrak{u} - c.$$

One reasonable assumption (true for waves which are not near breaking) is the absence of stagnation points in the flow. This assumption can be analiticaly written as

$$\mathfrak{u} < c \quad \text{throughout} \quad \text{the} \quad \text{fluid}, \tag{2.5}$$

Due to (2.5) we have cf. [3, 9] that the vorticity ω is a single-valued function of ψ, i.e.,

$$\omega(x, y) = \gamma(-\psi(x, y)),$$

which finally yields the reformulation of (2.1)-(2.4) as the free boundary value problem

$$\begin{cases} \Delta\psi &= \gamma(-\psi) & \text{in} & -d < y < \eta(x), \\ |\nabla\psi|^2 + 2g(y+d) &= Q & \text{on} & y = \eta(x), \\ \psi &= 0 & \text{on} & y = \eta(x), \\ \psi &= -p_0 & \text{on} & y = -d, \end{cases} \tag{2.6}$$

where Q is a constant related to the total head, and $p_0 < 0$ is a constant representing the relative mass flux, given by

$$p_0 = \int_{-d}^{\eta(x)} (\mathfrak{u}(x, y) - c)\, dy.$$

We aim to further simplify the problem (2.6) by transforming it into a problem in the fixed domain $\overline{\Omega} := [-\pi, \pi] \times [p_0, 0]$. The latter task is performed by means of the partial hodograph transform

$$q(x, y) = x, \quad p(x, y) = -\psi(y) \tag{2.7}$$

which, due to assumption (2.5), provides a diffeomorphism from the fluid domain to Ω and renders the problem (2.6) into the quasilinear elliptic boundary value problem

$$\begin{cases} (1 + h_q^2)h_{pp} - 2h_p h_q h_{pq} + h_p^2 h_{qq} - \gamma h_p^3 &= 0 & \text{in } \overline{\Omega}, \\ 1 + h_q^2 + (2gh - Q)h_p^2 &= 0 & \text{on } p = 0, \\ h &= 0 & \text{on } p = p_0, \end{cases} \tag{2.8}$$

where the unknown function h defined on $\overline{\Omega}$ by

$$h(q, p) := y + d$$

represents the height above the flat bed and is even and of period 2π in the q-variable. The absence of stagnation points is now equivalent to the elliptic non-degeneracy condition

$$h_p > 0 \quad \text{in} \quad \overline{\Omega}.$$

The discontinuous vorticity regime requires a week formulation of the above system as done in [11] where the authors showed that (2.8) is equivalent to the problem

$$\begin{cases} \left\{\frac{1+h_q^2}{2h_p^2}+\Gamma(p)\right\}_p-\left\{\frac{h_q}{h_p}\right\}_q &= 0 \quad \text{in} \quad \Omega, \\ \frac{1+h_q^2}{2h_p^2}+gh &= \frac{Q}{2} \quad \text{on} \quad p=0, \\ h &= 0 \quad \text{on} \quad p=0, \end{cases} \tag{2.9}$$

whereby Γ is defined by

$$\Gamma(p)=\int_0^p \gamma(s)\,ds, \quad p\in[p_0,0].$$

By a solution of (2.9) we understand a function $h\in W^{2,r}_{\text{per}}\subset C^{1,\alpha}_{\text{per}}$, with $r>\frac{2}{1-\alpha}$, (for a fixed $\alpha\in(1/3,1)$) that is a generalized solution cf. [17], Section 8. A family of laminar solutions, i.e. parallel shear flows with flat free surfaces, parametrized by $\lambda>2\max_{[p_0,0]}\Gamma$ is given by

$$H(p):=H(p,\lambda)=\int_0^p \frac{ds}{\sqrt{\lambda-2\Gamma(s)}}+\frac{Q-\lambda}{2g}\in C^{1,\alpha}([p_0,0]) \tag{2.10}$$

cf. [11]. The parameter λ is related to wave speed at the flat free surface $y=0$ of the laminar flow by the formula

$$\sqrt{\lambda}=(c-\mathfrak{u})|_{y=0}=\frac{1}{H_p(0)},$$

and to Q through the relation

$$\int_{p_0}^0 \frac{ds}{\sqrt{\lambda-2\Gamma(s)}}=\frac{Q-\lambda}{2g}.$$

The considerations in [11] show that the necessary and sufficient condition for the existence of waves of small amplitude that are perturbations of the laminar flow solutions (2.10) is that the Sturm-Liouville problem

$$\begin{cases} (a^3U_p)_p=aU & \text{in} \quad (p_0,0), \\ a^3U_p=gU & \text{at} \quad p=0, \\ U=0 & \text{at} \quad p=p_0 \end{cases} \tag{2.11}$$

has a nontrivial solution $U\in C^{1,\alpha}(p_0,0), U\not\equiv 0$. Here $a(\lambda,p)=\sqrt{\lambda-2\Gamma(p)}\in C^\alpha([p_0,0])$. We will study in the next section the problem (2.11) for the case when the water flow has a layer of constant non vanishing vorticity adjacent to the free surface above a rotational layer of a different vorticity adjacent to the flat bed.

