DISPERSION RELATIONS FOR ROTATIONAL GRAVITY WATER FLOWS HAVING TWO JUMPS IN THE VORTICITY DISTRIBUTION

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ABSTRACT. We derive the dispersion relation for periodic traveling waves propagating at the surface of water with a layer of constant non-zero vorticity situated between two layers of irrotational flow. Due to the complicated nature of the dispersion relation-a fourth order algebraic equation with intricate coefficients-we also give an estimate of a very simple form involving only the levels at which the vorticity has jumps. Our formula generalizes a corresponding one from [5].

1. INTRODUCTION

The subject of this paper lies within the broader area of nonlinear flows with vorticity whose relatively recent rigorous development is due to the complexity of the problem cf. [33]. Even in the framework of irrotational flows substantial achievements appeared just in the last decades with the extensive studies of the Stokes waves [34] and the flow beneath them concerning particle trajectories, behavior of the pressure [2, 3, 7, 12]. Allowing for vorticity in the problem not only adds flavor to the mathematical problem but also describes physically relevant phenomena like wave-current interactions cf. [4, 26, 33] whose study have not had a firm theoretical basis until relatively recently through [11] where the existence of small and large amplitude steady periodic gravity water waves with a general (regular) vorticity distribution was proved. The richness of the subject is highlighted by the numerous papers that followed [11] and treated topics ranging from symmetry [8, 9, 31], stability [14], regularity of the free surface and of the stream lines [10, 17, 21, 22, 37], stratifications [18, 25, 36] to features like stagnation points and critical layers [15, 16, 27, 28, 35] or the presence of a singular (merely bounded or piece-wise constant) vorticity distribution [13, 30, 32].

In this paper we address the case of an interior layer of constant non zero vorticity surrounded by irrotational flow. This situation corresponds to the case of a strong undercurrent that does not extend to the bed. The Equatorial Undercurrent in the Pacific Ocean is an example of such an occurrence - see [6] for a discussion of the suitability of the constant vorticity assumption in this setting.

The dispersion relation for small-amplitude waves indicates how the relative speed of the wave at the free surface varies with respect to certain parameters like the wave-length, the mean depth of the flow, and-in the case of a piece-wise constant vorticity like the one we consider here-the position of vorticity jumps. For the case of flows with constant vorticity the first accounts of the dispersion relation are due to Thompson [33] and Biesel [1]. As far as our paper is concerned we derive here the dispersion relation for small-amplitude two-dimensional steady periodic gravity water waves which propagate over a flat bed and

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have a discontinuous (piece-wice constant) vorticity distribution. The discontinuity we consider here has the following form: we assume that there is a uniform layer of constant non-zero vorticity which separates an irrotational flow adjacent to the free surface from another irrotational layer adjacent to the flat bed. The dispersion relation we obtain recovers the corresponding formula (23) from [5] for the case of a layer of constant non vanishing vorticity adjacent to the flat bed within an irrotational flow as well as the dispersion relation (81) from [13] in the context of a layer of constant non-zero vorticity adjacent to the free surface above fluid in irrotational flow. The existence of steady two-dimensional periodic water waves of small and large amplitude in a flow with an arbitrary bounded (but discontinuous) vorticity- thus, incorporating the kind of waves we consider here-was proven in [13] in the framework of a fixed mass flux and in [20] in the context of a fixed mean depth. Moreover, it was shown [23, 24] that the dispersion relations corresponding to the approach in [20] coincide with those in [13] and [5].
It is worth to mention that the dispersion relation for capillary-gravity waves for the situation of a layer of vorticity adjacent to the surface above irrotational fluid as well as for the case of an isolated layer of vorticity adjacent to the flat bed was obtained in [29].

2. Presentation of the problem

We will be working here with two-dimensional steady periodic waves which travel over a rotational, inviscid and incompressible fluid propagating in the positive x-direction over the flat bed $y=-d$ (for some $d>0$) and whose free surface $y=\eta(x)$ is a small perturbation of the flat free surface $y=0$. In a reference frame moving in the same direction as the wave with wave speed $c>0$ and assuming that the only restoring force acting upon the fluid is gravity, the equations of motion are the Euler's equations

$$\left\{\begin{array}{rcl}(\mathfrak{u}-c)\mathfrak{u}_x+\mathfrak{v}\mathfrak{u}_y &=& -P_x\\ (\mathfrak{u}-c)\mathfrak{v}_x+\mathfrak{v}\mathfrak{v}_y &=& -P_y-g,\end{array}\right. \tag{2.1}$$

together with the incompressibility condition

$$\mathfrak{u}_x+\mathfrak{v}_y=0, \tag{2.2}$$

whereby $(\mathfrak{u},\mathfrak{v})$ denotes the velocity field, P is the pressure and g is the gravitational constant. We will work under the assumption that $(\mathfrak{u},\mathfrak{v})$, P and the surface wave profile $x\to\eta(x)$ are periodic in the variable x and for simplicity we choose the period $L=2\pi$. The vorticity of the flow is

$$\omega:=\mathfrak{u}_y-\mathfrak{v}_x.$$

Associated with equations (2.1) and (2.2) are the kinematic boundary conditions and the dynamic boundary condition. The kinematic boundary conditions take the form

$$\left\{\begin{array}{l}\mathfrak{v}=(\mathfrak{u}-c)\eta_x \quad \text{on} \quad y=\eta(x)\\ \mathfrak{v}=0 \quad \text{on} \quad y=-d\end{array}\right. \tag{2.3}$$

and represent essentially a necessary and sufficient condition for the flow to move along a boundary but not across/through the boundary. The dynamic boundary condition

$$P=P_{atm} \quad \text{on} \quad y=\eta(x), \tag{2.4}$$

decouples the motion of the air above the free surface from that of the water. Here P represents pressure, P_{atm} being the constant atmospheric pressure. We refer the interested reader to [4] for details concerning the validity of (2.1)-(2.4). We seek further to reduce the number of unknowns and introduce the stream function ψ defined (up to a constant) by the relations

$$\psi_x = -\mathfrak{v}, \;\; \psi_y = \mathfrak{u} - c.$$

One more assumption, namely

$$\mathfrak{u} < c \quad \text{throughout} \quad \text{the} \quad \text{fluid}, \tag{2.5}$$

expressing the absence of stagnation points in the flow, ensures cf. [4, 11] that the vorticity ω is a single-valued function of ψ, i.e.,

