Local bifurcation and regularity for steady periodic capillary-gravity water waves with constant vorticity

C.-I. Martin

Email address: `calin.martin@univie.ac.at` (C.-I. Martin)

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C.-I. Martin

Faculty of Mathematics, University of Vienna, Nordbergstrasse 15, 1090 Vienna, Austria

Abstract

We study periodic capillary-gravity waves at the free surface of water in a flow with constant vorticity over a flat bed. Using bifurcation theory the local existence of waves of small amplitude is proved even in the presence of stagnation points in the flow. We also derive the dispersion relation. Moreover, we prove a regularity result for the free surface.

Keywords: capillary-gravity water waves, dispersion relation, bifurcation theory
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1. Introduction

The mathematical theory of periodic traveling waves was for a long time confined to the study of irrotational flows (see, e.g., [14, 26, 38]) which are suitable for waves traveling into still water. Nevertheless the papers [34, 35, 36] make clear that nonuniform currents give rise to water flows with vorticity.
It is worth to point out that zero vorticity means either no underlying current (a situation corresponding to swell due to a distant storm and entering a region of still water cf. the discussion in [8]) or a uniform underlying current (cf. the discussion in [10]), while constant vorticity is the hallmark of tidal currents cf. the discussion in [13, 35].
Lately a lot of research had been carried out in the field of water waves with

Email address: calin.martin@univie.ac.at (C.-I. Martin)

vorticity: see [3, 4, 5] for existence results, [17, 40] for matters of uniqueness, [12, 20, 33] for regularity results and [6, 7] concerning the symmetry of rotational water waves. However, the surface tension-a force per unit length due to a pressure difference across a curved surface- is neglected in most of the recent works on rotational water waves. The existence of pure capillary water waves with vorticity was proved in [41] and the existence of capillary-gravity waves with vorticity was shown in [42]. However in both of these two papers it was assumed that there are no **stagnation points** throughout the fluid domain, i.e., that **the wave speed c is strictly larger than the horizontal fluid velocity**. We established in [30] the local existence of capillary water waver with constant vorticity in the presence of stagnation points in the fluid domain whose free surface is not necessarily the graph of a function. In [31] we proved a regularity result for steady periodic travelling capillary waves of small amplitude at the free surface of water in a flow with constant vorticity over a flat bed. In the present paper we consider both gravity and surface tension i.e., we investigate capillary-gravity water waves with vorticity in the presence of stagnation points and of a free surface which admits overhanging profiles. The existence of small amplitude capillary and capillary-gravity water waves with stagnation points, for flows with varying vorticity and stratification, but with the free surface a graph, was proven in [25], following on work in [19]. We want to mention that capillary waves with arbitrary vorticity were considered in the paper [42] but under the assumption that there are no stagnation points in the fluid domain and that the free surface is always a graph. We base our approach on a method developed in [11] which uses a reformulation of the original problem as an equation for a function of one variable, giving the elevation of the free surface when the fluid domain is the conformal image of a half-plane. The reformulation is presented in Section 2. In Section 3 we use bifurcation theory from a simple eigenvalue in the spirit of Crandall-Rabinowitz to prove the existence of waves of small amplitude. Due to the length of the discussion we do not address here the issue of double bifurcation. In Section 4 we prove that any $C^{2,\alpha}$ free surface is in fact C^{∞}.

We now present the free-boundary value problem of steady periodic traveling capillary water waves with constant vorticity γ in a flow of finite depth. We consider two- dimensional waves propagating over water with a flat bed. The X variable will represent the direction of propagation, Y will be the height variable and $((V_1(X,Y,t), V_2(X,Y,t))$ denotes the velocity field. The water domain Ω in the XY-plane is bounded below by the impermeable flat bed

$$\mathcal{B} = \{(X, 0); X \in \mathbb{R}\},$$

and above by an a priori unknown curve

$$\mathcal{S}(t) = \{(u(t,s), v(t,s)); s \in \mathbb{R}\}, t \geq 0 \tag{1}$$

with

$$(u_s(t,s))^2 + (v_s(t,s))^2 > 0 \text{ for all } s \in \mathbb{R}, t \geq 0 \tag{2}$$

and

$$u(t, s+L) = u(t,s) + L,\ v(t, s+L) = v(t,s) \text{ for all } s \in \mathbb{R}, \tag{3}$$

representing the free surface of the water (not necessary the graph of a function), which is L-periodic in the horizontal direction. The equations of motion are the equation of mass conservation

