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DISPERSION RELATIONS FOR PERIODIC WATER WAVES WITH SURFACE TENSION AND DISCONTINUOUS VORTICITY

Calin Iulian Martin

Institut für Mathematik, Universität Wien
Nordbergstraße 15, 1090 Wien, Austria

Abstract. We derive the dispersion relation for water waves with surface tension and having a piecewise constant vorticity distribution. More precisely, we consider here two scenarios; the first one is that of a flow with constant non-zero vorticity adjacent to the flat bed while above this layer of vorticity we assume the flow to be irrotational. The second type of flow has a layer of non-vanishing vorticity adjacent to the free surface and is irrotational below.

1. **Introduction.** We expand upon some recent results obtained in [26] concerning local bifurcation for steady periodic travelling capillary-gravity water waves possessing a discontinuous (step function like) vorticity distribution. The relevance of this type of water flows can be justified on physical grounds by the fact that rotational waves generated by wind have a thin layer of high vorticity that is adjacent to the free surface, while in the near bed region there may exist currents resulting from sediment transport along the ocean bed.

The study of nonlinear periodic travelling waves was confined for centuries to irrotational flows with substantial progress within this framework occuring in the last decades, see for instance the case of the Stokes waves [30] and the flow beneath them (particle trajectories, behaviour of the pressure) cf.[1], [2], [5], [10]. The presence of vorticity in a flow not only complicates a mathematical problem but also accomodates concrete physical needs for it is well known that flows with vorticity describe wave-current interactions among other physically relevant phenomena [3], [22], [29]. In spite of the importance of the wave-current interactions [22] the difficulties that vorticity adds to the problem have prevented a rigorous mathematical development which appeared only relatively recently in [9] where the existence of small and large amplitude steady periodic gravity water waves with a general (continuous) vorticity distribution was proved. The paper [9] was followed by a bulk of mathematically rigorous results concerning stability [12], symmetry [6, 7, 27], regularity of the free surface and of the stream lines [8, 15, 17, 18, 34], adding complications such as stagnation points and critical layers [13, 14, 24, 25, 32], stratifications [16, 20, 33], or allowing for a discontinuous (piecewise constant) [26] or merely bounded ([11, 28]) vorticity distribution.

We shall devote this paper to obtaining the dispersion relation which is a formula giving the relative speed of the wave at the free surface in terms of the mean depth

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of the flow, the wave number, the vorticity distribution and-for the step function like vorticity distribution we are dealing in this paper-the position of the jump in vorticity. The existence of small-amplitude steady periodic capillary-gravity water waves with such a discontinuous vorticity was proven in [26] by associating to the water wave problem a diffraction problem with suitable transmission conditions on each line of discontinuity of the vorticity function. A similar analysis concerning the dispersion relation was performed in the case of pure gravity waves in [11] and [19]. The outline of the paper is as follow. In Section 2 we give a presentation of the water wave problem. Section 3 deals with the case of a layer of constant non-vanishing vorticity adjacent to the flat bed while Section 4 presents the situation when the layer of constant non-zero vorticity is adjacent to the free surface.

2. **The water wave problem.** We shall consider herein two-dimensional periodic waves over a rotational,inviscid and incompressible fluid which propagates in the positive x-direction over the flat bed $y = -d$ $(d > 0)$ and whose free surface is a small perturbation of the flat free surface $y = 0$. The restoring forces acting upon the fluid are gravity and surface tension. In a reference frame moving in the same direction with the wave and speed $c > 0$, the equations of motion are the Euler's equation

$$\begin{cases} (u-c)u_x + vu_y &= -P_x \\ (u-c)v_x + vv_y &= -P_y - g, \end{cases} \tag{1}$$

together with the incompressibility condition

$$u_x + v_y = 0, \tag{2}$$

whereby (u, v) denotes the velocity field, P is the pressure and g is the gravitational constant. The vorticity of the flow is

$$\omega := u_y - v_x.$$

The equations of motion are supplemented by the kinematic boundary conditions which require that the free surface $y = \eta(x)$ and the bed $y = -d$ always consist of the same fluid particles and therefore take the form

$$\begin{cases} v = (u-c)\eta_x \quad \text{on} \quad y = \eta(x) \\ v = 0 \quad \text{on} \quad y = -d \end{cases} \tag{3}$$

Moreover, the dynamic boundary condition

$$P = P_{atm} - \sigma \frac{\eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}}, \tag{4}$$

states that the pressure jump across an interface is proportional to its the mean curvature. We denoted by P_{atm} the constant atmospheric pressure and by $\sigma > 0$ the coefficient of surface tension. For a justification of the validity of (1)-(4) we refer the reader to [3]. We make one further assumption, namely that

$$u < c \quad \text{throughout} \quad \text{the} \quad \text{fluid}, \tag{5}$$

which expresses the absence of stagnation points in the flow and ensures cf. [3, 9] that the vorticity ω is a single-valued function of ψ, i.e.,

$$\omega(x, y) = \gamma(-\psi(x, y)).$$

Via the stream function ψ defined (up to an additive constant) by means of

$$\psi_x = -v, \;\; \psi_y = u - c$$

we reformulate (1)-(4) as the free boundary value problem

$$\begin{cases} \Delta\psi = \gamma(-\psi) & \text{in} \quad -d < y < \eta(x), \\ |\nabla\psi|^2 + 2g(y+d) - 2\sigma\dfrac{\eta_{xx}}{(1+\eta_x^2)^{3/2}} = Q & \text{on} \quad y = \eta(x), \\ \psi = 0 & \text{on} \quad y = \eta(x), \\ \psi = -p_0 & \text{on} \quad y = -d, \end{cases} \tag{6}$$

where Q is a constant related to the total head, and $p_0 < 0$ is a constant representing the relative mass flux, given by

$$p_0 = \int_{-d}^{\eta(x)} (u(x,y) - c)\, dy.$$

The absence of stagnation points also allows us to transform (6) by means of the partial hodograph transform