3. The dispersion relation

Let $p_1\in[p_0,0]$. We consider a water flow consisting of an rotational layer of vorticity γ_1 adjacent to the free surface corresponding to $p\in[p_1,0]$ and of a second rotational layer of vorticity γ_2 adjacent to the flat bed corresponding to $p\in[p_0,p_1]$. We have first that the function Γ is of the form

$$\Gamma(p)=\begin{cases} \gamma_1 p, & p\in[p_1,0], \\ \gamma_2 p+p_1(\gamma_1-\gamma_2), & p\in[p_0,p_1]. \end{cases} \tag{3.1}$$

Thus

$$a(\lambda, p) = \sqrt{\lambda - 2\Gamma(p)} = \begin{cases} \sqrt{\lambda - 2\gamma_1 p}, & p \in [p_1, 0], \\ \sqrt{\lambda - 2\gamma_2 p - 2p_1(\gamma_1 - \gamma_2)}, & p \in [p_0, p_1]. \end{cases} \tag{3.2}$$

The general theory, cf. [11], aims at finding a function $M \in C^{1,\alpha}(p_0, 0)$ subject to (2.11). Let us set

$$u := M|_{[p_0,p_1]}, \quad v := M|_{[p_1,0]}.$$

The functions u and v will have to satisfy

$$(a^3 u_p)_p = au, \quad \text{for} \quad p \in (p_0, p_1), \tag{3.3}$$

$$(a^3 v_p)_p = av, \quad \text{for} \quad p \in (p_1, 0), \tag{3.4}$$

together with the matching conditions

$$u(p_1) = v(p_1), \qquad u_p(p_1) = v_p(p_1), \tag{3.5}$$

and the boundary conditions

$$u(p_0) = 0, \tag{3.6}$$

$$(a^3 v_p)(0) = gv(0). \tag{3.7}$$

We attempt first to solve equation (3.3) for u. In doing so we set

$$u(p) = \frac{2\gamma_2}{a(p)} \tilde{u}\left(\frac{a(p)}{\gamma_2}\right), \quad p \in [p_0, p_1],$$

whereby $a(\lambda, p) = \sqrt{\lambda - 2\gamma_2 p - 2p_1(\gamma_1 - \gamma_2)}$. Since $a_p = -\frac{\gamma_2}{a(p)}$ we have

$$a^3(p) u_p = 2\gamma_2^2 \tilde{u}\left(\frac{a(p)}{\gamma_2}\right) - 2a(p)\gamma_2 \tilde{u}'\left(\frac{a(p)}{\gamma_2}\right).$$

The latter yields

$$(a^3 u_p)_p = 2\gamma_2 \tilde{u}''\left(\frac{a}{\gamma_2}\right),$$

which transforms the differential equation (3.3) into $\tilde{u}'' = \tilde{u}$, whereby $s = \frac{a}{\gamma_2}$. We then have $\tilde{u}(s) = b_1 \cosh(s) + b_2 \sinh(s)$, (for real constants b_1, b_2) and consequently the general solution of (3.3) is

$$u(p) = \frac{2\gamma_2}{a(p)} \left[b_1 \cosh\left(\frac{a(p)}{\gamma_2}\right) + b_2 \sinh\left(\frac{a(p)}{\gamma_2}\right) \right].$$

The bottom boundary condition (3.6) gives that $b_2 = -b_1 \tanh^{-1}\left(\frac{a(p_0)}{\gamma_2}\right)$. Hence, we find that the general solution of (3.3) has the shape

$$u(p) = \frac{2\gamma_2 c}{a(p) q(\lambda)} \sinh\left(\frac{a(p) - a(p_0)}{\gamma_2}\right), \tag{3.8}$$

whereby c is some real constant and $q(\lambda) = \sinh\left(\frac{a(p_0)}{\gamma_2}\right)$. Moreover,

$$u_p = -\frac{2\gamma_2 c}{a^2(p) q(\lambda)} \cosh\left(\frac{a(p) - a(p_0)}{\gamma_2}\right) + \frac{2\gamma_2^2 C}{a^3(p) q(\lambda)} \sinh\left(\frac{a(p) - a(p_0)}{\gamma_2}\right). \tag{3.9}$$

Employing a similar approach for equation (3.4) we find that

$$v(p) = \frac{2\gamma_1}{a(p)} \left[c_1 \cosh\left(\frac{a(p)}{\gamma_1}\right) + c_2 \sinh\left(\frac{a(p)}{\gamma_1}\right) \right], \tag{3.10}$$

with $a(p)=\sqrt{\lambda-2\gamma_1 p}$ and c_1, c_2 are some real constants. Furthermore,

$$v_p=\frac{2\gamma_1^2}{a^3(p)}\left[c_1\cosh\left(\frac{a(p)}{\gamma_1}\right)+c_2\sinh\left(\frac{a(p)}{\gamma_1}\right)\right]-\frac{2\gamma_1}{a^2(p)}\left[c_1\sinh\left(\frac{a(p)}{\gamma_1}\right)+c_2\cosh\left(\frac{a(p)}{\gamma_1}\right)\right]. \tag{3.11}$$

To deal with the matching conditions (3.5) we make the notations

$$\frac{a(p_1)-a(p_0)}{\gamma_2}=\theta, \quad \frac{a(p_1)}{\gamma_1}=\rho.$$

Setting also $C:=\frac{c}{q(\lambda)}$ we obtain the following system arising from (3.5)

$$\left\{\begin{array}{rcl} \gamma_2 C\sinh(\theta) & = & \gamma_1[c_1\cosh(\rho)+c_2\sinh(\rho)], \\ -\gamma_1\gamma_2\rho C\cosh(\theta)+\gamma_2^2 C\sinh(\theta) & = & \gamma_1^2[c_1\cosh(\rho)+c_2\sinh(\rho)] \\ & & -\rho\gamma_1^2[c_1\sinh(\rho)+c_2\cosh(\rho)], \end{array}\right. \tag{3.12}$$

which after replacing $c_1\cosh(\rho)+c_2\sinh(\rho)$ with $\frac{\gamma_2}{\gamma_1}C\sinh(\theta)$ in the second equation becomes equivalent to

$$\left\{\begin{array}{rcl} c_1\cosh(\rho)+c_2\sinh(\rho) & = & \frac{\gamma_2}{\gamma_1}C\sinh(\theta) \\ c_1\sinh(\rho)+c_2\cosh(\rho) & = & \frac{\gamma_2}{\gamma_1}C\cosh(\theta)-\frac{1}{\rho}\left(\frac{\gamma_2}{\gamma_1}\right)^2 C\sinh(\theta)+\frac{1}{\rho}\frac{\gamma_2}{\gamma_1}C\sinh(\theta). \end{array}\right. \tag{3.13}$$