$$V_{1X} + V_{2Y} = 0 \tag{4}$$

and Euler's equation

$$\begin{cases} V_{1t} + V_1 V_{1X} + V_2 V_{1Y} &= -P_X \\ V_{2t} + V_1 V_{2X} + V_2 V_{2Y} &= -P_Y - g \end{cases} \tag{5}$$

where $P(X,Y,t)$ denotes pressure and g is the gravitational constant of acceleration. The boundary conditions associated to (4) and (5) are of two types dynamic and kinematic boundary conditions. The dynamic boundary condition expresses the stresses that the atmosphere exerts on the fluid surface and takes therefore the form

$$P(u(t,s), v(t,s), t) = P_0 - \sigma \frac{u_s(t,s)v_{ss}(t,s) - u_{ss}(t,s)v_s(t,s)}{((u_s(t,s))^2 + (v_s(t,s))^2)^{3/2}} \text{ on } \mathcal{S}, \tag{6}$$

P_0 being the constant atmospheric pressure, $\sigma > 0$ the coefficient of surface tension which is a force per unit length due to a pressure difference across a curved surface, cf. [28], [1] and $\frac{u_s(t,s)v_{ss}(t,s)-u_{ss}(t,s)v_s(t,s)}{((u_s(t,s))^2+(v_s(t,s))^2)^{3/2}}$ representing the mean curvature of $\mathcal{S}$. The kinematic boundary conditions require that the free surface and the bed always consist of the same fluid particles. If $S_0(X,Y,t)=0$ is the implicit equation of the free surface, the kinematic boundary condition can be expressed as

$$S_{0t} + S_{0X}V_1 + S_{0Y}V_2 = 0 \text{ on } \mathcal{S}, \tag{7}$$

cf. [28], while the kinematic boundary condition on the bed is

$$V_2 = 0 \text{ on } \mathcal{B}. \tag{8}$$

The assumption of steady periodic traveling waves at speed $c > 0$ means that we have a space-time dependence of the form $X-ct$ for the free surface, the pressure, and for the velocity field. Then after the change of variables $x = X - ct, y = Y$ the equation of the free surface becomes $S(x,y) = 0$ for some S, and (5) – (8) are transformed to the stationary problem

$$\begin{cases} (V_1(x,y)-c)V_{1x}(x,y) + V_2(x,y)V_{1y}(x,y) = -P_x(x,y) \\ (V_1(x,y)-c)V_{2x}(x,y) + V_2(x,y)V_{2y}(x,y) = -P_y(x,y) - g \end{cases} \text{in } \Omega \tag{9}$$

and

$$\begin{cases} S_x(V_1-c) + S_yV_2 = 0 & \text{on} \quad \mathcal{S} \\ P = P_0 - \sigma\frac{u_sv_{ss}-u_{ss}v_s}{((u_s)^2+(v_s)^2)^{3/2}} & \text{on} \quad \mathcal{S}, \\ V_2 = 0 & \text{on} \quad \mathcal{B}. \end{cases} \tag{10}$$

We remark that if $S(x,y) = y - \eta(x)$ the first equation becomes $V_2 = \eta_x(V_1 - c)$ The equation of mass conservation (4) permits us to introduce the stream function ψ which satisfies:

$$\psi_x = -V_2 \text{ and } \psi_y = V_1 - c \tag{11}$$

[1] Surface tension is perpendicular to any line drawn in the surface having the same magnitude for all directions of the line and the same value at all points on the surface, cf. [13]

being defined through the line integral

$$\psi(x,y) = -m + \int_{(0,0)}^{(x,y)} (-V_2(x,y))dx + (V_1(x,y) - c)dy,$$

for some constant m. The equation of mass conservation $V_{1x} + V_{2y} = 0$ ensures the path-independence of the above line integral provided the path is in the simply connected domain Ω, where Ω is the set of all $(x,y) \in \mathbb{R}^2$ above the line $y = 0$ bounded above by the curve $\{(x,y) \in \mathbb{R}^2 : S(x,y) = 0\}$. We proved in [30] that

$$\psi = 0 \text{ on } \mathcal{S}$$

and

$$\psi = -m \text{ on } \mathcal{B},$$

where m is the relative mass flux defined through

$$m = \int_0^{v(t,s_0)} \left[V_1(u(t,s_0) - ct, y) - c\right] dy,$$

with $(u(t,s_0 - ct, v(t,s_0))$ being the wave trough. For the fact that m is constant we refer the reader to [30]. We can then write (5) – (8) in terms of ψ as follows

$$\begin{cases} \psi_y\psi_{xy} - \psi_x\psi_{yy} = -P_x \\ -\psi_y\psi_{xx} + \psi_x\psi_{xy} = -P_y - g \end{cases} \quad \text{in } \Omega \tag{12}$$

and

$$\begin{cases} \psi = 0 & \text{on} \quad \mathcal{S}, \\ P = P_0 - \sigma \frac{u_s(t,s)v_{ss}(t,s) - u_{ss}(t,s)v_s(t,s)}{((u_s(t,s))^2 + (v_s(t,s))^2)^{3/2}} & \text{on} \quad \mathcal{S}, \\ \psi = -m & \text{on} \quad \mathcal{B}. \end{cases} \tag{13}$$