$$q(x,y) = x, \;\; p(x,y) = -\psi(x,y) \tag{7}$$

into the quasilinear elliptic boundary value problem

$$\begin{cases} (1+h_q^2)h_{pp} - 2h_p h_q h_{pq} + h_p^2 h_{qq} - \gamma h_p^3 = 0 & \text{in } \overline{\Omega}, \\ 1 + h_q^2 + (2gh - Q)h_p^2 - 2\sigma\dfrac{h_p^2 h_{qq}}{(1+h_q^2)^{3/2}} = 0 & \text{on } p = 0, \\ h = 0 & \text{on } p = p_0, \end{cases} \tag{8}$$

where the unknown function h defined on $\overline{\Omega} := [-\pi, \pi] \times [p_0, 0]$ by

$$h(q,p) := y + d$$

represents the height above the flat bed and is even and of period 2π in the q-variable. The condition of no stagnation points in the fluid is equivalent to the elliptic nondegeneracy condition

$$h_p > 0 \quad \text{in} \quad \overline{\Omega}. \tag{9}$$

We are interested in this paper in the situation when the range of the vorticity function consists of two values. More precisely, we assume that there are $\gamma_1, \gamma_2 \in \mathbb{R}$ with $\gamma_1 \neq \gamma_2$ and $p_1 \in (p_0, 0)$ such that

$$\gamma(p) = \begin{cases} \gamma_1, & \text{for} \quad p \in [p_0, p_1] \\ \gamma_2, & \text{for} \quad p \in (p_1, 0). \end{cases}$$

The above vorticity distribution requires the consideration of a further so called diffraction (or transmission) problem. Namely, setting $\Omega_1 = (-\pi,\pi) \times (p_0, p_1), \Omega_2 = (-\pi, \pi) \times (p_1, 0)$ we look for $(\mathfrak{u}, \mathfrak{U}) \in C^{2+\alpha}(\overline{\Omega}_1) \times C^{2+\alpha}(\overline{\Omega}_2)$ satisfying

$$\begin{cases} (1+\mathfrak{u}_q^2)\mathfrak{u}_{pp} - 2\mathfrak{u}_p\mathfrak{u}_q\mathfrak{u}_{pq} + \mathfrak{u}_p^2\mathfrak{u}_{qq} - \gamma_1\mathfrak{u}_p^3 = 0 & \text{in } \Omega_1, \\ (1+\mathfrak{U}_q^2)\mathfrak{U}_{pp} - 2\mathfrak{U}_p\mathfrak{U}_q\mathfrak{U}_{pq} + \mathfrak{U}_p^2\mathfrak{U}_{qq} - \gamma_2\mathfrak{U}_p^3 = 0 & \text{in } \Omega_2, \\ 1 + \mathfrak{U}_q^2 + (2g\mathfrak{U} - Q)\mathfrak{U}_p^2 - 2\sigma\dfrac{\mathfrak{U}_p^2\mathfrak{U}_{qq}}{(1+\mathfrak{U}_q^2)^{3/2}} = 0 & \text{on } p = 0, \\ \mathfrak{u} = \mathfrak{U} & \text{on } p = p_1, \\ \mathfrak{u}_p = \mathfrak{U}_p & \text{on } p = p_1, \\ \mathfrak{u} = 0 & \text{on } p = p_0, \end{cases} \tag{10}$$

Using Crandall-Rabinowitz theory we proved [26] the existence of local bifurcation curves consisting of solutions of (10) and (9) and emerging from laminar (trivial) solutions of (10). The laminar solutions represent water waves with a flat free surface and parallel streamlines. If $(\overline{u}, \overline{U})$ denotes a laminar flow solution which depends only on the variable p then it solves (10) and (9) if and only if it solves the system

$$\begin{cases} \overline{u}'' = \gamma_1 \overline{u}'^3 & \text{in } p_0 < p < p_1, \\ \overline{U}'' = \gamma_2 \overline{U}'^3 & \text{in } p_1 < p < 0, \\ 1 + (2g\overline{U}(0) - Q)\overline{U}'^2(0) = 0, \\ \overline{u}(p_1) = \overline{U}(p_1), \\ \overline{u}'(p_1) = \overline{U}'(p_1), \\ \overline{u}(p_0) = 0. \end{cases} \tag{11}$$

Solving (11) we obtain a family of laminar solutions parametrized by $\lambda > 2\max_{[p_0,0]}\Gamma$, given by

$$\begin{aligned} \overline{u}(p) := \overline{u}(p;\lambda) := \int_{p_0}^{p} \frac{1}{\sqrt{\lambda - 2\Gamma(s)}}\, ds, \qquad p \in [p_0, p_1], \\ \overline{U}(p) := \overline{U}(p;\lambda) := \int_{p_0}^{p} \frac{1}{\sqrt{\lambda - 2\Gamma(s)}}\, ds, \qquad p \in [p_1, 0]. \end{aligned} \tag{12}$$

Moreover, the constant λ is related to the speed at the surface of the laminar flow in the following way

$$\sqrt{\lambda} = \frac{1}{\overline{U}_p(0)} = (c-u)|_{y=\eta(x)}.$$

Let us set $a(\lambda, p) = \sqrt{\lambda - 2\Gamma(p)}, p \in [p_0, 0]$ and let k be the wave number. The necessary and sufficient conditions for the existence of waves of small amplitude obtained as perturbations of the laminar flows (12) is (see [26]) that the problem

$$(a^3 v_p)_p = k^2 a v \quad \text{on} \quad (p_0, p_1), \tag{13}$$

$$(a^3 V_p)_p = k^2 a V \quad \text{on} \quad (p_1, 0), \tag{14}$$

$$v = V \quad \text{on} \quad p = p_1, \tag{15}$$

$$v_p = V_p \quad \text{on} \quad p = p_1, \tag{16}$$

$$a^3 V_p = (g + \sigma k^2) V \quad \text{at} \quad p = 0, \tag{17}$$

$$v = 0 \quad \text{at} \quad p = p_0, \tag{18}$$

has a non-trivial solution $(v, V) \in C^{2+\alpha}([p_0, p_1]) \times C^{2+\alpha}([p_1, 0])$. In sections 3 and 4 we will solve (13)-(18) in the two special cases that we described before. Moreover, we will also derive in each case the dispersion relation.