Solving (3.13) for c_1 and c_2 we find

$$c_1=\frac{\gamma_2}{\gamma_1}C\sinh(\theta-\rho)+\frac{1}{\rho}\left(\frac{\gamma_2}{\gamma_1}-1\right)\frac{\gamma_2}{\gamma_1}C\sinh(\theta)\sinh(\rho), \tag{3.14}$$

and

$$c_2=\frac{\gamma_2}{\gamma_1}C\cosh(\theta-\rho)+\frac{1}{\rho}\left(1-\frac{\gamma_2}{\gamma_1}\right)\frac{\gamma_2}{\gamma_1}C\sinh(\theta)\cosh(\rho). \tag{3.15}$$

We will find below formulas for θ and ρ in terms of $\sqrt{\lambda}$, of the average mean depth d and of the depth at which the jump in the vorticity distributions occurs. Let d_0 to be the average depth corresponding to p_1. At the bifurcation point we have a horizontal fluid velocity $\mathfrak{u}$ that is only a function of y, with $\mathfrak{u}_y=\omega$ and $(c-\mathfrak{u})(0)=\sqrt{\lambda}$. Therefore, $(c-\mathfrak{u})(y)-(c-\mathfrak{u})(0)=\int_0^y(-\omega(s))\,ds$. Thus,

$$(c-\mathfrak{u})(y)=\left\{\begin{array}{ll} \sqrt{\lambda}-\gamma_1 y, & y\in[-d_0,0], \\ \sqrt{\lambda}+\gamma_1 d_0-\gamma_2(y+d_0), & y\in[-d,-d_0] \end{array}\right. \tag{3.16}$$

We compute now the relative mass fluxes from the formula (3.16) and obtain

$$-p_1=\int_{-d_0}^0(c-u)(y)=\int_{-d_0}^0(\sqrt{\lambda}-\gamma_1 y)\,dy=\sqrt{\lambda}d_0+\frac{\gamma_1}{2}d_0^2. \tag{3.17}$$

Considering (3.17) as an equation of second order in the unknown d_0 we have

$$\frac{\gamma_1}{2}d_0^2+\sqrt{\lambda}d_0+p_1=0 \tag{3.18}$$

with solutions

$$d_0=\frac{-\sqrt{\lambda}\pm\sqrt{\lambda-2\gamma_1 p_1}}{\gamma_1}.$$

Since $d_0 > 0$ and because the expression $\frac{-\sqrt{\lambda}+\sqrt{\lambda-2\gamma_1 p_1}}{\gamma_1}$ is positive irrespective of the sign of γ_1 we have that

$$d_0 = \frac{-\sqrt{\lambda}+\sqrt{\lambda-2\gamma_1 p_1}}{\gamma_1} = \frac{a(p_1)-a(0)}{\gamma_1}, \tag{3.19}$$

where the last equality is true from the formula (3.2) for the function $a(p,\lambda)$. Moreover, we have from (3.19) that

$$\rho = \frac{a(p_1)}{\gamma_1} = \frac{\sqrt{\lambda}}{\gamma_1} + d_0. \tag{3.20}$$

Similarly, we compute

$$\begin{aligned} -p_0 &= -\int_{-d}^{-d_0} [\sqrt{\lambda} + \gamma_1 d_0 - \gamma_2(y+d_0)]\,dy + \int_{-d_0}^{0} (\sqrt{\lambda} - \gamma_1 y)\,dy \\ &= \sqrt{\lambda} d + \gamma_1 d_0 d - \frac{\gamma_1}{2} d_0^2 + \frac{\gamma_2}{2}(d-d_0)^2. \end{aligned} \tag{3.21}$$

Putting the above in the form of a second order equation in d we obtain

$$\frac{\gamma_2}{2} d^2 + [\sqrt{\lambda} + (\gamma_1 - \gamma_2) d_0] d + \frac{\gamma_2 - \gamma_1}{2} d_0^2 + p_0 = 0. \tag{3.22}$$

The discriminant of the above equation is

$$\Delta = \lambda + (\gamma_1 - \gamma_2) d_0 (2\sqrt{\lambda} + \gamma_1 d_0) - 2\gamma_2 p_0 \tag{3.23}$$

$$= \lambda - 2p_1(\gamma_1 - \gamma_2) - 2\gamma_2 p_0 = a^2(p_0), \tag{3.24}$$

whereby for the second equality we have used equation (3.18) and the third equality follows from the formula for $a(p)$. Hence,

$$d = -\frac{\sqrt{\lambda}}{\gamma_2} + \frac{(\gamma_2 - \gamma_1) d_0}{\gamma_2} \pm \frac{\sqrt{\lambda - 2p_1(\gamma_1 - \gamma_2) - 2\gamma_2 p_0}}{\gamma_2},$$

which implies that

$$d - d_0 = -\frac{\sqrt{\lambda} + \gamma_1 d_0}{\gamma_2} \pm \frac{\sqrt{\lambda + 2\gamma_2(p_1 - p_0) - 2p_1\gamma_1}}{\gamma_2}. \tag{3.25}$$