If $\gamma = V_{2X} - V_{1Y}$ is the vorticity we obtain from (9) and properties of ψ Bernoulli's law, which says that

$$E = \frac{(V_1 - c)^2 + V_2^2}{2} + P + gy + \gamma\psi$$

is constant throughout the fluid domain. On the free surface we have

$$E = \frac{(V_1 - c)^2 + V_2^2}{2} + P_0 - \sigma \frac{u_s(t,s)v_{ss}(t,s) - u_{ss}(t,s)v_s(t,s)}{((u_s(t,s))^2 + (v_s(t,s))^2)^{3/2}} + gv.$$

Therefore setting $Q = 2(E - P_0)$ we obtain

$$\psi_x^2 + \psi_y^2 - 2\sigma \frac{u_s(t,s)v_{ss}(t,s) - u_{ss}(t,s)v_s(t,s)}{((u_s(t,s))^2 + (v_s(t,s))^2)^{3/2}} + 2gv = Q$$

on the free surface. We therefore have that the stream function ψ satisfies the following boundary value problem:

$$\begin{aligned} \Delta\psi &= -\gamma && \text{in } \Omega, \\ \psi &= -m && \text{on } \mathcal{B}, \\ \psi &= 0 && \text{on } \mathcal{S}, \\ |\nabla\psi|^2 - 2\sigma \tfrac{u_s(t,s)v_{ss}(t,s) - u_{ss}(t,s)v_s(t,s)}{((u_s(t,s))^2 + (v_s(t,s))^2)^{3/2}} + 2gv &= Q && \text{on } \mathcal{S}. \end{aligned} \tag{14}$$

Choosing a parametrization so that u and v are independent of t in the moving frame leads to the following free boundary value problem

$$\begin{aligned} \Delta\psi &= -\gamma && \text{in } \Omega, \\ \psi &= -m && \text{on } \mathcal{B}, \\ \psi &= 0 && \text{on } \mathcal{S}, \\ |\nabla\psi|^2 - 2\sigma \tfrac{u_s v_{ss} - u_{ss} v_s}{(u_s^2 + v_s^2)^{3/2}} + 2gv &= Q && \text{on } \mathcal{S}. \end{aligned} \tag{15}$$

We will prove in the paper the local existence of waves of small amplitude to the problem (15) and a regularity result for the free surface. Moreover we derive the dispersion relation, i.e., a formula which gives the speed of the bifurcating laminar flow (see the proof of Theorem 8) in terms of the depth, the period and the vorticity. This dispersion relation is obtained even in the presence of stagnation points in the flow, feature that is not allowed in the paper [42]. We show that for flows with sufficiently small wave-length L there do not exist stagnation points while, if the vorticity is big enough we do have stagnation points.

Our investigation opens up possibilities for the detailed examination of the flow pattern, in the same vein to the ones pursued in [8, 10] for irrotational gravity water waves and in [21] for linear periodic capillary and capillary-gravity water waves. Concerning the particle flow patterns we would also like to mention the papers [9], [32] and [22].

2. Equivalence of the free boundary problem with a problem with a fixed domain

We give in this section a reformulation of the boundary value problem (15) as a quasilinear equation in a fixed domain for a periodic function of one variable. We need first a few notations and some preliminary results; for the proofs of these results we refer the reader to [11].

For an integer $p \geq 0$ and for $\alpha \in (0,1)$ we denote $C^{p,\alpha}$ the standard space of functions whose partial derivatives up to order p are Hölder continuous with exponent α over their domain of definition. $C^{p,\alpha}_{\text{loc}}$ will denote the set of functions of class $C^{p,\alpha}$ over any compact subset of their domain of definition. By $C^{p,\alpha}_{2\pi}$ we denote the space of functions of one real variable which are 2π periodic and of class $C^{p,\alpha}_{\text{loc}}$ in $\mathbb{R}$. By $C^{p,\alpha}_{2\pi,o}$ we denote the functions that are in $C^{p,\alpha}_{2\pi}$ and have zero mean over one period. For any $d > 0$ let

$$\mathcal{R}_d = \{(x,y) \in \mathbb{R}^2 : -d < y < 0\}.$$

For any $w \in C^{p,\alpha}_{2\pi}$ let $W \in C^{p,\alpha}(\overline{R}_d)$ be the unique solution of

$$\begin{aligned} \Delta W &= 0 \text{ in } \mathcal{R}_d, \\ W(x,-d) &= 0,\ x \in \mathbb{R}, \\ W(x,0) &= w(x),\ x \in \mathbb{R}. \end{aligned} \tag{16}$$