3. **The case of a layer of constant non vanishing vorticity adjacent to the flat bed.**

Let $p_1 \in (p_0, 0)$ denote the height of the discontinuity in a flow with constant non-zero vorticity in the lower layer $[p_0, p_1)$ and zero vorticity in the upper layer $[p_1, 0]$. We aim at finding functions $(v, V) \in C^{2+\alpha}([p_0, p_1]) \times C^{2+\alpha}([p_1, 0])$ such that

$$(a^3 v_p)_p = k^2 a v \quad \text{on} \quad (p_0, p_1) \tag{19}$$

$$(a^3 V_p)_p = k^2 a V \quad \text{on} \quad (p_1, 0) \tag{20}$$

$$v = V \quad \text{on} \quad p = p_1 \tag{21}$$

$$v_p = V_p \quad \text{on} \quad p = p_1 \tag{22}$$

$$a^3 V_p = (g + \sigma k^2) V \quad \text{at} \quad p = 0, \tag{23}$$

$$v = 0 \quad \text{at} \quad p = p_0 \tag{24}$$

where k denotes the wave number and $a(\lambda,p)=\sqrt{\lambda-2\Gamma(p)}, p\in[p_0,0]$, with

$$\Gamma(p)=\begin{cases}0 & p\in[p_1,0],\\ \gamma(p-p_1) & p\in[p_0,p_1)\end{cases}.$$

We first determine the general solution of (19) on the interval (p_0,p_1) and satisfying the boundary condition (24) by setting

$$v(p)=\frac{2\gamma}{\sqrt{\lambda-2\gamma(p-p_1)}}\widetilde{v}\left(k\frac{\sqrt{\lambda-2\gamma(p-p_1)}}{\gamma}\right), p\in[p_0,p_1].$$

A routine calculation shows that the latter ansatz transforms the equation (19) in $\widetilde{v}''(s)=\widetilde{v}(s)$ for s between $s_1=k\frac{\sqrt{\lambda}}{\gamma}$ and $s_2=k\frac{\sqrt{\lambda-2\gamma(p_0-p_1)}}{\gamma}$. Together with $\widetilde{v}(s_2)=0$ leads to

$$\widetilde{v}(s)=\beta\sinh(s-s_2),$$

for s between s_1 and s_2 and for some constant $\beta\in\mathbb{R}$. We then obtain

$$v(p)=\frac{2\gamma\beta}{\sqrt{\lambda-2\gamma(p-p_1)}}\sinh\left(k\frac{\sqrt{\lambda-2\gamma(p-p_1)}-\sqrt{\lambda-2\gamma(p_0-p_1)}}{\gamma}\right), p\in[p_0,p_1]. \tag{25}$$

Since $a(p,\lambda)=\sqrt{\lambda}$ for $p\in(p_1,0)$ we have to solve the equation $V_{pp}=k^2\lambda^{-1}V$ and find the general solution

$$V(p)=\alpha_1\cosh\left(\frac{k}{\sqrt{\lambda}}p\right)+\alpha_2\sinh\left(\frac{k}{\sqrt{\lambda}}p\right), p\in[p_1,0], \tag{26}$$

with constants $\alpha_1,\alpha_2\in\mathbb{R}$. The matching conditions (21), (22) along the discontinuity line $p=p_1$ hold true if and only if

$$\begin{cases}\alpha_1\cosh\left(k\frac{p_1}{\sqrt{\lambda}}\right)+\alpha_2\sinh\left(k\frac{p_1}{\sqrt{\lambda}}\right) = \frac{2\gamma\beta}{\sqrt{\lambda}}\sinh\left(k\frac{\sqrt{\lambda}-\sqrt{\lambda-2\gamma(p_0-p_1)}}{\gamma}\right)\\ \frac{\alpha_1 k}{\sqrt{\lambda}}\sinh\left(k\frac{p_1}{\sqrt{\lambda}}\right)+\frac{\alpha_2 k}{\sqrt{\lambda}}\cosh\left(k\frac{p_1}{\sqrt{\lambda}}\right) = \frac{2\gamma^2\beta}{\lambda\sqrt{\lambda}}\sinh\left(k\frac{\sqrt{\lambda}-\sqrt{\lambda-2\gamma(p_0-p_1)}}{\gamma}\right)\\ \qquad -\frac{2\gamma\beta k}{\lambda}\cosh\left(k\frac{\sqrt{\lambda}-\sqrt{\lambda-2\gamma(p_0-p_1)}}{\gamma}\right)\end{cases} \tag{27}$$

Before we solve the previous system for α_1 and α_2 we prove in the sequel that

$$d_0=-\frac{p_1}{\sqrt{\lambda}},\qquad d_0-d=\frac{\sqrt{\lambda}-\sqrt{\lambda-2\gamma(p_0-p_1)}}{\gamma}, \tag{28}$$

whereby d_0 represents the average depth of the discontinuity curve of the vorticity and as before d stands for the average depth of the water. The horizontal velocity of the trivial fluid flow equals

$$(c-u)(y)=\begin{cases}\sqrt{\lambda}, & -d_0\le y\le 0,\\ \sqrt{\lambda}-\gamma(y+d_0), & -d\le y\le -d_0.\end{cases} \tag{29}$$