To decide the sign in (3.25) we use (3.18) to conclude that $\lambda - 2p_1\gamma_1 = (\sqrt{\lambda} + \gamma_1 d_0)^2$. Therefore, assuming that $\gamma_2 > 0$, we have

$$\sqrt{\lambda + 2\gamma_2(p_1 - p_0) - 2p_1\gamma_1} = \sqrt{(\sqrt{\lambda} + \gamma_1 d_0)^2 + 2\gamma_2(p_1 - p_0)} > |\sqrt{\lambda} + \gamma_1 d_0| = \sqrt{\lambda} + \gamma_1 d_0,$$

where in the last equality we have used relation (3.16). The above inequality shows that

$$\frac{\sqrt{\lambda + 2\gamma_2(p_1 - p_0) - 2p_1\gamma_1} - (\sqrt{\lambda} + \gamma_1 d_0)}{\gamma_2} > 0 \quad \text{for} \quad \gamma_2 > 0.$$

One can show in a similar manner that

$$\frac{\sqrt{\lambda + 2\gamma_2(p_1 - p_0) - 2p_1\gamma_1} - (\sqrt{\lambda} + \gamma_1 d_0)}{\gamma_2} > 0 \quad \text{for} \quad \gamma_2 < 0.$$

Since $d > d_0$ we infer from the above considerations that

$$d - d_0 = \frac{\sqrt{\lambda + 2\gamma_2(p_1 - p_0) - 2p_1\gamma_1} - (\sqrt{\lambda} + \gamma_1 d_0)}{\gamma_2} = -\frac{\gamma_1}{\gamma_2} d_0 + \frac{a(p_0) - a(0)}{\gamma_2}. \tag{3.26}$$

The last equality in (3.26) implies that $\gamma_1 d_0 + \gamma_2(d - d_0) = a(p_0) - a(0)$ which together with (3.19) gives

$$\theta = \frac{a(p_1) - a(p_0)}{\gamma_2} = d_0 - d. \tag{3.27}$$

We can now return to the surface boundary condition (3.7) which can be expressed in a transparent way. Due to (3.10) and (3.11) we can write equation (3.7) as

$$\begin{aligned} 2\gamma_1^2\left[c_1\cosh\left(\frac{\sqrt{\lambda}}{\gamma_1}\right) + c_2\sinh\left(\frac{\sqrt{\lambda}}{\gamma_1}\right)\right] - 2\gamma_1\sqrt{\lambda}\left[c_1\sinh\left(\frac{\sqrt{\lambda}}{\gamma_1}\right) + c_2\cosh\left(\frac{\sqrt{\lambda}}{\gamma_1}\right)\right] \\ = \frac{2\gamma_1 g}{\sqrt{\lambda}}\left[c_1\cosh\left(\frac{\sqrt{\lambda}}{\gamma_1}\right) + c_2\sinh\left(\frac{\sqrt{\lambda}}{\gamma_1}\right)\right]. \end{aligned} \tag{3.28}$$

Using formulas (3.14) and (3.15) we find that

$$\begin{aligned} &c_1\cosh\left(\frac{\sqrt{\lambda}}{\gamma_1}\right) + c_2\sinh\left(\frac{\sqrt{\lambda}}{\gamma_1}\right) \\ &= \frac{\gamma_2}{\gamma_1}C\sinh\left(\theta - \rho + \frac{\sqrt{\lambda}}{\gamma_1}\right) + \frac{1}{\rho}\left(\frac{\gamma_2}{\gamma_1} - 1\right)\frac{\gamma_2}{\gamma_1}C\sinh(\theta)\sinh\left(\rho - \frac{\sqrt{\lambda}}{\gamma_1}\right) \\ &= -\frac{\gamma_2}{\gamma_1}C\sinh(d) + \frac{1}{\rho}\left(\frac{\gamma_2}{\gamma_1} - 1\right)\frac{\gamma_2}{\gamma_1}C\sinh(d_0 - d)\sinh(d_0), \end{aligned} \tag{3.29}$$

whereby the last equality is a consequence of formulas (3.20) and (3.27). Analogously, we find that

$$\begin{aligned} &c_1\sinh\left(\frac{\sqrt{\lambda}}{\gamma_1}\right) + c_2\cosh\left(\frac{\sqrt{\lambda}}{\gamma_1}\right) \\ &= \frac{\gamma_2}{\gamma_1}C\cosh\left(\theta - \rho + \frac{\sqrt{\lambda}}{\gamma_1}\right) + \frac{1}{\rho}\left(1 - \frac{\gamma_2}{\gamma_1}\right)\frac{\gamma_2}{\gamma_1}C\sinh(\theta)\cosh\left(\rho - \frac{\sqrt{\lambda}}{\gamma_1}\right) \\ &= \frac{\gamma_2}{\gamma_1}C\cosh(d) + \frac{1}{\rho}\left(1 - \frac{\gamma_2}{\gamma_1}\right)\frac{\gamma_2}{\gamma_1}C\sinh(d_0 - d)\cosh(d_0), \end{aligned} \tag{3.30}$$