The function $(x,y) \to W(x,y)$ is 2π-periodic in x throughout $\mathcal{R}_d$. For $p \in \mathbb{Z}, p \geq 1$, and $\alpha \in (0,1)$ we define the *periodic Dirichlet-Neumann operator for a strip* $\mathcal{G}_d$ by

$$\mathcal{G}_d(w)(x) = W_y(x,0),\ x \in \mathbb{R}.$$

The operator $\mathcal{G}_d : C^{p,\alpha}_{2\pi} \to C^{p-1,\alpha}_{2\pi}$ is a bounded linear operator. If the function w takes the constant value c then

$$\mathcal{G}_d(c) = \frac{c}{d}. \tag{17}$$

Let Z be the unique (up to a constant) harmonic function in $\mathcal{R}_d$, such that $Z + iW$ is holomorphic in $\mathcal{R}_d$. If $w \in C^{p,\alpha}_{2\pi,o}$ it follows from the discussion in

Section 2 of [11] that the function $(x,y) \to Z(x,y)$ is 2π-periodic in x throughout $\mathcal{R}_d$. We specify the constant in the definition of Z by asking that $x \to Z(x,0)$ has zero mean over one period. We define $\mathcal{C}_d(w)$ by

$$\mathcal{C}_d(w)(x) = Z(x,0),\ x \in \mathbb{R}.$$

The obtained mapping $\mathcal{C}_d : C^{p,\alpha}_{2\pi,o} \to C^{p,\alpha}_{2\pi,o}$ is a bounded linear operator and is called the *periodic Hilbert transform for a strip.* If $w \in C^{p,\alpha}_{2\pi,o}$ for $p \geq 1$ we have

$$\mathcal{G}_d(w) = (\mathcal{C}_d(w))' = \mathcal{C}_d(w'). \tag{18}$$

It also follows (see [11]) that for $p \geq 1$,

$$\mathcal{G}_d(w) = \frac{[w]}{d} + \mathcal{C}_d(w'), \tag{19}$$

where $[w]$ denotes the average of w over one period.

Definition 1. We say that a solution (Ω, ψ) of the water wave equation (15) is of class $C^{2,\alpha}$ if the free surface satifies (1), (2) and (3) with $u, v \in C^{2,\alpha}$ and $\psi \in C^2(\Omega) \cap C^{2,\alpha}(\overline{\Omega})$.

Definition 2.

- We say that $\Omega \subset \mathbb{R}^2$ is an *L-periodic strip like domain* if it is contained in the upper half (X,Y)-plane and if its boudary consists of the real axis $\mathcal{B}$ and a parametric curve $\mathcal{S}$ defined by (1) and which satisfies (2) and (3).

- For any such domain, the *conformal mean depth* is defined to be unique positive number h such that there exists an onto conformal mapping $\tilde{U} + i\tilde{V} : \mathcal{R}_h \to \Omega$ which admits an extension between the closures of these domains, with onto mappings

  $$\{(x,0) : x \in \mathbb{R}\} \to \mathcal{S},$$

  and

  $$\{(x,-h) : x \in \mathbb{R}\} \to \mathcal{B},$$

  and such that

$$\begin{array}{rcl} \tilde{U}(x+L,y) & = & \tilde{U}(x,y)+L, \\ \tilde{V}(x+L,y) & = & \tilde{V}(x,y), \end{array} \quad (x,y)\in\mathcal{R}_h \tag{20}$$

The existence and uniqueness of such an h was proved in Appendix A of the paper [11].

We are now able to formulate the equivalence of (15) with a quasilinear equation for a periodic function of one variable in a fixed domain.