By the definition of the mass flux we have

$$-p_0=\int_{-d}^{0}(c-u)\,dy=\int_{-d}^{-d_0}(\sqrt{\lambda}-\gamma(y+d_0))\,dy+\int_{-d_0}^{0}\sqrt{\lambda}\,dy=d\sqrt{\lambda}+\frac{\gamma}{2}(d-d_0)^2,$$

and

$$-p_1=\int_{-d_0}^{0}(c-u)\,dy=d_0\sqrt{\lambda}.$$

We now see that the latter relation already implies the first equation in (28). Looking at the relation defining p_0 as being an equation of degree two in d we get

$$d = d_0 - \frac{\sqrt{\lambda} \pm \sqrt{\lambda - 2\gamma(p_0 + d_0\sqrt{\lambda})}}{\gamma} = d_0 - \frac{\sqrt{\lambda} \pm \sqrt{\lambda - 2\gamma(p_0 - p_1)}}{\gamma}.$$

Since $d_0 - d < 0$ and $\frac{\sqrt{\lambda} - \sqrt{\lambda - 2\gamma(p_0 - p_1)}}{\gamma} < 0$ for both choices of the sign of γ we conclude that

$$d_0 - d = \frac{\sqrt{\lambda} - \sqrt{\lambda - 2\gamma(p_0 - p_1)}}{\gamma},$$

which is in fact the second relation in (28). We can now rewrite the system (27) as

$$\begin{cases} \alpha_1 \cosh(kd_0) - \alpha_2 \sinh(kd_0) & = \frac{2\gamma\beta}{\sqrt{\lambda}} \sinh(k(d_0 - d)) \\ -\frac{\alpha_1 k}{\sqrt{\lambda}} \sinh(kd_0) + \frac{\alpha_2 k}{\sqrt{\lambda}} \cosh(kd_0) & = \frac{2\gamma^2\beta}{\lambda\sqrt{\lambda}} \sinh(k(d_0 - d)) - \frac{2\gamma\beta k}{\lambda} \cosh(k(d_0 - d)) \end{cases} \tag{30}$$

Solving the latter system for α_1 and α_2 we get

$$\begin{cases} \alpha_1 & = -\frac{2\gamma\beta}{\sqrt{\lambda}} \sinh(kd) - \frac{2\gamma^2\beta}{k\lambda} \sinh(k(d - d_0)) \sinh(kd_0) \\ \alpha_2 & = -\frac{2\gamma\beta}{\sqrt{\lambda}} \cosh(kd) - \frac{2\gamma^2\beta}{k\lambda} \sinh(k(d - d_0)) \cosh(kd_0) \end{cases} \tag{31}$$

The top boundary condition (23) becomes

$$k\alpha_2\lambda = (g + \sigma k^2)\alpha_1,$$

which can be rewritten as

$$\begin{aligned} k\cosh(kd)\lambda^{3/2} + \gamma\cosh(kd_0)\sinh(k(d - d_0))\lambda - (g + \sigma k^2)\sinh(kd)\lambda^{1/2} \\ - \frac{(g + \sigma k^2)\gamma}{k} \sinh(kd_0)\sinh(k(d - d_0)) = 0. \end{aligned} \tag{32}$$

Using addition formulæ for hyperbolic functions we find that (32) takes the form

$$\begin{aligned} k\lambda^{\frac{3}{2}} + \frac{\gamma}{2}\left(\tanh(kd) + \frac{\sinh(kd - 2kd_0)}{\cosh(kd)}\right)\lambda - (g + \sigma k^2)\tanh(kd)\sqrt{\lambda} \\ - \frac{(g + \sigma k^2)\gamma}{2k}\left(1 - \frac{\cosh(kd - 2kd_0)}{\cosh(kd)}\right) = 0, \end{aligned} \tag{33}$$

equation called the *dispersion relation* giving the wave speed $\sqrt{\lambda}$ at the flat free surface of the laminar flow at the bifurcation point.

Remark 1. (i) When taking the limit $d_0 \to d$ we recover the dispersion relation for irrotational capillary-gravity water waves

$$\sqrt{\lambda} = \sqrt{\frac{(g + \sigma k^2)}{k} \tanh(kd)}$$

cf. [21],[23].

(ii) Looking at the limit $d_0 \to 0$ we see that equation (33) becomes

$$k\lambda^{\frac{3}{2}} + \gamma\tanh(kd)\lambda - (g + \sigma k^2)\tanh(kd)\lambda^{\frac{1}{2}} = 0$$

having the unique positive solution

$$\sqrt{\lambda} = -\frac{\gamma\tanh(kd)}{2k} + \sqrt{\frac{\gamma^2\tanh^2(kd)}{4k^2} + \left(\frac{g}{k} + \sigma k\right)\tanh(kd)}, \tag{34}$$

recovering the dispersion relation for flows with surface tension, constant vorticity γ and without stagnation points from [31]. Note that in the context of flows allowing for stagnation points and constant vorticity formula (34) was obtained in [24].

We are returning to find necessary and sufficient conditions for local bifurcation to occur. The analysis is splitted in two cases.
Let us first set

$$\begin{aligned} p(x) &= kx^3 + \frac{\gamma}{2}\left(\tanh(kd) + \frac{\sinh(kd-2kd_0)}{\cosh(kd)}\right)x^2 - (g+\sigma k^2)\tanh(kd)x \\ &\quad - \frac{(g+\sigma k^2)\gamma}{2k}\left(1 - \frac{\cosh(kd-2kd_0)}{\cosh(kd)}\right), \end{aligned} \tag{35}$$

Since the discriminant of

$$p'(x) = 3kx^2 + \gamma\left(\tanh(kd) + \frac{\sinh(kd-2kd_0)}{\cosh(kd)}\right)x - (g+\sigma k^2)\tanh(kd)$$

is strictly positive and the product of its two real roots is strictly negative we conclude that the equation $p'(x) = 0$ has one negative real root $x_{neg} < 0$ and one positive real root $x_{pos} > 0$. Therefore the polynomial $p(x)$ is strictly increasing on $(-\infty, x_{neg}) \cup (x_{pos}, \infty)$ and strictly decreasing on (x_{neg}, x_{pos}).