$$\omega(x, y) = \gamma(-\psi(x, y)),$$

which finally yields the reformulation of (2.1)-(2.4) as the free boundary value problem

$$\begin{cases} \Delta\psi &= \gamma(-\psi) & \text{in} & -d < y < \eta(x), \\ |\nabla\psi|^2 + 2g(y+d) &= Q & \text{on} & y = \eta(x), \\ \psi &= 0 & \text{on} & y = \eta(x), \\ \psi &= -p_0 & \text{on} & y = -d, \end{cases} \tag{2.6}$$

where Q is a constant related to the total head, and $p_0 < 0$ is a constant representing the relative mass flux, given by

$$p_0 = \int_{-d}^{\eta(x)} (\mathfrak{u}(x, y) - c)\, dy.$$

It is now customary to transform the free boundary value problem (2.6) into a problem in the fixed domain $\overline{\Omega} := [-\pi, \pi] \times [p_0, 0]$. This is achieved by means of the partial hodograph transform

$$q(x, y) = x, \quad p(x, y) = -\psi(y) \tag{2.7}$$

which, due to assumption (2.5), provides a diffeomorphism from the fluid domain to Ω and renders the problem (2.6) into the quasilinear elliptic boundary value problem

$$\begin{cases} (1 + h_q^2)h_{pp} - 2h_p h_q h_{pq} + h_p^2 h_{qq} - \gamma h_p^3 &= 0 \;\; \text{in } \overline{\Omega}, \\ 1 + h_q^2 + (2gh - Q)h_p^2 &= 0 \;\; \text{on } p = 0, \\ h &= 0 \;\; \text{on } p = p_0, \end{cases} \tag{2.8}$$

where the unknown function h defined on $\overline{\Omega}$ by

$$h(q, p) := y + d$$

represents the height above the flat bed and is even and of period 2π in the q-variable. The discontinuous vorticity we work with here requires the reformulation of (2.8) in a week form. This was achieved in [13] by putting (2.8) in the week formulation

$$\begin{cases} \left\{ \frac{1+h_q^2}{2h_p^2} + \Gamma(p) \right\}_p - \left\{ \frac{h_q}{h_p} \right\}_q &= 0 \quad \text{in} \quad \Omega, \\ \frac{1+h_q^2}{2h_p^2} + gh &= \frac{Q}{2} \quad \text{on} \quad p = 0, \\ h &= 0 \quad \text{on} \quad p = 0, \end{cases} \tag{2.9}$$

whereby Γ is defined by

$$\Gamma(p) = \int_0^p \gamma(s)\, ds, \quad p \in [p_0, 0].$$

Remark 2.1. In the formula above the division by h_p is possible because of the elliptic non-degeneracy condition

$$h_p > 0 \quad \text{in} \quad \overline{\Omega},$$

which is in fact equivalent to the absence of stagnation points expressed by (2.5).

By a solution of (2.9) we understand a function $h \in W^{2,r}_{\text{per}} \subset C^{1,\alpha}_{\text{per}}$, with $r > \frac{2}{1-\alpha}$, (for a fixed $\alpha \in (1/3, 1)$) that is a generalized solution cf. [19], Section 8. A family of laminar solutions, i.e. parallel shear flows with flat free surfaces, parametrized by $\lambda > 2 \max_{[p_0,0]} \Gamma$ is given by

$$H(p) := H(p, \lambda) = \int_0^p \frac{ds}{\sqrt{\lambda - 2\Gamma(s)}} + \frac{Q - \lambda}{2g} \in C^{1,\alpha}([p_0, 0]) \tag{2.10}$$

cf. [13]. The parameter λ is related to wave speed at the flat free surface $y = 0$ of the laminar flow by the formula

$$\sqrt{\lambda} = (c - \mathfrak{u})|_{y=0} = \frac{1}{H_p(0)},$$

and to Q through the relation

$$\int_{p_0}^0 \frac{ds}{\sqrt{\lambda - 2\Gamma(s)}} = \frac{Q - \lambda}{2g}.$$

The considerations in [13] show that the necessary and sufficient condition for the existence of waves of small amplitude that are perturbations of the laminar flow solutions (2.10) is that the Sturm-Liouville problem

$$\begin{cases} (a^3 U_p)_p = aU & \text{in} \quad (p_0, 0), \\ a^3 U_p = gU & \text{at} \quad p = 0, \\ U = 0 & \text{at} \quad p = p_0 \end{cases} \tag{2.11}$$

has a nontrivial solution $U \in C^{1,\alpha}(p_0, 0), U \not\equiv 0$. Here $a(\lambda, p) = \sqrt{\lambda - 2\Gamma(p)} \in C^{\alpha}([p_0, 0])$. We will study in the next section the problem (2.11) for the case when the flow has a layer of constant non vanishing vorticity between two other layers of irrotational flow.

3. The dispersion relation

Let $p_1, p_2 \in [p_0, 0]$ such that $p_1 < p_2$. We consider a current of constant vorticity $\gamma \neq 0$ in the middle layer corresponding to $p \in [p_1, p_2]$ and zero vorticity in the lower and upper layers corresponding to $p \in [p_0, p_1]$ and to $p \in [p_2, 0]$, respectively. We then have

$$\Gamma(p) = \begin{cases} 0, & p \in [p_2, 0], \\ \gamma(p - p_2), & p \in [p_1, p_2], \\ \gamma(p_1 - p_2), & p \in [p_0, p_1], \end{cases} \tag{3.1}$$

and therefore

$$a(p) = \begin{cases} \sqrt{\lambda}, & p \in [p_2, 0], \\ \sqrt{\lambda - 2\gamma(p - p_2)}, & p \in [p_1, p_2], \\ \sqrt{\lambda - 2\gamma(p_1 - p_2)}, & p \in [p_0, p_1]. \end{cases} \tag{3.2}$$