Theorem 1. *If* (Ω,ψ) *of class* $C^{2,\alpha}$ *is a solution of* (15) *then there exists a positive number* h*, a function* $v\in C^{2,\alpha}_{2\pi}$ *and a constant* $a\in\mathbb{R}$ *such that*

$$\begin{array}{l} \left\{\frac{m}{kh}+\gamma\left(\mathcal{G}_{kh}(\frac{v^2}{2})-v\mathcal{G}_{kh}(v)\right)\right\}^2=\left(Q+2\sigma\frac{\mathcal{G}_{kh}(v)v''-\mathcal{G}_{kh}(v')v'}{(v'^2+\mathcal{G}_{kh}(v)^2)^{3/2}}-2gv\right)\left(v'^2+\mathcal{G}_{kh}(v)^2\right) \\ [v]=h \\ v(x)>0 \text{ for all } x\in\mathbb{R}, \\ \text{the mapping } x\to\left(\frac{x}{k}+\mathcal{C}_{kh}(v-h)(x),v(x)\right) \text{ is injective on } \mathbb{R}, \\ v'(x)^2+\mathcal{G}_{kh}(v)(x)^2\neq 0 \text{ for all } x\in\mathbb{R}, \end{array} \tag{21}$$

where $k=\frac{2\pi}{L}$*. Moreover*

$$\mathcal{S}=\left\{\left(a+\frac{x}{k}+\mathcal{C}_{kh}(v-h)(x),v(x)\right):x\in\mathbb{R}\right\}. \tag{22}$$

Conversely, let $h>0$ *and* $v\in C^{2,\alpha}_{2\pi}$ *be such that* (21) *holds. Assume also that* $\mathcal{S}$ *is defined by* (22)*, let* Ω *be the domain whose boundary consists of* $\mathcal{S}$ *and of the real axis* $\mathcal{B}$ *and let* $a\in\mathbb{R}$ *be arbitrary. Then there exists a function* ψ *in* Ω *such that* (Ω,ψ) *is a solution of* (15) *of class* $C^{2,\alpha}$*.*

Proof. We first prove the necessity. Let (Ω,ψ) be a solution of class $C^{2,\alpha}$ of (15). Then we denote by h the conformal mean depth of Ω and by $\tilde{U}+i\tilde{V}$ the conformal mapping associated to Ω. If we consider the mapping $U+iV:\mathcal{R}_{kh}\to\Omega$ given by

$$\begin{array}{l} U(x,y)=\tilde{U}(\frac{x}{k},\frac{y}{k}), \\ V(x,y)=\tilde{V}(\frac{x}{k},\frac{y}{k}), \end{array} \quad (x,y)\in\mathcal{R}_{kh}, \tag{23}$$

where $k=\frac{2\pi}{L}$ then following the proof of Theorem 2.2 in [11] we see that $U,V\in C^{2,\alpha}(\overline{\mathcal{R}_h})$ and $U+iV$ is a conformal mapping from $\mathcal{R}_{kh}$ onto Ω which extends homeomorphically to the closures of these domains, with onto mappings

$$\{(x,0) : x \in \mathbb{R}\} \to \mathcal{S},$$

and

$$\{(x,-kh) : x \in \mathbb{R}\} \to \mathcal{B}.$$

Moreover,

$$U_x^2(x,0) + V_x^2(x,0) \neq 0 \text{ for all } x \in \mathbb{R}. \tag{24}$$

Let us set

$$v(x) = V(x,0) \text{ for all } x \in \mathbb{R}, \quad u(x) = U(x,0) \text{ for all } x \in \mathbb{R} \tag{25}$$

We then have that

$$u = \mathcal{C}_{kh}(v),$$

and from (18) it is immediate that

$$u' = \mathcal{G}_{kh}(v) \text{ and } u'' = \mathcal{G}_{kh}(v') \tag{26}$$

It also follows [11] that $v \in C^{2,\alpha}_{2\pi}$ and

$$[v] = h \tag{27}$$

$$v(x) > 0 \text{ for all } x \in \mathbb{R}, \tag{28}$$

$$\text{the mapping } x \to \left(\frac{x}{k} + \mathcal{C}_{kh}(v-h)(x), v(x)\right) \text{ is injective on } \mathbb{R}, \tag{29}$$

$$\mathcal{S} = \left\{\left(a + \frac{x}{k} + \mathcal{C}_{kh}(v-h)(x), v(x)\right) : x \in \mathbb{R}\right\}, \tag{30}$$

for some $a \in \mathbb{R}$, whose presence in the formula (30) is due to the invariance of problem (15) to horizontal translations. From (24) and the Cauchy-Riemann equations it follows that

$$v'(x)^2 + \mathcal{G}_{kh}(v)(x)^2 \neq 0 \text{ for all } x \in \mathbb{R}. \tag{31}$$