A calculation shows that the surface condition (3.28) can be rewritten after using (3.29) and (3.30) and replacing $\rho = \frac{\sqrt{\lambda}}{\gamma_1} + d_0$ as

$$\begin{aligned} &\cosh(d)\lambda^{\frac{3}{2}} - [(\gamma_1 - \gamma_2)\sinh(d - d_0)\cosh(d_0) - \gamma_1 d_0\cosh(d) - \gamma_1\sinh(d)]\lambda \\ &\quad - [\gamma_1(\gamma_1 - \gamma_2)\sinh(d - d_0)\sinh(d_0) + g\sinh(d) - \gamma_1^2 d_0\sinh(d)]\sqrt{\lambda} \\ &\quad - g[\gamma_1 d_0\sinh(d) - (\gamma_1 - \gamma_2)\sinh(d - d_0)\sinh(d_0)] = 0. \end{aligned} \tag{3.31}$$

Equation (3.31) is called the dispersion relation. It gives the wave speed at the free surface of the bifurcating laminar flow provided it has a unique positive solution. In what follows we show that under certain conditions the equation (3.31) has indeed a unique positive solutions.

Before undertaking the above mentioned task we consider two limit scenarios which will prove that our analysis generalizes the ones in [4] and in Chapter 8 of [11]

Remark 3.1. We set $\gamma_1 = 0$ in formula (3.31), i.e. we are in the case of a water flow with a rotational layer adjacent to the flat bed below irrotational fluid, situation considered in [4]. The equation (3.31) becomes

$$\cosh(d)\lambda^{\frac{3}{2}} + \gamma_2 \sinh(d-d_0)\cosh(d_0)\lambda - g\sinh(d)\sqrt{\lambda} - g\gamma_2 \sinh(d-d_0)\sinh(d_0) = 0, \tag{3.32}$$

which coincides with formula (23) in [4].

Remark 3.2. Setting now $\gamma_2 = 0$ we are placing ourselves in the scenario of [11] where the situation of a fluid with a rotational layer adjacent to the free surface above irrotational flow was considered. Formula (3.31) displays now like

$$\begin{aligned}\cosh(d)\lambda^{\frac{3}{2}} &- \gamma_1[\sinh(d-d_0)\cosh(d_0) - \sinh(d) - d_0\cosh(d)]\lambda \\ &- [\gamma_1^2 \sinh(d-d_0)\sinh(d_0) + g\sinh(d) - \gamma_1^2 d_0 \sinh(d)]\sqrt{\lambda} \\ &- g\gamma_1[d_0\sinh(d) - \sinh(d-d_0)\sinh(d_0)] = 0, \end{aligned} \tag{3.33}$$

which can be rewritten as

$$\begin{aligned}\lambda^{\frac{3}{2}} &+ \gamma_1\left[\frac{\sinh(d_0)\cosh(d-d_0)}{\cosh(d)} + d_0\right]\lambda \\ &- \tanh(d)\left[g - \gamma_1^2\left(d_0 - \frac{\sinh(d-d_0)\sinh(d_0)}{\sinh(d)}\right)\right]\sqrt{\lambda} \\ &- \gamma_1 g\tanh(d)\left[d_0 - \frac{\sinh(d-d_0)\sinh(d_0)}{\sinh(d)}\right] = 0, \end{aligned} \tag{3.34}$$

recovering formula (81) from [11].

Lemma 3.3. *Assume that $\gamma_1 > 0$ and $\gamma_2 > 0$. Then local bifurcation always occurs.*

Proof. Denote with $p(x)$ the polynomial obtained by setting $\sqrt{\lambda} = x$ in the expression in the left hand side of (3.31). We claim first that $p(0) < 0$. Indeed,

$$p(0) = -g[\gamma_1 d_0 \sinh(d) - (\gamma_1 - \gamma_2)\sinh(d-d_0)\sinh(d_0)] < 0$$

since

$$\gamma_1 d_0 \sinh(d) - (\gamma_1-\gamma_2)\sinh(d-d_0)\sinh(d_0) > \gamma_1(d_0\sinh(d) - \sinh(d-d_0)\sinh(d_0)) > 0$$

since the function

$$d_0 \to d_0\sinh(d) - \sinh(d-d_0)\sinh(d_0)$$

has the value 0 at $d_0 = 0$ and its derivative is

$$\sinh(d) - \sinh(d-2d_0) > 0 \quad \text{for all} \quad \mathrm{d} > \mathrm{d_0}.$$

The claim is thus proved and entails the existence of a positive root x_0 of p. We prove now that x_0 is the unique positive root of p. Assuming that p has a second positive root then the third root of p would be real and positive, since by Viète's relations the product of the three roots of p equals $-p(0) > 0$. But their sum equals

$$\begin{aligned}(\gamma_1-\gamma_2)\sinh(d-d_0)\cosh(d_0) &- \gamma_1 d_0\cosh(d) - \gamma_1\sinh(d) \\ &< \gamma_1\sinh(d-d_0)\cosh(d_0) - \gamma_1 d_0\cosh(d) - \gamma_1\sinh(d) \\ &= \gamma_1(\sinh(d-d_0)\cosh(d_0) - \sinh(d)) - \gamma_1 d_0\cosh(d) \\ &= -\gamma_1\sinh(d_0)\cosh(d-d_0) - \gamma_1 d_0\cosh(d) < 0,\end{aligned}$$

and therefore p can not have more than one positive root. The above considerations show that x_0 is the only positive root of p giving the dispersion relation. $\square$