We are looking for a function $U \in C^{1,\alpha}(p_0, 0)$ subject to (2.11). Denoting

$$u := U|_{[p_0,p_1]}, \quad v := U|_{[p_1,p_2]} \quad w := U|_{[p_2,0]}$$

it follows that the functions u, v, w will have to satisfy

$$(a^3 u_p)_p = au, \quad \text{for} \quad p \in (p_0, p_1), \tag{3.3}$$

$$(a^3 v_p)_p = av, \quad \text{for} \quad p \in (p_1, p_2), \tag{3.4}$$

$$(a^3 w_p)_p = aw, \quad \text{for} \quad p \in (p_2, 0), \tag{3.5}$$

together with the compatibility conditions

$$u(p_1) = v(p_1), \qquad u_p(p_1) = v_p(p_1), \tag{3.6}$$

$$v(p_2) = w(p_2), \qquad v_p(p_2) = w_p(p_2), \tag{3.7}$$

and the boundary conditions

$$u(p_0) = 0, \tag{3.8}$$

$$(a^3 w_p)(0) = gw(0). \tag{3.9}$$

Since a is constant on $[p_0, p_1]$, equation (3.3) becomes $u_{pp} = a^{-2} u$ with the general solution

$$u(p) = q_1 \cosh\left(\frac{p}{\sqrt{\lambda - 2\gamma(p_1 - p_2)}}\right) + q_2 \sinh\left(\frac{p}{\sqrt{\lambda - 2\gamma(p_1 - p_2)}}\right),$$

for some constants $q_1, q_2 \in \mathbb{R}$. From the boundary condition (3.8) we deduce that

$$u(p) = c \sinh\left(\frac{p - p_0}{\sqrt{\lambda - 2\gamma(p_1 - p_2)}}\right), \tag{3.10}$$

with c being a constant.
To solve equation (3.4) for v on $[p_1, p_2]$ we set

$$v = \frac{2\gamma}{a(p)} \tilde{v}\left(\frac{a(p)}{\gamma}\right), \quad p \in [p_1, p_2].$$

We find first that

$$v_p = \frac{2\gamma^2}{a^3(p)} \tilde{v}\left(\frac{a(p)}{\gamma}\right) - \frac{2\gamma}{a^2(p)} \tilde{v}'\left(\frac{a(p)}{\gamma}\right), \tag{3.11}$$

and

$$(a^3 v_p)_p = 2\gamma \tilde{v}''\left(\frac{a(p)}{\gamma}\right).$$

Therefore, the equation (3.4) yields $\tilde{v}''(s) = \tilde{v}(s)$ for $s = \frac{a(p)}{\gamma}$. Therefore,

$$v(p) = \frac{2\gamma}{a(p)}\left(c_1 \cos\left(\frac{a(p)}{\gamma}\right) + c_2 \sinh\left(\frac{a(p)}{\gamma}\right)\right), \quad p \in [p_1, p_2], \tag{3.12}$$

for some constants $c_1, c_2 \in \mathbb{R}$, whereby $a(p) = \sqrt{\lambda - 2\gamma(p - p_2)}$ for all $p \in [p_1, p_2]$.
Solving for w in (3.5) on $[p_2, 0]$ we find the general solution

$$w(p) = \alpha_1 \cosh\left(\frac{p}{\sqrt{\lambda}}\right) + \alpha_2 \sinh\left(\frac{p}{\sqrt{\lambda}}\right), \tag{3.13}$$

for some constants $\alpha_1, \alpha_2 \in \mathbb{R}$.
The compatibility conditions (3.7) can be written as:

$$\left\{\begin{array}{rcl} \alpha_1 \cosh\left(\frac{p_2}{\sqrt{\lambda}}\right) + \alpha_2 \sinh\left(\frac{p_2}{\sqrt{\lambda}}\right) & = & \frac{2\gamma}{\sqrt{\lambda}}\left[c_1 \cosh\left(\frac{\sqrt{\lambda}}{\gamma}\right) + c_2 \sinh\left(\frac{\sqrt{\lambda}}{\gamma}\right)\right] \\ \frac{\alpha_1}{\sqrt{\lambda}} \sinh\left(\frac{p_2}{\sqrt{\lambda}}\right) + \frac{\alpha_2}{\sqrt{\lambda}} \cosh\left(\frac{p_2}{\sqrt{\lambda}}\right) & = & \frac{2\gamma^2}{\lambda\sqrt{\lambda}}\left[c_1 \cosh\left(\frac{\sqrt{\lambda}}{\gamma}\right) + c_2 \sinh\left(\frac{\sqrt{\lambda}}{\gamma}\right)\right] \\ & & -\frac{2\gamma}{\lambda}\left[c_1 \sinh\left(\frac{\sqrt{\lambda}}{\gamma}\right) + c_2 \cosh\left(\frac{\sqrt{\lambda}}{\gamma}\right)\right] \end{array}\right. \tag{3.14}$$

and the compatibility conditions (3.6) read as

$$\left\{\begin{array}{rcl} c \sinh\left(\frac{p_1 - p_0}{a(p_1)}\right) & = & \frac{2\gamma}{a(p_1)}\left[c_1 \cosh\left(\frac{a(p_1)}{\gamma}\right) + c_2 \sinh\left(\frac{a(p_1)}{\gamma}\right)\right], \\ \frac{c}{a(p_1)} \cosh\left(\frac{p_1 - p_0}{a(p_1)}\right) & = & \frac{2\gamma^2}{a^3(p_1)}\left[c_1 \cosh\left(\frac{a(p_1)}{\gamma}\right) + c_2 \sinh\left(\frac{a(p_1)}{\gamma}\right)\right], \\ & & -\frac{2\gamma}{a^2(p_1)}\left[c_1 \sinh\left(\frac{a(p_1)}{\gamma}\right) + c_2 \cosh\left(\frac{a(p_1)}{\gamma}\right)\right], \end{array}\right. \tag{3.15}$$

which after setting

$$\theta := \frac{a(p_1)}{\gamma}, \quad \rho := \frac{p_1 - p_0}{a(p_1)}$$

becomes

$$\left\{\begin{array}{rcl} c \sinh(\rho) & = & \frac{2}{\theta}[c_1 \cosh(\theta) + c_2 \sinh(\theta)] \\ c \cosh(\rho) & = & \frac{2}{\theta^2}[c_1 \cosh(\theta) + c_2 \sinh(\theta)] - \frac{2}{\theta}[c_1 \sinh(\theta) + c_2 \cosh(\theta)]. \end{array}\right. \tag{3.16}$$

Solving for c_1, c_2 in (3.16) we find

$$c_1 = \frac{c}{2}[\theta \sinh(\rho + \theta) - \sinh(\rho) \sinh(\theta)] \tag{3.17}$$

$$c_2 = \frac{c}{2}[-\theta \cos(\rho + \theta) + \sinh(\rho) \cosh(\theta)] \tag{3.18}$$