Now let $\xi : \mathcal{R}_{kh} \to \mathbb{R}$ be defined by

$$\xi(x,y) = \psi(U(x,y), V(x,y)),\ (x,y) \in \mathcal{R}_{kh}. \tag{32}$$

The harmonicity in Ω of the function $(x,y) \to \psi(x,y) + \frac{\gamma}{2}y^2$ and the invariance of harmonic functions under conformal mappings imply that

$$\xi + \frac{\gamma}{2}V^2 \text{ is harmonic in } \mathcal{R}_{kh}. \tag{33}$$

The chain rule and the Cauchy-Riemann equations imply that

$$\xi_x^2 + \xi_y^2 = (\psi_x^2(U,V) + \psi_y^2(U,V))(V_x^2 + V_y^2) \text{ in } \overline{\mathcal{R}_{kh}}.$$

From the last equation in (15) and (26) it follows that

$$\xi_x^2 + \xi_y^2 = \left(Q + 2\sigma \frac{\mathcal{G}_{kh}(v)v'' - \mathcal{G}_{kh}(v')v'}{(v'^2 + \mathcal{G}_{kh}(v)^2)^{3/2}} - 2gv\right)\left(v'^2 + \mathcal{G}_{kh}(v)^2\right) \tag{34}$$

Define $\zeta : \mathcal{R}_{kh} \to \mathbb{R}$ through

$$\zeta = \xi + m + \frac{\gamma}{2}V^2. \tag{35}$$

Using the boundary conditions from (15) we obtain the following

$$\begin{aligned}
\Delta\zeta &= 0 \text{ in } \mathcal{R}_{kh},\\
\zeta(x,-kh) &= 0 \text{ for all } x \in \mathbb{R},\\
\zeta(x,0) &= m + \tfrac{\gamma}{2}v^2(x) \text{ for all } x \in \mathbb{R},\\
(\zeta_y - \gamma V V_y)^2 &= \left(Q + 2\sigma \tfrac{\mathcal{G}_{kh}(v)v'' - \mathcal{G}_{kh}(v')v'}{(v'^2 + \mathcal{G}_{kh}(v)^2)^{3/2}} - 2gv\right)\left(v'^2 + \mathcal{G}_{kh}(v)^2\right) \text{ at } (x,0) \text{ for all } x \in \mathbb{R}.
\end{aligned} \tag{36}$$

The system (36) can be reformulated by using the Dirichlet-Neumann operator and (17) as

$$\left\{\frac{m}{kh} + \gamma\left(\mathcal{G}_{kh}(\frac{v^2}{2}) - v\mathcal{G}_{kh}(v)\right)\right\}^2 = \left(Q + 2\sigma \frac{\mathcal{G}_{kh}(v)v'' - \mathcal{G}_{kh}(v')v'}{(v'^2 + \mathcal{G}_{kh}(v)^2)^{3/2}} - 2gv\right)\left(v'^2 + \mathcal{G}_{kh}(v)^2\right) \tag{37}$$

For the sufficiency suppose that the positive number h and the function $v \in C_{2\pi}^{2,\alpha}$ satisfy (21). Let V be the harmonic function on $\mathcal{R}_{kh}$ which satisfies

$$V(x,-kh) = 0$$

and

$$V(x,0) = v(x) \text{ for all } x \in \mathbb{R},$$

and let $U : \mathcal{R}_{kh} \to \mathcal{R}$ be such that $U + iV$ is holomorphic. An application of Lemma 2.1 from [11] yields that $U + iV \in C^{2,\alpha}(\overline{\mathcal{R}_{kh}})$. From $[v] = h$ we obtain

$$\begin{cases} U(x+2\pi, y) = U(x,y) + \frac{2\pi}{k}, \\ V(x+2\pi, y) = V(x,y), \end{cases} \quad (x,y) \in \mathcal{R}_{kh}. \tag{38}$$

The injectivity of the mapping $x \to \left(\frac{x}{k} + \mathcal{C}_{kh}(v-h)(x), v(x)\right)$ gives that the curve (30) is non-self-intersecting and from $v(x) > 0$ we have that (30) is contained in the upper half-plane. If Ω denotes the domain whose boundary consists of $\mathcal{S}$ and $\mathcal{B}$, it follows from Theorem 3.4 in [39] that $U + iV$ is a conformal mapping from $\mathcal{R}_{kh}$ onto Ω, which extends to a homeomorphism between the closures of these domains, with onto mappings

$$\{(x,0) : x \in \mathbb{R}\} \to \mathcal{S},$$

and

$$\{(x,-kh) : x \in \mathbb{R}\} \to \mathcal{B}.$$

Together with (38) this implies that Ω is a L-periodic strip-like domain, with $L = 2\pi/k$. The conformal mean depth of Ω is h as it can be seen from the properties of the mapping $\tilde{U} + i\tilde{V} : \mathcal{R}_h \to \Omega$, where $\tilde{U}, \tilde{V}$ are given by (23). Let ζ be defined as the unique solution of the first three equations of (36). Then $\zeta \in C^{2,\alpha}(\overline{\mathcal{R}_{kh}}) \cap C^\infty(\mathcal{R}_{kh})$. Now, let ξ be defined by (35) and ψ by (32). We obtain that ψ satisfies the first three equations in (15). From the first equation in (21) we also have that the last equation from (15) holds. $\square$