Lemma 3.4. *Assume that* $\gamma_1 < 0$ *and* $\gamma_2 > 0$. *Then local bifurcation occurs if and only if*

$$\frac{g}{\gamma_1^2} > d_0^2 \frac{\cosh(d_0)}{\sinh(d_0)} - d_0. \tag{3.35}$$

Proof. We assume first that (3.35) holds and we prove that bifurcation occurs. Since $\sqrt{\lambda} + \gamma_1 d_0 > 0$, cf.(3.16), we make the substitution $\sqrt{\lambda} = -\gamma_1(x + d_0)$ in equation (3.31). A tedious calculation leads then to the equation $q(x) = 0$ where

$$\begin{aligned} q(x) = \cosh(d)x^3 + &\left[2d_0\cosh(d) - \sinh(d_0)\cosh(d-d_0) - \frac{\gamma_2}{\gamma_1}\sinh(d-d_0)\cosh(d_0)\right]x^2 \\ &\big[d_0^2\cosh(d) - \sinh(d-d_0)\sinh(d_0) + d_0(\sinh(d-d_0)\cosh(d_0) - \sinh(d_0)\cosh(d-d_0)) \\ &+\frac{\gamma_2}{\gamma_1}\sinh(d-d_0)(\sinh(d_0) - 2d_0\cosh(d_0)) - \frac{g}{\gamma_1^2}\sinh(d)\bigg]x \\ &+\sinh(d-d_0)\left(\left(1-\frac{\gamma_2}{\gamma_1}\right)[d_0^2\cosh(d_0) - d_0\sinh(d_0)] - \frac{g}{\gamma_1^2}\sinh(d_0) + \frac{g\gamma_2}{\gamma_1^3}\sinh(d_0)\right) \end{aligned} \tag{3.36}$$

Note now that $q(0) < 0$ is equivalent to

$$\frac{g}{\gamma_1^2}\left(1-\frac{\gamma_2}{\gamma_1}\right) > d_0^2\left(1-\frac{\gamma_2}{\gamma_1}\right)\frac{\cosh(d_0)}{\sinh(d_0)} - d_0\left(1-\frac{\gamma_2}{\gamma_1}\right)$$

which, since $1 - \frac{\gamma_2}{\gamma_1} > 0$, is in turn equivalent to our assumption (3.35). Therefore, q has a root $x_+ > 0$. We will prove that x_+ is the only positive root of q. To see this notice first that the coefficient of x^2 in q is positive. Indeed,

$$2d_0\cosh(d) - \sinh(d_0)\cosh(d-d_0) \geq d_0\cosh(d) > 0, \quad \text{for} \quad \mathrm{d} > \mathrm{d}_0 > 0$$

cf. relation (84) in [11]. Moreover, $-\frac{\gamma_2}{\gamma_1}\sinh(d-d_0)\cosh(d_0) > 0$ as one can easily see. Therefore, the sum of the roots of q is negative, while their product equals $-q(0) > 0$. These latter facts ensure, just as in the previous lemma that q can not have more than one positive root.

To prove the necessity, we assume ab absurdum that

$$\frac{g}{\gamma_1^2} \leq d_0^2\frac{\cosh(d_0)}{\sinh(d_0)} - d_0 := f_1(d_0), \tag{3.37}$$

and we will prove that q has no positive roots, situation equivalent with the absence of bifurcation, obtaining thus a contradiction. To proceed, note that (3.37) is equivalent to $q(0) > 0$. Moreover, the coefficient of x from q is positive if and only if

$$\begin{aligned} \frac{g}{\gamma_1^2} < &d_0^2\frac{\cosh(d)}{\sinh(d)} + d_0\frac{\sinh(d-d_0)\cosh(d_0) - \sinh(d_0)\cosh(d-d_0)}{\sinh(d)} \\ &- \frac{\sinh(d-d_0)\sinh(d_0)}{\sinh(d)} + \frac{\gamma_2}{\gamma_1}\frac{\sinh(d-d_0)(\sinh(d_0) - 2d_0\cosh(d_0))}{\sinh(d)} := f_2(d_0). \end{aligned} \tag{3.38}$$

We show now that

$$f_1(d_0) < f_2(d_0), \quad \text{for} \quad \mathrm{d}_0 > 0. \tag{3.39}$$

The latter and (3.37) would imply that the coefficient of x is indeed positive. To prove (3.39), we see that after multiplying it by $\sinh(d)\sinh(d_0)$ it becomes equivalent to

$$d_0^2 - 2d_0\sinh(d_0)\cosh(d_0) + \sinh^2(d_0) < \frac{\gamma_2}{\gamma_1}\sinh(d_0)(\sinh(d_0) - 2d_0\cosh(d_0)). \quad (3.40)$$

The function

$$d_0 \to d_0^2 - 2d_0\sinh(d_0)\cosh(d_0) + \sinh^2(d_0)$$

is 0 at $d_0 = 0$ and its derivative is

$$\begin{aligned}2d_0 - &2\sinh(d_0)\cosh(d_0) - 2d_0\cosh^2(d_0) - 2d_0\sinh^2(d_0) + 2\sinh(d_0)\cosh(d_0)\\ &= -4d_0\sinh^2(d_0) < 0,\end{aligned}$$

for all $d_0 < 0$. Therefore, the left hand side of (3.40) is strictly negative for $d_0 > 0$. Moreover, the function $d_0 \to \sinh(d_0) - 2d_0\cosh(d_0)$ is strictly negative for $d_0 > 0$ since it equals 0 at $d_0 = 0$ and its derivative is

$$\cosh(d_0) - 2\cosh(d_0) - 2d_0\sinh(d_0) < 0,$$

for all $d_0 > 0$. Therefore, the right hand side of (3.40) is proved since $\frac{\gamma_2}{\gamma_1} < 0$. Hence, inequality (3.39) is proved and, consequently, $q'(x) > 0$ for all $x > 0$. Thus, $q(x) > q(0) > 0$ for all $x > 0$, contradicting the occurrence of local bifurcation. □

Remark 3.5. Is it possible to write a precise formula for the positive solution of (3.31) by means of the Cardano's formula. Its intricacy makes it of little relevance. We prefer to give instead an estimate on the positive solution $\sqrt{\lambda}$ of (3.31).