We intend now to express the compatibility conditions in a more transparent way. We set d_1 to be the average depth corresponding to p_1 and d_2 the average depth corresponding to p_2. At the bifurcation point we have a horizontal fluid velocity $\mathfrak{u}$ that is only a function of y, with $\mathfrak{u}_y = \omega$ and $(c - \mathfrak{u})(0) = \sqrt{\lambda}$. Therefore, $(c - \mathfrak{u})(y) - (c - \mathfrak{u})(0) = \int_0^y (-\omega(s))\, ds$. Thus,

$$(c - \mathfrak{u})(y) = \left\{\begin{array}{ll} \sqrt{\lambda}, & y \in [-d_2, 0], \\ \sqrt{\lambda} - \gamma(y + d_2), & y \in [-d_1, -d_2] \\ \sqrt{\lambda} - \gamma(-d_1 + d_2), & y \in [-d, -d_1]. \end{array}\right. \tag{3.19}$$

We use now equation (3.19) in the formulas for the relative mass fluxes and obtain

$$-p_2 = \int_{-d_2}^0 (c - \mathfrak{u})(y)\, dy = \int_{-d_2}^0 \sqrt{\lambda}\, dy = \sqrt{\lambda} d_2. \tag{3.20}$$

$$\begin{aligned} -p_1 &= \int_{-d_1}^0 (c - \mathfrak{u})(y)\, dy = \int_{-d_1}^{-d_2} [\sqrt{\lambda} - \gamma(y + d_2)]\, dy + \int_{-d_2}^0 \sqrt{\lambda}\, dy \\ &= \sqrt{\lambda} d_1 + \frac{\gamma}{2}(d_1 - d_2)^2. \end{aligned} \tag{3.21}$$

Writing (3.21) as an equation of degree two in d_1 we obtain

$$\frac{\gamma}{2} d_1^2 + (\sqrt{\lambda} - \gamma d_2) d_1 + \frac{\gamma}{2} d_2^2 + p_1 = 0 \tag{3.22}$$

with the discriminant $\Delta = \lambda - 2\gamma(p_1 - p_2)$. Thus,

$$d_1 = \frac{-\sqrt{\lambda} + \gamma d_2 \pm \sqrt{\lambda - 2\gamma(p_1 - p_2)}}{\gamma},$$

which implies that

$$d_1 - d_2 = \frac{-\sqrt{\lambda} \pm \sqrt{\lambda - 2\gamma(p_1 - p_2)}}{\gamma}. \tag{3.23}$$

Using that $p_1 < p_2$ we have that $\frac{\sqrt{\lambda - 2\gamma(p_1-p_2)} - \sqrt{\lambda}}{\gamma} > 0$ irrespective of the sign of γ. Since $d_1 > d_2$ only the plus sign is possible in relation (3.23). Therefore

$$d_1 - d_2 = \frac{\sqrt{\lambda - 2\gamma(p_1 - p_2)} - \sqrt{\lambda}}{\gamma} = \frac{a(p_1) - a(p_2)}{\gamma} \tag{3.24}$$

A calculation reveals that the total mass flux equals

$$-p_0 = \int_{-d}^{0} (c - \mathfrak{u})(y)\, dy = \sqrt{\lambda} d - \gamma(d_2 - d_1)(d - d_1) + \frac{\gamma}{2}(d_2 - d_1)^2. \tag{3.25}$$

From (3.21) and (3.25) we obtain now that

$$p_1 - p_0 = [\sqrt{\lambda} + \gamma(d_1 - d_2)](d - d_1) = \sqrt{\lambda - 2\gamma(p_1 - p_2)}(d - d_1),$$

whereby the last equality above follows from (3.24). Hence, it follows from the above relation that

$$\frac{p_1 - p_0}{a(p_1)} = d - d_1. \tag{3.26}$$

Using that $\frac{p_2}{\sqrt{\lambda}} = -d_2$ we see that the system (3.14) has the simpler form

$$\left\{ \begin{array}{rcl} \alpha_1 \cosh(d_2) - \alpha_2 \sinh(d_2) & = & \frac{2\gamma}{\sqrt{\lambda}} \left[c_1 \cosh\left(\frac{\sqrt{\lambda}}{\gamma}\right) + c_2 \sinh\left(\frac{\sqrt{\lambda}}{\gamma}\right) \right] \\ -\alpha_1 \sinh(d_2) + \alpha_2 \cosh(d_2) & = & \frac{2\gamma^2}{\lambda} \left[c_1 \cosh\left(\frac{\sqrt{\lambda}}{\gamma}\right) + c_2 \sinh\left(\frac{\sqrt{\lambda}}{\gamma}\right) \right] \\ & & -\frac{2\gamma}{\sqrt{\lambda}} \left[c_1 \sinh\left(\frac{\sqrt{\lambda}}{\gamma}\right) + c_2 \cosh\left(\frac{\sqrt{\lambda}}{\gamma}\right) \right] \end{array} \right. \tag{3.27}$$

Multiplying above the first equation by $\cosh(d_2)$, the second by $\sinh(d_2)$ and adding the results we obtain

$$\begin{aligned} \alpha_1 = & \frac{2\gamma}{\sqrt{\lambda}} \left[c_1 \cosh\left(d_2 - \frac{\sqrt{\lambda}}{\gamma}\right) - c_2 \sinh\left(d_2 - \frac{\sqrt{\lambda}}{\gamma}\right) \right] \\ & + \frac{2\gamma^2}{\lambda} \sinh(d_2) \left[c_1 \cosh\left(\frac{\sqrt{\lambda}}{\gamma}\right) + c_2 \sinh\left(\frac{\sqrt{\lambda}}{\gamma}\right) \right] \end{aligned} \tag{3.28}$$

and similarly

$$\begin{aligned} \alpha_2 = & \frac{2\gamma}{\sqrt{\lambda}} \left[c_1 \sinh\left(d_2 - \frac{\sqrt{\lambda}}{\gamma}\right) - c_2 \cosh\left(d_2 - \frac{\sqrt{\lambda}}{\gamma}\right) \right] \\ & + \frac{2\gamma^2}{\lambda} \cosh(d_2) \left[c_1 \cosh\left(\frac{\sqrt{\lambda}}{\gamma}\right) + c_2 \sinh\left(\frac{\sqrt{\lambda}}{\gamma}\right) \right] \end{aligned} \tag{3.29}$$