3. Local bifurcation

This section is devoted to proving the existence of solutions to (21). The relation $[v] = h$ makes natural to set

$$v = w + h \tag{39}$$

Equation (39) implies immediately that $[w] = 0$. We then use (19) to find that

$$\mathcal{G}_{kh}(w+h) = \mathcal{G}_{kh}(w) + \mathcal{G}_{kh}(h) = \frac{[w]}{kh} + \mathcal{C}_{kh}(w') + \frac{h}{kh} = \frac{1}{k} + \mathcal{C}_{kh}(w')$$

and

$$\mathcal{G}_{kh}(v') = \mathcal{G}_{kh}(w') = \frac{[w']}{kh} + \mathcal{C}_{kh}(w'') = \mathcal{C}_{kh}(w''),$$

since w is periodic. Therefore we can rewrite (21) as

$$\begin{aligned}
&\left\{\tfrac{m}{kh} + \gamma\left(\tfrac{[w^2]}{2kh} - \tfrac{w}{k} - \tfrac{h}{2k} + \mathcal{C}_{kh}(ww') - w\mathcal{C}_{kh}(w')\right)\right\}^2 \\
&= \left\{Q + 2\sigma\frac{\frac{w''}{k} + w''\mathcal{C}_{kh}(w') - w'\mathcal{C}_{kh}(w'')}{\left(w'^2 + \left(\frac{1}{k} + \mathcal{C}_{kh}(w')\right)^2\right)^{3/2}} - 2gh - 2gw\right\}\left\{w'^2 + \left(\tfrac{1}{k} + \mathcal{C}_{kh}(w')\right)^2\right\} \\
&[w] = 0 \\
&w(x) > -h \text{ for all } x \in \mathbb{R}, \\
&\text{the mapping } x \to \left(\tfrac{x}{k} + \mathcal{C}_{kh}(w)(x), w(x) + h\right) \text{ is injective on } \mathbb{R}, \\
&w'(x)^2 + \left(\tfrac{1}{k} + \mathcal{C}_{kh}(w')(x)\right)^2 \neq 0 \text{ for all } x \in \mathbb{R},
\end{aligned} \tag{40}$$

We will regard m and Q as parameters and will prove the existence of solutions $w \in C^{1,\alpha}_{2\pi}$ to the problem (40) for all $\gamma \in \mathbb{R}$, $k > 0$, and $h > 0$ fixed.

We observe that $w = 0 \in C^{1,\alpha}_{2\pi,o}$ is a solution of (40) if and only if

$$Q = 2gh + \left(\frac{m}{h} - \frac{\gamma h}{2}\right)^2.$$

This suggests setting

$$\begin{aligned}
\lambda &= \tfrac{m}{h} - \tfrac{\gamma h}{2}, \\
\mu &= Q - 2gh - \left(\tfrac{m}{h} - \tfrac{\gamma h}{2}\right)^2
\end{aligned} \tag{41}$$

Note that the mapping $(m, Q) \to (\lambda, \mu)$ is a bijection from $\mathbb{R}^2$ onto itself. Using (41) we see that the equation (40) can be rewritten as

$$\begin{aligned}
&\left\{\tfrac{\lambda}{k} + \gamma\left(\tfrac{[w^2]}{2kh} - \tfrac{w}{k} + \mathcal{C}_{kh}(ww') - w\mathcal{C}_{kh}(w')\right)\right\}^2 \\
&= \left\{\lambda^2 + \mu + 2\sigma\frac{\frac{w''}{k} + w''\mathcal{C}_{kh}(w') - w'\mathcal{C}_{kh}(w'')}{\left(w'^2 + \left(\frac{1}{k} + \mathcal{C}_{kh}(w')\right)^2\right)^{3/2}} - 2gw\right\}\left\{w'^2 + \left(\tfrac{1}{k} + \mathcal{C}_{kh}(w')\right)^2\right\}
\end{aligned} \tag{42}$$

with $w \in C^{1,\alpha}_{2\pi,o}$, $\mu \in \mathbb{R}$ and $\lambda \in \mathbb{R}$. It is clear that $w = 0 \in C^{1,\alpha}_{2\pi,o}$ and $\mu = 0$ is a solution of (42) for all $\lambda \in \mathbb{R}$. We now apply the Crandall-Rabinowitz theorem [15] on bifurcation from a simple eingevalue in order to prove existence of non-trivial solutions to equation (42).