Lemma 3.6. *Assume that $\gamma_1 < \gamma_2$ and $d_0 < d$. Then*

$$\lambda_+(d_0) < \sqrt{\lambda} < \lambda_+(d), \quad (3.41)$$

whereby $\lambda_+(d)$ is the positive solution of the equation

$$\lambda + \gamma_1\tanh(d)\sqrt{\lambda} - g\tanh(d) = 0$$

and $\lambda_+(d_0)$ is the positive solution of the equation

$$\lambda + \gamma_1\tanh(d_0)\sqrt{\lambda} - g\tanh(d_0) = 0.$$

Proof. We show first that the function $d \to \lambda_+(d)$ is strictly increasing. Indeed, we have

$$\lambda_+(d) = \frac{-\gamma_1\tanh(d) + \sqrt{\gamma_1^2\tanh^2(d) + 4g\tanh(d)}}{2},$$

and hence

$$\lambda_+'(d) = \frac{1}{\cosh^2(d)}\left(-\gamma_1 + \frac{\gamma_1^2\tanh(d) + 2g}{\sqrt{\gamma_1^2\tanh^2(d) + 4g\tanh(d)}}\right)$$

is positive, as one can easily see. We thus have that $0 < \lambda_+(d_0) < \lambda_+(d)$. Note now that (3.31) can be rewritten as

$$\begin{aligned}\frac{1}{\cosh(d_0)}&[\lambda + \gamma_1\tanh(d)\sqrt{\lambda} - g\tanh(d)](\sqrt{\lambda} + \gamma_1 d_0)\\ &= (\gamma_1 - \gamma_2)\frac{\sinh(d - d_0)}{\cosh(d)}[\lambda + \gamma_1\tanh(d_0)\sqrt{\lambda} - g\tanh(d_0)]. \quad (3.42)\end{aligned}$$

We are now ready to prove estimate (3.41). We proceed by contradiction. Assuming that $\sqrt{\lambda} \geq \lambda_+(d)$ it follows that $\lambda + \gamma_1 \tanh(d)\sqrt{\lambda} - g\tanh(d) \geq 0$, which implies via (3.42) and (3.16) that $\lambda + \gamma_1 \tanh(d_0)\sqrt{\lambda} - g\tanh(d_0) \leq 0$. The latter yields that $\sqrt{\lambda} \in (0, \lambda_+(d_0)]$ which is a contradiction with the assumption that $\sqrt{\lambda} \geq \lambda_+(d)$. We have thus proved that

$$\sqrt{\lambda} < \lambda_+(d).$$

It remains to prove the left hand side of (3.41). The above proven inequality implies that $\lambda + \gamma_1 \tanh(d)\sqrt{\lambda} - g\tanh(d) < 0$ and employing again (3.42) and (3.16) we obtain that $\lambda + \gamma_1 \tanh(d_0)\sqrt{\lambda} - g\tanh(d_0) > 0$ which is possible if and only if $\sqrt{\lambda} > \lambda_+(d_0)$. □

Remark 3.7. Setting $\gamma_1 = 0$ we have that $\lambda_+(d_0) = \sqrt{g\tanh(d_0)}$, $\lambda_+(d) = \sqrt{g\tanh(d)}$ and inequality (3.41) recovers inequality (27) from [4].

Lemma 3.8. *Assume $\gamma_1 > \gamma_2$ and that $d > d_0$. Then*

$$\sqrt{\lambda} \in (0, \lambda_+(d_0)) \quad \text{or} \quad \sqrt{\lambda} > \lambda_+(\mathrm{d}).$$

Proof. The proof follows from the facts established in the proof of the previous lemma noticing that the expressions

$$\lambda + \gamma_1 \tanh(d)\sqrt{\lambda} - g\tanh(d)$$

and

$$\lambda + \gamma_1 \tanh(d_0)\sqrt{\lambda} - g\tanh(d_0)$$

need to have the same sign. □

References

[1] D. Clamond and A. Constantin. Recovery of steady periodic wave profiles from pressure measurements at the bed. *J. Fluid Mech.*, 714:463–475, 2013.

[2] A. Constantin. The trajectories of particles in Stokes waves. *Invent. Math.*, 166(3):523–535, 2006.

[3] A. Constantin. *Nonlinear water waves with applications to wave-current interactions and tsunamis*, volume 81 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011.

[4] A. Constantin. Dispersion relations for periodic traveling water waves in flows with discontinuous vorticity. *Commun. Pure Appl. Anal.*, 11(4):1397–1406, 2012.

[5] A. Constantin. Mean velocities in a Stokes wave. *Arch. Ration. Mech. Anal.*, 207(3):907–917, 2013.

[6] A. Constantin, M. Ehrnström, and E. Wahlén. Symmetry of steady periodic gravity water waves with vorticity. *Duke Math. J.*, 140(3):591–603, 2007.

[7] A. Constantin and J. Escher. Symmetry of steady periodic surface water waves with vorticity. *J. Fluid Mech.*, 498:171–181, 2004.

[8] A. Constantin and J. Escher. Analyticity of periodic traveling free surface water waves with vorticity. *Ann. of Math.*, 173:559–568, 2011.