Replacing c_1 and c_2 from (3.17) and (3.18) we obtain the following

$$c_1 \cosh\left(\frac{\sqrt{\lambda}}{\gamma}\right) + c_2 \sinh\left(\frac{\sqrt{\lambda}}{\gamma}\right) = \frac{c\theta}{2} \sinh\left(\rho + \theta - \frac{\sqrt{\lambda}}{\gamma}\right) + \frac{c}{2} \sinh(\rho) \sinh\left(\frac{\sqrt{\lambda}}{\gamma}\right) \tag{3.30}$$

$$\begin{aligned} c_1 \cosh\left(d_2 - \frac{\sqrt{\lambda}}{\gamma}\right) - c_2 \sinh\left(d_2 - \frac{\sqrt{\lambda}}{\gamma}\right) = & \frac{c\theta}{2} \sinh\left(\rho + \theta - \frac{\sqrt{\lambda}}{\gamma} + d_2\right) \\ & + \frac{c}{2} \sinh(\rho) \sinh\left(\frac{\sqrt{\lambda}}{\gamma} - \theta - d_2\right) \end{aligned} \tag{3.31}$$

$$c_1\sinh\left(d_2-\tfrac{\sqrt{\lambda}}{\gamma}\right)-c_2\cosh\left(d_2-\tfrac{\sqrt{\lambda}}{\gamma}\right) = \begin{array}{l}\tfrac{c\theta}{2}\cosh\left(\rho+\theta-\tfrac{\sqrt{\lambda}}{\gamma}+d_2\right)\\ -\tfrac{c}{2}\sinh(\rho)\cosh\left(\tfrac{\sqrt{\lambda}}{\gamma}-\theta-d_2\right)\end{array} \tag{3.32}$$

Recall now that

$$\rho = d - d_1 \tag{3.33}$$

by (3.26). Moreover,

$$\theta-\frac{\sqrt{\lambda}}{\gamma}=\frac{a(p_1)-a(p_2)}{\gamma}=d_1-d_2, \tag{3.34}$$

by (3.24). In view of (3.30)-(3.32), (3.33) and (3.34) we can rewrite (3.28) and (3.29) as

$$\begin{aligned}\alpha_1 =& \tfrac{2\gamma}{\sqrt{\lambda}}\left[\tfrac{c\theta}{2}\sinh(d)-\tfrac{c}{2}\sinh(d-d_1)\sinh(d_1)\right]\\ &+\tfrac{2\gamma^2}{\lambda}\sinh(d_2)\left[\tfrac{c\theta}{2}\sinh(d-d_2)+\tfrac{c}{2}\sinh(d-d_1)\sinh(d_2-d_1)\right],\end{aligned} \tag{3.35}$$

and

$$\begin{aligned}\alpha_2 =& \tfrac{2\gamma}{\sqrt{\lambda}}\left[\tfrac{c\theta}{2}\cosh(d)-\tfrac{c}{2}\sinh(d-d_1)\cosh(d_1)\right]\\ &+\tfrac{2\gamma^2}{\lambda}\cosh(d_2)\left[\tfrac{c\theta}{2}\sinh(d-d_2)+\tfrac{c}{2}\sinh(d-d_1)\sinh(d_2-d_1)\right].\end{aligned} \tag{3.36}$$

We return now to the top boundary condition (3.9) which due to (3.13) is equivalent to

$$\lambda\alpha_2 = g\alpha_1.$$

Using the formulas (3.35) and (3.36) in the latter relation we obtain (after division by $2\gamma c$) the following

$$\begin{aligned}&\lambda^{\frac{3}{2}}\left[\theta\cosh(d)\right.\left.-\sinh(d-d_1)\cos(d_1)\right]\\ &\qquad+\gamma\cosh(d_2)\left[\theta\sinh(d-d_2)+\sinh(d-d_1)\sinh(d_2-d_1)\right]\lambda\\ =&\sqrt{\lambda}g\left[\theta\sinh(d)\right.\left.-\sinh(d-d_1)\sinh(d_1)\right]\\ &\qquad+\gamma g\sinh(d_2)\left[\theta\sinh(d-d_2)+\sinh(d-d_1)\sinh(d_2-d_1)\right]\end{aligned} \tag{3.37}$$

After replacing $\theta=\frac{\sqrt{\lambda}}{\gamma}+d_1-d_2$ in (3.37) we have

$$\begin{aligned}&\cosh(d)\lambda^2+[(d_1-d_2)\cosh(d)+\sinh(d_1-d_2)\cosh(d-d_1-d_2)]\gamma\lambda^{\frac{3}{2}}\\ &+\left\{[(d_1-d_2)\sinh(d-d_2)+\sinh(d-d_1)\sinh(d_2-d_1)]\cosh(d_2)\gamma^2-g\sinh(d)\right\}\lambda\\ &-[(d_1-d_2)\sinh(d)+\sinh(d_2-d_1)\sinh(d-d_1-d_2)]g\gamma\sqrt{\lambda}\\ &-[(d_1-d_2)\sinh(d-d_2)+\sinh(d-d_1)\sinh(d_2-d_1)]g\sinh(d_2)\gamma^2=0\end{aligned} \tag{3.38}$$

Before we study in detail formula (3.38) we analyze two limit cases.

Remark 3.1. We set $d_2=0$ i.e. we are in the situation of a layer of constant non-zero vorticity adjacent to the free surface above fluid in irrotational flow. Equation (3.38) becomes in this case (after division by $\sqrt{\lambda}$)

$$\begin{aligned}&\cosh(d)\lambda^{\frac{3}{2}}+[d_1\cosh(d)+\sinh(d_1)\cosh(d-d_1)]\gamma\lambda\\ &+\left\{[d_1\sinh(d)-\sinh(d_1)\sinh(d-d_1)]\gamma^2-g\sinh(d)\right\}\sqrt{\lambda}\\ &-[d_1\sinh(d)-\sinh(d_1)\sinh(d-d_1)]g\gamma=0,\end{aligned} \tag{3.39}$$

which after a further division by $\cosh(d)$ reads as

$$\begin{aligned}\lambda^{\frac{3}{2}} + \left[d_1 + \frac{\sinh(d_1)\sinh(d-d_1)}{\cosh(d)}\right]\gamma\lambda \\ +\left\{\left[d_1 - \frac{\sinh(d_1)\sinh(d-d_1)}{\sinh(d)}\right]\gamma^2 - g\right\}\sqrt{\lambda}\tanh(d) \\ -\left[d_1 - \frac{\sinh(d_1)\sinh(d-d_1)}{\sinh(d)}\right]g\gamma\tanh(d) = 0,\end{aligned} \tag{3.40}$$

i.e. we recover relation (81) from [13].