Lemma 3.1. *If $\gamma > 0$ local bifurcation always occurs.*

Proof. If p is the polynomial defined before we see that $p(0) < 0$ and setting

$$x_0 = -\frac{\gamma}{2k\tanh(kd)}\left(1 - \frac{\cosh(kd-2kd_0)}{\cosh(kd)}\right) < 0,$$

we have

$$\begin{aligned} p(x_0) &= kx_0^3 + \frac{\gamma}{2}x_0^2\left(\tanh(kd) + \frac{\sinh(kd-2kd_0)}{\cosh(kd)}\right) \\ &= \frac{\gamma}{2}x_0^2\left(\tanh(kd) + \frac{\sinh(kd-2kd_0)}{\cosh(kd)} - \frac{1}{\tanh(kd)} + \frac{\cosh(kd-2kd_0)}{\sinh(kd)}\right) \\ &= \frac{\gamma}{2}x_0^2\frac{\cosh(2kd-2kd_0)-1}{\sinh(kd)\cosh(kd)} > 0, \end{aligned} \tag{36}$$

relation which coroborated with the monotonicity properties of p shows that the equation $p(x) = 0$ has exactely one strictly positive root $\sqrt{\lambda}$ and two strictly negative roots. Because of the intricate nature of the Cardano's formula we refrain here from presenting the explicit formula for $\sqrt{\lambda}$. We derive for it a useful estimate instead. In doing so we write relation (32) as

$$k\sqrt{\lambda}\left(\lambda - \frac{g+\sigma k^2}{k}\tanh(kd)\right) = \frac{\gamma\sinh(k(d-d_0))\cosh(kd_0)}{\cosh(kd)}\left(\frac{g+\sigma k^2}{k}\tanh(kd_0) - \lambda\right).$$

Since $0 < d_0 < d$ we infer from the above relation that

$$\sqrt{\frac{g+\sigma k^2}{k}\tanh(kd_0)} < \sqrt{\lambda} < \sqrt{\frac{g+\sigma k^2}{k}\tanh(kd)}.$$

□

Lemma 3.2. *If $\gamma < 0$ then the necessary and sufficient condition for local bifurcation to occur at the largest positive root of p is*

$$|\gamma| < \frac{\sqrt{(g+\sigma k^2)\left[(d-d_0)\sinh(kd) - \frac{1}{k}\sinh(kd_0)\sinh(kd-kd_0)\right]}}{(d-d_0)\sqrt{(d-d_0)k\cosh(kd) - \cosh(kd_0)\sinh(kd-kd_0)}} \tag{37}$$

Proof. Defining x_0 as before we have now that $x_0 > 0$ while $p(x_0) < 0$. The latter facts together with $p(0) > 0$ give that the equation $p(x) = 0$ has one positive root in $(0, x_0)$ and another one in (x_0, ∞). From $p(x_0) < 0$ and $p(0) > 0$ we infer that at the local maximum point x_{neg} we have $p(x_{neg}) > 0$ and at the local minimum point x_{pos} we have $p(x_{pos}) < 0$. We will show that only the root from (x_0, ∞) is relevant from the point of view of the dispersion relation. This is so because the assumption of no stagnation points in the flow implies that $\sqrt{\lambda} > \gamma(d_0 - d)$. Therefore, only the root larger than $\gamma(d_0 - d)$ is of relevance since $x_0 \leq \gamma(d_0 - d)$. The last inequality is equivalent to

$$2(kd - kd_0)(e^{kd} - e^{-kd}) - e^{kd} - e^{-kd} + e^{kd-2kd_0} + e^{2kd_0-kd} \geq 0,\ d \geq d_0 \geq 0,$$

which is in turn equivalent to the inequality (28) from [4]. We proceed further by noticing that the necessary and sufficient condition for the absence of stagnation points in the flow (29) can be restated as $p(\gamma(d_0 - d)) < 0$. Indeed, if $p(\gamma(d_0 - d)) \geq 0$, then $\gamma(d_0 - d) > x_{pos}$ since the monotonicity properties of p imply that p is strictly negative on (x_0, x_{pos}). Therefore p is strictly increasing on $(\gamma(d_0 - d), \infty)$ which implies that $p(x) \geq 0$ for $x \in (\gamma(d_0 - d), \infty)$. The latter is possible only if $\gamma(d_0 - d) \geq \sqrt{\lambda}$ which is a contradiction with the assumption of no stagnation points in the flow (29). The claim $p(\gamma(d_0 - d)) < 0$ is thus proved. Drawing the graph of p we then conclude that p has only one root in $(\gamma(d_0 - d), \infty)$. To prove (37) we compute from (35)

$$\begin{aligned}
&\cosh(kd)p(\gamma(d_0-d)) \\
&\qquad = k\gamma^3(d_0-d)^3\cosh(kd) + \gamma^3(d_0-d)^2\cosh(kd_0)\sinh(kd-kd_0) \\
&\qquad - \gamma(g+\sigma k^2)(d_0-d)\sinh(kd) - \gamma\frac{g+\sigma k^2}{k}\sinh(kd_0)\sinh(kd-kd_0) < 0.
\end{aligned} \tag{38}$$

It remains to prove that $(d-d_0)k\cosh(kd) - \cosh(kd_0)\sinh(kd-kd_0) > 0$ for $d > d_0$. Let $D := kd, D_0 := kd_0$. Then using summation formulas for hyperbolic functions the asserted inequality is equivalent to

$$f(D) := (D - D_0)\cosh(D) - \frac{\sinh(D) + \sinh(D - 2D_0)}{2} > 0$$

for $D > D_0$. But $F(D_0) = 0$ and $f'(D) = \sinh(D)(D - D_0) + \frac{\cosh(D) - \cosh(D-2D_0)}{2} > 0$ for $D > D_0 \geq 0$. □

Remark 2. Note that setting $d_0 = 0$ in formula (37) we obtain the necessary and sufficient condition

$$(g + k^2\sigma + \gamma^2 d)\tanh(kd) > k\gamma^2 d^2$$

for bifurcation in the absence of stagnation points, cf. relation (79) in [24]. For the case of pure gravity waves (i.e. $\sigma = 0$) one obtains after setting $d_0 = 0$ in (37) the necessary and sufficient condition

$$(g + \gamma^2 d)\tanh(kd) > k\gamma^2 d^2$$

for local bifurcation in the absence of stagnation points, cf. relation (5.19) in [13].