[9] A. Constantin and W. Strauss. Exact steady periodic water waves with vorticity. *Comm. Pure Appl. Math.*, 57(4):481–527, 2004.

[10] A. Constantin and W. Strauss. Pressure beneath a Stokes wave. *Comm. Pure Appl. Math.*, 63(4):533–557, 2010.

[11] A. Constantin and W. Strauss. Periodic traveling gravity water waves with discontinuous vorticity. *Arch. Ration. Mech. Anal.*, 202(1):133–175, 2011.

[12] A. Constantin and W. A. Strauss. Stability properties of steady water waves with vorticity. *Comm. Pure Appl. Math.*, 60(6):911–950, 2007.

[13] A. Constantin and E. Varvaruca. Steady periodic water waves with constant vorticity: regularity and local bifurcation. *Arch. Ration. Mech. Anal.*, 199(1):33–67, 2011.

[14] M. Ehrnström, J. Escher, and E. Wahlén. Steady water waves with multiple critical layers. *SIAM J. Math. Anal.*, 43(3):1436–1456, 2011.

[15] J. Escher. Regularity of rotational travelling water waves. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 370(1964):1602–1615, 2012.
[16] J. Escher, A.-V. Matioc, and B.-V. Matioc. On stratified steady periodic water waves with linear density distribution and stagnation points. *J. Differential Equations*, 251(10):2932–2949, 2011.
[17] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order.* Springer Verlag, 2001.
[18] D. Henry. Steady periodic waves bifurcating for fixed-depth rotational flows with discontinuous vorticity. *preprint.*
[19] D. Henry. Analyticity of the streamlines for periodic travelling free surface capillary-gravity water waves with vorticity. *SIAM J. Math. Anal*, 42(6):3103–3111, 2010.
[20] D. Henry. Analyticity of the Free Surface for Periodic Travelling Capillary-Gravity Water Waves with Vorticity. *J. Math. Fluid Mech.*, 14(2):249–254, 2012.
[21] D. Henry. Dispersion relations for steady periodic water waves of fixed mean-depth with an isolated bottom vorticity layer. *J. Nonlinear Math. Phys.*, 19(suppl. 1):1240007, 14, 2012.
[22] D. Henry. Dispersion relations for steady periodic water waves with an isolated layer of vorticity at the surface. *Nonlinear Anal. Real World Appl.*, 14(2):1034–1043, 2013.
[23] D. Henry and B.-V. Matioc. On the regularity of steady periodic stratified water waves. *Commun. Pure Appl. Anal.*, 11(4):1453–1464, 2012.
[24] I. G. Jonsson. *Wave-current interactions. In: The Sea* . Wiley, New York, 1990.
[25] P. Karageorgis. Dispersion relation for water waves with non-constant vorticity. *Eur. J. Mech. B Fluids*, 34:7–12, 2012.
[26] C. I. Martin. Local bifurcation and regularity for steady periodic capillary–gravity water waves with constant vorticity. *Nonlinear Anal. Real World Appl.*, (14):131–149, 2013.
[27] C. I. Martin. Local bifurcation for steady periodic capillary water waves with constant vorticity. *J. Math. Fluid Mech.*, 15(1):155–170, 2013.
[28] C. I. Martin. Dispersion relations for periodic water waves with surface tension and discontinuous vorticity. *Discrete Cont. Dyn. Syst. Ser. A*, accepted for publication.
[29] C. I. Martin and B.-V. Matioc. Existence of capillary-gravity water waves with piecewise constant vorticity. arXiv:1302.5523.
[30] A.-V. Matioc and B.-V. Matioc. On the symmetry of periodic gravity water waves with vorticity. *Differential Integral Equations*, 26(1-2):129–140, 2013.
[31] B.-V. Matioc. Analyticity of the streamlines for periodic traveling water waves with bounded vorticity. *Int. Math. Res. Not.*, 17:3858–3871, 2011.
[32] B.-V. Matioc and A.-V. Matioc. Capillary-gravity water waves with discontinuous vorticity: existence and regularity results. *Comm. Math. Phys.*, 2013.
[33] K. Okuda. Internal flow structure of short wind waves. *Journal of the Oceanographical Society of Japan*, 38:28–42, 1982.
[34] C. Pattiaratchi and M. Collins. Sediment transport under waves and tidal currents: A case study from the northern Bristol Channel, U.K. *Marine Geology*, 56(1-4):27–40, 1984.
[35] D. H. Peregrine. Interactions of water waves and currents. *Adv. Appl. Mech.*, 16:9–117, 1976.
[36] O. M. Phillips and M. L. Banner. Wave breaking in the presence of wind drift and swell. *J. Fluid Mech.*, 66:625–640, 1974.
[37] G. Thomas and G. Klopman. *Wave-current interactions in the nearshore region.* WIT, Southampton, United Kingdom, 1997.
[38] J. F. Toland. Stokes waves. *Topol. Methods Nonlinear Anal.*, 7(1):1–48, 1996.
[39] E. Wahlén. Steady water waves with a critical layer. *J. Differential Equations*, 246(6):2468–2483, 2009.
[40] S. Walsh. Stratified steady periodic water waves. *SIAM J. Math. Anal.*, 41(3):1054–1105, 2009.
[41] L.-J. Wang. Regularity of traveling periodic stratified water waves with vorticity. *Nonlinear Anal.*, 81:247–263, 2013.

Institut für Mathematik, Universität Wien, Nordbergstrasse 15, 1090 Wien, Austria
E-mail address: calin.martin@univie.ac.at