Remark 3.2. We put now $d_1 := d$ in formula (3.38), which means that we are in the situation of a layer of constant non-zero vorticity adjacent to the flat bed below fluid in irrotational flow. Equation (3.38) is then transformed in

$$\begin{aligned}\cosh(d)\lambda^{\frac{3}{2}}[\sqrt{\lambda}+\gamma(d-d_2)] + [\sqrt{\lambda}+\gamma(d-d_2)]\lambda\gamma\cosh(d_2)\sinh(d-d_2) \\ -[\sqrt{\lambda}+\gamma(d-d_2)]g\sinh(d)\sqrt{\lambda} - [\sqrt{\lambda}+\gamma(d-d_2)]g\gamma\sinh(d_2)\sinh(d-d_2) = 0.\end{aligned} \tag{3.41}$$

By relation (3.19) we have that $\sqrt{\lambda}+\gamma(d-d_2) > 0$. After dividing by $\sqrt{\lambda}+\gamma(d-d_2) > 0$ above we are left with

$$\cosh(d)\lambda^{\frac{3}{2}} + \gamma\cosh(d_2)\sinh(d-d_2)\lambda - g\sinh(d)\sqrt{\lambda} - g\gamma\sinh(d_2)\sinh(d-d_2) = 0, \tag{3.42}$$

which recovers formula (23) in [5].

Proposition 3.3. *If $\gamma > 0$ then the local bifurcation always holds.*

Proof. Let $p(x)$ be the polynomial obtained by replacing $\sqrt{\lambda}$ by x in the left hand side of (3.38), i.e.

$$\begin{aligned}p(x) = \cosh(d)x^4 + [(d_1-d_2)\cosh(d) + \sinh(d_1-d_2)\cosh(d-d_1-d_2)]\gamma x^3 \\ +\big\{[(d_1-d_2)\sinh(d-d_2) + \sinh(d-d_1)\sinh(d_2-d_1)]\cosh(d_2)\gamma^2 - g\sinh(d)\big\}x^2 \\ -[(d_1-d_2)\sinh(d) + \sinh(d_2-d_1)\sinh(d-d_1-d_2)]g\gamma x \\ -[(d_1-d_2)\sinh(d-d_2) + \sinh(d-d_1)\sinh(d_2-d_1)]g\sinh(d_2)\gamma^2 = 0\end{aligned} \tag{3.43}$$

Note that $p(0) = -H(d_1)\gamma^2 g\sinh(d_2) < 0$ by Lemma 4.1 (c). This implies that p has at least one positive root. We will show in the sequel that this is the only positive root. For this we analyze the derivative of p. We have

$$\begin{aligned}p'(x) = 4\cosh(d)x^3 + 3[(d_1-d_2)\cosh(d) + \sinh(d_1-d_2)\cosh(d-d_1-d_2)]\gamma x^2 \\ + 2\big\{[(d_1-d_2)\sinh(d-d_2) + \sinh(d-d_1)\sinh(d_2-d_1)]\cosh(d_2)\gamma^2 - g\sinh(d)\big\}x \\ - [(d_1-d_2)\sinh(d) + \sinh(d_2-d_1)\sinh(d-d_1-d_2)]g\gamma.\end{aligned} \tag{3.44}$$

We notice first that $p'(0) = -G(d_1)g\gamma < 0$ cf. Lemma 4.1 (b). This implies that p' has at least one root $y_0 > 0$. We will show that y_0 is in fact the unique positive root of p'. Indeed, the existence of a second positive root would imply that the third root is also real and positive, since by Viète's relations the product of the three roots equals $-\frac{p'(0)}{4\cosh(d)} > 0$. But this is impossible since the sum of the three roots is $-\frac{3F(d_1)\gamma}{4\cosh(d)} < 0$ by Lemma 4.1 (a). We conclude now from the previous discussion that $p'(x) < 0$ on $[0, y_0)$ and $p'(x) > 0$ on (y_0, ∞). Therefore, p is strictly decreasing on $[0, y_0)$ and strictly increasing on (y_0, ∞).

Since $p(0) < 0$ we can conclude, by inspecting the graph of p, that p has indeed a unique positive solution x_+ giving the dispersion relation. $\square$

Remark 3.4. One can write an exact formula for $\sqrt{\lambda}$, but its intricacy makes it of little use. We prefer instead to give in the following proposition an estimate on $\sqrt{\lambda}$ which in fact generalizes the estimate (27) from [5], obtained in the context of a layer of constant non-zero vorticity adjacent to the flat bed below fluid in irrotational flow . The latter mentioned situation can be derived from our setting by puting $d_1 = d$.

Proposition 3.5. *If $\gamma > 0$ and $\frac{d_1+d_2}{2} < d < d_1 + d_2$ then*

$$\sqrt{g \tanh(d_1 + d_2 - d)} < \sqrt{\lambda} < \sqrt{g \tanh(d)}. \tag{3.45}$$

Proof. Note that the dispersion relation (3.38) can be written

$$\begin{aligned}
&(\lambda + (d_1 - d_2)\gamma\sqrt{\lambda}[\lambda \cosh(d) - g \sinh(d)] \\
&+ [(d_1 - d_2)\sinh(d - d_2) + \sinh(d - d_1)\sinh(d_2 - d_1)]\gamma^2(\lambda \cosh(d_2) - g \sinh(d_2)] \\
&+ \sinh(d_1 - d_2)\gamma\sqrt{\lambda}[\lambda \cosh(d_1 + d_2 - d) - g \sinh(d_1 + d_2 - d)] = 0.
\end{aligned} \tag{3.46}$$