4. **The case of a layer of constant non-zero vorticity adjacent to the free surface.** We denote by $p_1 \in (p_0, 0)$ the height of the discontinuity line in a flow having constant vorticity $\gamma \neq 0$ in the upper layer corrresponding to $p \in [p_1, 0]$, while in the lower layer corresponding to $[p_0, p_1)$ the vorticity is assumed to be zero. As before, we are looking for a pair of functions $(v, V) \in C^{2+\alpha}([p_0, p_1]) \times C^{2+\alpha}([p_1, 0])$ such that

$$(a^3 v_p)_p = k^2 a v \quad \text{on} \quad (p_0, p_1) \tag{39}$$
$$(a^3 V_p)_p = k^2 a V \quad \text{on} \quad (p_1, 0) \tag{40}$$
$$v = V \quad \text{on} \quad p = p_1 \tag{41}$$
$$v_p = V_p \quad \text{on} \quad p = p_1 \tag{42}$$
$$a^3 V_p = (g + \sigma k^2) V \quad \text{at} \quad p = 0, \tag{43}$$
$$v = 0 \quad \text{at} \quad p = p_0 \tag{44}$$

where k denotes the wave number and $a(\lambda, p) = \sqrt{\lambda - 2\Gamma(p)}, p \in [p_0, 0]$ whereby

$$\Gamma(p) = \begin{cases} \gamma p & \text{if} \quad p_1 \leq p \leq 0 \\ \gamma p_1 & \text{if} \quad p_0 \leq p \leq p_1. \end{cases}$$

Since a is const on (p_0, p_1) the equation (39) becomes $v_{pp} = k^2 a^{-2} v$ and due to the boundary condition on $p = p_0$ its solution is

$$v(p) = \beta \sinh\left(k \frac{p - p_0}{\sqrt{\lambda - 2\gamma p_1}}\right), \ p_0 \leq p \leq p_1,$$

for some constant $\beta \in \mathbb{R}$. In order to find the solution of (40) on $[p_1, 0]$ we follow an approach from [11] and make the ansatz $V(p) = \frac{2\gamma}{a(p)} V_0 \left(k \frac{a(p)}{\gamma}\right)$. We find first that $(a^3 V_p)_p = 2\gamma k^2 V_0'' \left(k \frac{a}{\gamma}\right)$. Thus,

$$V(p) = \frac{2\gamma}{\sqrt{\lambda - 2\gamma p}} \left(\beta_1 \cosh\left(k \frac{\sqrt{\lambda - 2\gamma p}}{\gamma}\right) + \beta_2 \sinh\left(k \frac{\sqrt{\lambda - 2\gamma p}}{\gamma}\right)\right), \ p_1 \leq p \leq 0$$

for some constants β_1, β_2. Setting

$$\frac{p_1 - p_0}{\sqrt{\lambda - 2\gamma p_1}} = \rho, \ \frac{\sqrt{\lambda - 2\gamma p_1}}{\gamma} = \theta \tag{45}$$

we can express the compatibility conditions (41) and (42) along the line $p = p_1$ as

$$\begin{cases} \beta \sinh(k\rho) = \frac{2}{\theta}[\beta_1 \cosh(k\theta) + \beta_2 \sinh(k\theta)], \\ \beta k \cosh(k\rho) = \frac{2}{\theta^2}[\beta_1 \cosh(k\theta) + \beta_2 \sinh(k\theta)] - \frac{2k}{\theta}[\beta_1 \sinh(k\theta) + \beta_2 \cosh(k\theta)]. \end{cases} \tag{46}$$

Solving the above system for β_1 and β_2 we find

$$\beta_1 = \frac{\beta}{2}\left(\theta \sinh(k(\rho + \theta)) - \frac{1}{k} \sinh(k\rho) \sinh(k\theta)\right) \tag{47}$$

$$\beta_2 = \frac{\beta}{2}\left(-\theta \cosh(k(\rho + \theta)) + \frac{1}{k} \sinh(k\rho) \cosh(k\theta)\right). \tag{48}$$

Using the latter formulas for β_1 and β_2 in (43) we find the dispersion relation

$$(\gamma\sqrt{\lambda}-(g+\sigma k^2))\left[\theta\sinh\left(k\left(\theta+\rho-\frac{\sqrt{\lambda}}{\gamma}\right)\right)-\frac{1}{k}\sinh(k\rho)\sinh\left(k\left(\theta-\frac{\sqrt{\lambda}}{\gamma}\right)\right)\right]$$
$$=k\lambda\left[-\theta\cosh\left(k\left(\theta+\rho-\frac{\sqrt{\lambda}}{\gamma}\right)\right)+\frac{1}{k}\sinh(k\rho)\cosh\left(k\left(\theta-\frac{\sqrt{\lambda}}{\gamma}\right)\right)\right]. \tag{49}$$

We claim that

$$\theta+\rho-\frac{\sqrt{\lambda}}{\gamma}=d, \qquad \theta-\frac{\sqrt{\lambda}}{\gamma}=d_0, \qquad \rho=d-d_0, \tag{50}$$

where d_0 is the average depth of the discontinuity curve of vorticity. For laminar flows we have that the horizontal fluid velocity u depends only on y, and moreover $u_y=\gamma$ and $(c-u)(0)=\sqrt{\lambda}$. Thus

$$(c-u)(y)=\begin{cases}\sqrt{\lambda}-\gamma y, & -d_0\le y\le 0,\\ \sqrt{\lambda}+\gamma d_0, & -d\le y\le -d_0.\end{cases} \tag{51}$$