We prove first the second inequality in (3.45). Indeed, assuming that $\lambda \geq g \tanh(d)$ we obtain that $\lambda > g \tanh(d_2)$, since $d > d_2$, and $\lambda > g \tanh(d_1 + d_2 - d)$, since $2d > d_1 + d_2$. Using the latter inequalities, Lemma 4.1 (c) together with $\gamma > 0$ yields that the left hand side of (3.46) is strictly positive, which is a contradiction. To prove the first inequality in (3.45) we assume for the sake of contradiction that

$$\lambda \leq g \tanh(d_1 + d_2 - d). \tag{3.47}$$

Inequality (3.47) implies immediately that

$$\lambda < g \tanh(d_2). \tag{3.48}$$

The positivity of γ, Lemma 4.1 (c) and inequalities (3.47) and (3.48) imply now that

$$\begin{aligned}
&[(d_1 - d_2)\sinh(d - d_2) + \sinh(d - d_1)\sinh(d_2 - d_1)]\gamma^2(\lambda \cosh(d_2) - g \sinh(d_2)] \\
&+ \sinh(d_1 - d_2)\gamma\sqrt{\lambda}[\lambda \cosh(d_1 + d_2 - d) - g \sinh(d_1 + d_2 - d)] < 0.
\end{aligned} \tag{3.49}$$

We see now from (3.46) and (3.49) that the first expression in (3.46) has to be non-negative, which implies that $\lambda \geq g \tanh(d)$. But the latter is a contradiction with the already proven second inequality in (3.45) $\square$

Proposition 3.6. *Let $\gamma < 0$. Assume that*

$$\frac{g}{\gamma^2} > (d_1 - d_2)^2 \frac{(d_1 - d_2)\cosh(d_1) + \sinh(d_2 - d_1)\cosh(d_2)}{(d_1 - d_2)\sinh(d_1) + \sinh(d_2 - d_1)\sinh(d_2)} \tag{3.50}$$

and that

$$\begin{aligned}
\frac{g}{\gamma^2} > &\frac{(d_1 - d_2)^2[(d_1 - d_2)\cosh(d) - 3\sinh(d_1 - d_2)\cosh(d - d_1 - d_2)]}{[(d_1 - d_2)\sinh(d) + \sinh(d_1 - d_2)\sinh(d - d_1 - d_2)]} \\
&+ \frac{2(d_1 - d_2)[(d_1 - d_2)\sinh(d - d_2) + \sinh(d - d_1)\sinh(d_2 - d_1)]\cosh(d_2)}{[(d_1 - d_2)\sinh(d) + \sinh(d_1 - d_2)\sinh(d - d_1 - d_2)]}.
\end{aligned} \tag{3.51}$$

Then local bifurcation occurs.

Proof. Since we have from (3.19) that $\sqrt{\lambda}+\gamma(d_1-d_2) > 0$ we make in (3.38) the substitution $\sqrt{\lambda} = -\gamma(x + d_1 - d_2)$ and obtain the equation

$$\begin{aligned}
&\gamma^4 \cosh(d)x^4 + (4\gamma^4(d_1 - d_2)\cosh(d) - f_1\gamma^4)x^3 \\
&+ (6\gamma^4(d_1 - d_2)^2 \cosh(d) - 3f_1\gamma^4(d_1 - d_2) + f_2\gamma^2)x^2 \\
&+ (4\gamma^4(d_1 - d_2)^3 \cosh(d) - 3f_1\gamma^4(d_1 - d_2)^2 + 2f_2\gamma^2(d_1 - d_2) + f_3\gamma^2)x \\
&+ \gamma^4(d_1 - d_2)^4 \cosh(d) - f_1\gamma^4(d_1 - d_2)^3 + f_2\gamma^2(d_1 - d_2)^2 + f_3\gamma^2(d_1 - d_2) - f_4\gamma^2 = 0
\end{aligned} \tag{3.52}$$

whereby

$$f_1 = (d_1 - d_2)\cosh(d) + \sinh(d_1 - d_2)\cosh(d - d_1 - d_2),$$
$$f_2 = [(d_1 - d_2)\sinh(d - d_2) + \sinh(d - d_1)\sinh(d_2 - d_1)]\cosh(d_2)\gamma^2 - g\sinh(d),$$
$$f_3 = [(d_1 - d_2)\sinh(d) + \sinh(d_2 - d_1)\sinh(d - d_1 - d_2)]g,$$

and

$$f_4 = [(d_1 - d_2)\sinh(d - d_2) + \sinh(d - d_1)\sinh(d_2 - d_1)]g\sinh(d_2).$$

□

If we denote by $q(x)$ the polynomial obtained by dividing by γ^4 the left-hand side of the equation (3.52) we have that

$$\begin{aligned}
q(0) = &-(d_1 - d_2)^3 \sinh(d_1 - d_2)\cosh(d - d_1 - d_2) \\
&+ (d_1 - d_2)^2[(d_1 - d_2)\sinh(d - d_2) + \sinh(d - d_1)\sinh(d_2 - d_1)]\cosh(d_2) \\
&+ \frac{g}{\gamma^2}(d_1 - d_2)\sinh(d_2 - d_1)\sinh(d - d_1 - d_2) \\
&- \frac{g}{\gamma^2}([(d_1 - d_2)\sinh(d - d_2) + \sinh(d - d_1)\sinh(d_2 - d_1)]\sinh(d_2).
\end{aligned} \tag{3.53}$$

Using that

$$\begin{aligned}
\sinh(d - d_2)&\cosh(d_2) - \sinh(d_1 - d_2)\cosh(d - d_1 - d_2) \\
&= \frac{1}{2}(\sinh(d) + \sinh(d - 2d_2)) - \frac{1}{2}(\sinh(d - 2d_2) + \sinh(2d_1 - d)) \\
&= \frac{1}{2}(\sinh(d) - \sinh(2d_1 - d)) \\
&= \sinh(d - d_1)\cosh(d_1)
\end{aligned}$$

and

$$\begin{aligned}
\sinh(d_2 - d_1)&\sinh(d - d_1 - d_2) - \sinh(d - d_2)\sinh(d_2) = \\
&= \frac{1}{2}(\cosh(d - 2d_1) - \cosh(d - 2d_2)) - \frac{1}{2}(\cosh(d) - \cosh(d - 2d_2)) \\
&= \frac{1}{2}(\cosh(d - 2d_1) - \cosh(d)) \\
&= \sinh(d - d_1)\sinh(-d_1)
\end{aligned}$$

we have that

$$\begin{aligned}
q(0) = &(d_1 - d_2)^2 \sinh(d - d_1)[(d_1 - d_2)\cosh(d_1) + \sinh(d_2 - d_1)\cosh(d_2)] \\
&- \frac{g}{\gamma^2}\sinh(d - d_1)[(d_1 - d_2)\sinh(d_1) + \sinh(d_2 - d_1)\sinh(d_2)] < 0
\end{aligned}$$

by (3.50) and Lemma 4.2. Note also that

$$\begin{aligned} q'(0) &= 4(d_1-d_2)^3\cosh(d) - 3f_1(d_1-d_2)^2 + 2\frac{f_2}{\gamma^2}(d_1-d_2) + \frac{f_3}{\gamma^2} \\ &= (d_1-d_2)^2[(d_1-d_2)\cosh(d) - 3\sinh(d_1-d_2)\cosh(d-d_1-d_2)] \\ &+ 2(d_1-d_2)[(d_1-d_2)\sinh(d-d_2) + \sinh(d-d_1)\sinh(d_2-d_1)]\cosh(d_2) \\ &- \frac{g}{\gamma^2}[(d_1-d_2)\sinh(d) + \sinh(d_1-d_2)\sinh(d-d_1-d_2)] < 0 \end{aligned}$$

by (3.51) and Lemma 4.3.
Note now that the sum of roots of the polynomial $q'(x)$ equals

$$\begin{aligned} &-\frac{3}{4}\left(4(d_1-d_2) - \frac{f_1}{\cosh(d)}\right) \\ &= -\frac{3}{4}\left(4(d_1-d_2) - (d_1-d_2) - \frac{\sinh(d_1-d_2)\cosh(d-d_1-d_2)}{\cosh d}\right) \\ &= -\frac{3}{4}\left(2(d_1-d_2) + (d_1-d_2) - \frac{\sinh(d_1-d_2)\cosh(d-d_1-d_2)}{\cosh d}\right). \end{aligned} \tag{3.54}$$

The expression in the bracket above is positive since the function

$$d_1 \to (d_1-d_2) - \frac{\sinh(d_1-d_2)\cosh(d-d_1-d_2)}{\cosh d}$$

is zero at d_2 and its derivative is $1-\frac{\cosh(d-2d_1)}{\cosh(d)} > 0$ for $d > d_1 > 0$. Equation (3.54) ensures now that the sum of roots of $q'(x)$ is negative. Using the latter and that $q(0) < 0$, $q'(0) < 0$ we employ an argument similar to that in the proof of Proposition 3.3 and conclude that the polynomial $q(x)$ has a unique positive root x_0 giving the dispersion relation by means of the formula $\sqrt{\lambda} = -\gamma(x_0 + d_1 - d_2)$.

4. Appendix

Lemma 4.1. (a) *Let*

$$F(d_1) := (d_1-d_2)\cosh(d) + \sinh(d_1-d_2)\cosh(d-d_1-d_2).$$

Then $F(d_1) > 0$ for all $d_1 > d_2$.

(b) *Let*

$$G(d_1) := (d_1-d_2)\sinh(d) + \sinh(d_2-d_1)\sinh(d-d_1-d_2).$$

Then $G(d_1) > 0$ for all $d_1 > d_2$.

(c) *Let*

$$H(d_1) := (d_1-d_2)\sinh(d-d_2) + \sinh(d-d_1)\sinh(d_2-d_1).$$

Then $H(d_1) > 0$ for all $d_1 > d_2$.

Proof. We see first that $F(d_1)|_{d_1:=d_2} = 0$. Secondly,

$$\begin{aligned} F'(d_1) &= 1 + \cosh(d_1-d_2)\cosh(d-d_1-d_2) - \sinh(d_1-d_2)\sinh(d-d_1-d_2) \\ &= 1 + \cosh(2d_1-d) > 0, \end{aligned}$$

and therefore the claim in (a) is proved.

For (b) note that $G(d_1)|_{d_1:=d_2} = 0$ and

$$\begin{aligned} G'(d_1) &= \sinh(d) - \cosh(d_2 - d_1)\sinh(d - d_1 - d_2) - \sinh(d_2 - d_1)\cosh(d - d_1 - d_2) \\ &= \sinh(d) - \sinh(d - 2d_1) > 0, \end{aligned}$$

since $d > d - 2d_1$. Thus, the assertion in (b) is proved.

Similarly, in (c) we have that $H(d_1)|_{d_1:=d_2} = 0$ and

$$\begin{aligned} H'(d_1) &= \sinh(d - d_2) - \cosh(d - d_1)\sinh(d_2 - d_1) - \sinh(d - d_1)\cosh(d_2 - d_1) \\ &= \sinh(d - d_2) - \sinh(d + d_2 - 2d_1) = 2\sinh(d_1 - d_2)\cosh(d - d_1) > 0 \end{aligned}$$

for all $d_1 > d_2$. □

Lemma 4.2. *Let*

$$E(d_1) = (d_1 - d_2)\sinh(d_1) - \sinh(d_1 - d_2)\sinh(d_2).$$

Then $E(d_1) > 0$ *for all* $d_1 > d_2 > 0$.

Proof. We see that $E(d_1)|_{d_1:=d_2} = 0$ and

$$\begin{aligned} E'(d_1) &= \sinh(d_1) + d_1\cosh(d_1) - \sinh(d_2)\cosh(d_1 - d_2) && (4.1) \\ &= \sinh(d_1) + d_1\cosh(d_1) - \frac{1}{2}\sinh(d_1) - \frac{1}{2}\sinh(2d_2 - d_1) && (4.2) \\ &= d_1\cosh(d_1) + \frac{1}{2}[\sinh(d_1) - \sinh(2d_2 - d_1)] && (4.3) \\ &= d_1\cosh(d_1) + \sinh(d_1 - d_2)\cosh(d_2) > 0 && (4.4) \end{aligned}$$

for all $d_1 > d_2 > 0$. □

Lemma 4.3. *Let*

$$K(d_1) = (d_1 - d_2)\sinh(d) + \sinh(d_1 - d_2)\sinh(d - d_1 - d_2).$$

Then $K(d_1) > 0$ *for all* $d > d_1 > d_2 > 0$.

Proof. Notice that $K(d_1)|_{d_1:=d_2} = 0$ and

$$\begin{aligned} K'(d_1) &= \sinh(d) + \cosh(d_2 - d_1)\sinh(d - d_1 - d_2) + \sinh(d_2 - d_1)\cosh(d - d_1 - d_2) \\ &= \sinh(d) + \sinh(d - 2d_1) = 2\sinh(d - d_1)\cosh(d_1) > 0 \end{aligned}$$

for all $d > d_1 > 0$. □

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