From $\int_{-d_0}^{0}(c-u)(y)\,dy=-p_1$ and $\int_{-d}^{-d_0}(c-u)(y)\,dy=-p_0+p_1$ we obtain by means of (51) that

$$\frac{\gamma}{2}d_0^2+\sqrt{\lambda}d_0+p_1=0, \tag{52}$$

and

$$(\sqrt{\lambda}+\gamma d_0)(d-d_0)=-p_0+p_1 \tag{53}$$

We eliminate p_1 between (52) and (53) and obtain

$$\sqrt{\lambda}=-\frac{p_0}{d}-\gamma d_0+\gamma\frac{d_0^2}{2d} \tag{54}$$

Eliminating λ from (52) and (54) and adding the resulting expression for $-2\gamma p_1$ to the square of (54) we obtain that

$$\lambda-2\gamma p_1=\frac{1}{d^2}\left(p_0-\frac{\gamma}{2}d_0^2\right)^2. \tag{55}$$

Therefore, equation (52), considered as a second degree equation in the unknown d_0 possesses the real solutions

$$d_0=\frac{-\sqrt{\lambda}\pm\sqrt{\lambda-2\gamma p_1}}{\gamma}. \tag{56}$$

If $\gamma>0$ then only the sum in (56) produces a positive number and is therefore the right choice. If $\gamma<0$ then both formulas for d_0 in (56) are positive, but since the flow (51) is assumed to be free of stagnation points we have the condition

$$\gamma d_0>-\sqrt{\lambda},$$

which allows only for the sum in (56) as the correct expression for d_0. Thus, independently of the sign of γ, we established that

$$d_0=\frac{-\sqrt{\lambda}+\sqrt{\lambda-2\gamma p_1}}{\gamma}. \tag{57}$$

The absence of stagnation points implies via (51) and (54) that $p_0-\frac{\gamma}{2}d_0^2<0$. Therefore, equation (55) implies that

$$\sqrt{\lambda-2\gamma p_1}=\frac{1}{d}\left(-p_0+\frac{\gamma}{2}d_0^2\right).$$

Thus,

$$\theta = \frac{\sqrt{\lambda - 2\gamma p_1}}{\gamma} = -\frac{p_0}{\gamma d} + \frac{d_0^2}{2d}.$$

The latter equality implies by means of (54) that

$$\frac{\sqrt{\lambda}}{\gamma} = \theta - d_0. \tag{58}$$

Relations (45), (53) and (58) ensure that

$$\rho = \frac{p_1 - p_0}{\gamma\theta} = \frac{(d-d_0)(\sqrt{\lambda} + \gamma d_0)}{\gamma\theta} = d - d_0.$$

Adding the expressions for θ and ρ from the previous two relations we obtain

$$\rho + \theta - \frac{\sqrt{\lambda}}{\gamma} = d,$$

and thus the claim (50) is proved. Hence the dispersion relation (49) becomes

$$\begin{aligned}(\gamma\sqrt{\lambda} - (g+\sigma k^2)) &\left[\left(\frac{\sqrt{\lambda}}{\gamma} + d_0\right)\sinh(kd) - \frac{1}{k}\sinh(k(d-d_0))\sinh(kd_0)\right] \\ &= k\lambda\left[-\left(\frac{\sqrt{\lambda}}{\gamma} + d_0\right)\cosh(kd) + \frac{1}{k}\sinh(k(d-d_0))\cosh(kd_0)\right],\end{aligned}$$

which is equivalent to

$$\begin{aligned}k\lambda^{\frac{3}{2}} &+ \left[kd_0 + \frac{\sinh(kd_0)\cosh(k(d-d_0))}{\cosh(kd)}\right]\gamma\lambda \\ &- \sqrt{\lambda}\tanh(kd)\left\{(g+\sigma k^2) - \gamma^2\left[d_0 - \frac{1}{k}\frac{\sinh(k(d-d_0))\sinh(kd_0)}{\sinh(kd)}\right]\right\} \\ &- \gamma(g+\sigma k^2)\tanh(kd)\left[d_0 - \frac{1}{k}\frac{\sinh(k(d-d_0))\sinh(kd_0)}{\sinh(kd)}\right] = 0.\end{aligned} \tag{59}$$

Recalling that $\sqrt{\lambda} = (c-u)(0)$ is the speed at the free surface of the laminar flow, we are looking for positive roots for $\sqrt{\lambda}$ of (59). The latter would entail the occurrence of local bifurcation. Before we go into details we analyze a few special cases.

Remark 3. (i) Setting $d_0 = 0$ in (59) we obtain the case of *irrotational flow*. Therefore, the dispersion relation (59) becomes

$$k\lambda^{\frac{3}{2}} - (g+\sigma k^2)\tanh(kd)\sqrt{\lambda} = 0,$$

yielding the positive solution

$$\sqrt{\lambda} = \sqrt{\frac{g+\sigma k^2}{k}\tanh(kd)}$$

which we also obtained in Remark 1, and which stands for the dispersion relation for irrotational flows, cf. [21],[23].

(ii) Puting $d_0 = d$ in (59) we are in the case of a flow of *constant vorticity* γ in which the dispersion relation (59) becomes

$$k\lambda^{\frac{3}{2}} + (kd + \tanh(kd))\gamma\lambda - \tanh(kd)(g+\sigma k^2 - \gamma^2 d)\sqrt{\lambda} - \gamma(g+\sigma k^2)d\tanh(kd) = 0,$$

equation which can be rewritten as

$$k\lambda(\sqrt{\lambda} + \gamma d) + \gamma\sqrt{\lambda}\tanh(kd)(\sqrt{\lambda}+\gamma d) - (g+\sigma k^2)\tanh(kd)(\sqrt{\lambda}+\gamma d) = 0.$$

The absence of stagnation points ensures that $\sqrt{\lambda}+\gamma d>0$, and thus the dispersion relation becomes

$$k\lambda+\gamma\tanh(kd)\sqrt{\lambda}-(g+\sigma k^2)\tanh(kd)=0.$$

The unique positive solution of the avove equation is

$$\sqrt{\lambda}=-\frac{\gamma}{2k}\tanh(kd)+\frac{1}{2}\sqrt{\frac{\gamma^2\tanh^2(kd)}{k^2}+4\left(\frac{g}{k}+\sigma k\right)\tanh(kd)},$$

giving the speed at the free surface of the laminar flow solutions for capillary-gravity water waves of constant vorticity, without stagnation points, see [31]. For the case of pure gravity waves we refer the reader to [9].

We now return to find necessary and sufficient conditions for local bifurcation to occur which is equivalent to asking that (59) has one positive real root. We split the discussion into two cases according to the sign of the vorticity. Denote by $p(x)$ the polynomial obtained by setting $x=\sqrt{\lambda}$ in the left hand side of (59). First we notice that

$$d_0-\frac{1}{k}\frac{\sinh(k(d-d_0))\sinh(kd_0)}{\sinh(kd)}>0,\quad d_0\in(0,d]. \tag{60}$$

Note that the latter equals

$$\frac{1}{k}\left(D_0-\frac{\sinh(D-D_0)\sinh(D_0)}{\sinh(D)}\right),$$

for $D_0:=kd_0$ and $D:=kd$. Considering the bracket in the right hand side above as a function g of $D_0\in[0,D]$ we have that $g(0)=0$ and

$$g'(D_0)=1-\frac{\cosh(D_0)\sinh(D-D_0)-\sinh(D_0)\cosh(D-D_0)}{\sinh(D)}=1-\frac{\sinh(D-2D_0)}{\sinh(D)}>0,$$

for $D_0\in(0,D]$.

Proposition 1. *If $\gamma>0$, then local bifurcation always occurs.*

Proof. Viète's relations ensure that the sum of the three roots of p is negative, while their product is positive in view of (60). Hence, equation (59) has exactely one positive root that gives the dispersion relation. □

Proposition 2. *If $\gamma<0$, then local bifurcation occurs if and only if the inequality* (63) *holds true.*

Proof. We perform in (59) the substitution $\sqrt{\lambda}=-\gamma(x+d_0)$ and obtain the polynomial equation

$$\begin{aligned}p_1(x)=kx^3&+\left(2kd_0-\frac{\sinh(kd_0)\cosh(k(d-d_0))}{\cosh(kd)}\right)x^2\\&+\left(kd_0^2-\frac{1}{k}\frac{\sinh(kd_0)\sinh(k(d-d_0))}{\cosh(kd)}-\frac{g+\sigma k^2}{\gamma^2}\tanh(kd)+d_0\frac{\sinh(k(d-2d_0))}{\cosh(kd)}\right)x\\&+\frac{\sinh(k(d-d_0))}{\cosh(kd)}\left(d_0^2\cosh(kd_0)-\frac{d_0}{k}\sinh(kd_0)-\frac{1}{k}\frac{g+\sigma k^2}{\gamma^2}\sinh(kd_0)\right)=0.\end{aligned} \tag{61}$$

Denoting again $D:=kd$, $D_0:=kd_0$ we have that the coefficient of x^2 is

$$2D_0-\frac{\sinh(D_0)\cosh(D-D_0)}{\cosh(D)}\geq D_0>0, \tag{62}$$

by formula (84) in [11]. We remark now that $p_1(0) < 0$ if and only if

$$\frac{g+\sigma k^2}{k\gamma^2} > d_0^2 \frac{\cosh(kd_0)}{\sinh(kd_0)} - \frac{d_0}{k} := f_1(d_0). \tag{63}$$

Also, note that the $p_1'(0) > 0$ if and only if

$$\frac{g+\sigma k^2}{k\gamma^2} < d_0^2 \frac{\cosh(kd)}{\sinh(kd)} - \frac{1}{k^2} \frac{\sinh(k(d-d_0))\sinh(kd_0)}{\sinh(kd)} + \frac{d_0}{k} \frac{\sinh(k(d-2d_0))}{\sinh(kd)} := f_2(d_0). \tag{64}$$

The following inequality involving $f_1(d_0)$ and $f_2(d_0)$ will prove to be useful in establishing our claim. Namely,

$$0 < f_1(d_0) < f_2(d_0), \quad d_0 > 0. \tag{65}$$

The first inequality in (65) is equivalent to $\tanh(kd_0) - kd_0 < 0$ which is true since the function $y \to \tanh(y) - y$ has the value 0 at 0 while its derivative is strictly negative for $y > 0$. After multiplying both sides in (65) with $\sinh(kd_0)\sinh(kd)$, using addition formulas for $\sinh(k(d-2d_0))$ and writing $d = d_0 + (d - d_0)$ in $\sinh(kd)$ we see that the second inequality is equivalent with

$$(kd_0)^2 - 2kd_0 \sinh(kd_0)\cosh(kd_0) + \sinh^2(kd_0) < 0.$$

Denoting $kd_0 := \alpha$ and viewing the left side above as a polynomial of degree two in α, the inequality is true if and only if

$$\frac{1-e^{-2\alpha}}{2} < \alpha < \frac{e^{2\alpha}-1}{2}.$$

The latter inequalities are easy to check. We are now ready to prove that bifurcation occurs if and only if (63) holds. To prove the necessity note that if (63) holds then $p_1(0) < 0$. This shows that p_1 has at least one root $x_{pos} > 0$. We show that x_{pos} is the only positive root. Indeed, since the product of the three roots equals (by Viète's relations) $-p_1(0) > 0$, the existence of a second positive root would imply that the third root is also real and positive. But the latter is impossible since the sum of the three roots is negative by (62). For the sufficiency we remark that if $\frac{g+\sigma k^2}{k\gamma^2} \le f_1(d_0)$ then p_1 has no positive roots. Indeed, under this assumption we have $p_1(0) \ge 0$ and $p_1'(0) > 0$ from (64) and (65). But an inspection of the coefficients of p_1 and together with $p_1'(0) > 0$ and (62) gives that $p_1'(x) > 0$ for $x > 0$. Hence $p_1(x) > 0$ for $x > 0$. We have thus showed that (63) is the necessary and sufficient condition for local bifurcation. □

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E-mail address: calin.martin@univie.ac.at