Theorem 2. *Let X and Y be Banach spaces, I be an open interval in $\mathbb{R}$ containing λ^*, and $F : I \times X \to Y$ be a continous map satisfying the following properties:*

1. *$F(\lambda, 0) = 0$ for all $\lambda \in I$;*
2. *$\partial_\lambda F$, $\partial_u F$ and $\partial^2_{\lambda,u} F$ exist and are continuous;*
3. *$\mathcal{N}(\partial_u F(\lambda^*, 0))$ and $Y/\mathcal{R}(\partial_u F(\lambda^*, 0))$ are one-dimensional, with the null space generated by u^*;*
4. *$\partial^2_{\lambda,u} F(\lambda^*, 0)(1, u^*) \notin \mathcal{R}(\partial_u F(\lambda^*, 0))$*

Then there exists a continuous local bifurcation curve $\{(\lambda(s), u(s) : |s| < \varepsilon\}$ with $\varepsilon > 0$ sufficiently small such that $(\lambda(0), u(0)) = (\lambda^*, 0)$ and there exists a neighbourhood $\mathcal{O}$ of $(\lambda^*, 0) \in I \times X$ such that

$$\{(\lambda, u) \in \mathcal{O} : u \neq 0, F(\lambda, u) = 0\} = \{(\lambda(s), u(s) : \ 0 < |s| < \varepsilon\}.$$

Moreover, we have

$$u(s) = su^* + o(s) \ \text{in } X, \ |s| < \varepsilon.$$

If $\partial^2_u F$ is also continuous, then the curve is of class C^1.

In order to apply the local bifurcation theorem 2 to (42) we set

$$X = \mathbb{R} \times C^{p+1,\alpha}_{2\pi,o,e}, \ Y = C^{p,\alpha}_{2\pi,e},$$

where for any integer $p \geq 0$ we denote:

$$C^{p,\alpha}_{2\pi,e} = \{f \in C^{p,\alpha}_{2\pi} : f(x) = f(-x) \text{ for all } x \in \mathbb{R}\},$$

$$C^{p,\alpha}_{2\pi,o,e} = \{f \in C^{p,\alpha}_{2\pi,o} : f(x) = f(-x) \text{ for all } x \in \mathbb{R}\}.$$

Equation (42) can be written as $F(\lambda, (\mu, w)) = 0$ where $F : \mathbb{R} \times X \to Y$ is given by

$$
\begin{aligned}
F(\lambda,(\mu,w)) = {} & \gamma^2\left(\mathcal{C}_{kh}(ww') - w\mathcal{C}_{kh}(w') - \frac{w}{k} + \frac{[w^2]}{2kh}\right)^2 \\
& + \frac{2\lambda\gamma}{k}\left(\mathcal{C}_{kh}(ww') - w\mathcal{C}_{kh}(w') - \frac{w}{k} + \frac{[w^2]}{2kh}\right) \\
& - \left(\mu + 2\sigma\frac{\frac{w''}{k} + w''\mathcal{C}_{kh}(w') - w'\mathcal{C}_{kh}(w'')}{\left(w'^2 + \left(\frac{1}{k} + \mathcal{C}_{kh}(w')\right)^2\right)^{3/2}} - 2gw\right)\left(w'^2 + \left(\frac{1}{k} + \mathcal{C}_{kh}(w')\right)^2\right) \\
& - \lambda^2\left(w'^2 + \frac{2}{k}\mathcal{C}_{kh}(w') + \left(\mathcal{C}_{kh}(w')\right)^2\right).
\end{aligned}
\tag{43}
$$

Since $F(\lambda,(0,0)) = 0$ the first condition from the local bifurcation theorem 2 is verified. We now compute

$$
\partial_{(\mu,w)}F(\lambda,(0,0))(\nu,f) = \lim_{t\to 0}\frac{F(\lambda,t(\nu,f)) - F(\lambda,(0,0))}{t}.
$$

Using Lemma 12 it turns out that

$$
\begin{aligned}
\partial_{(\mu,w)}F(\lambda,(0,0))(\nu,f) &= -\frac{2\lambda\gamma}{k^2}f + \frac{2gf}{k^2} - \frac{\nu}{k^2} - 2\sigma f'' - 2\frac{\lambda^2}{k}\mathcal{C}_{kh}(f') \\
&= -\frac{2}{k^2}\left(\lambda\gamma f - gf + \lambda^2 k\mathcal{C}_{kh}(f') + \sigma k^2 f''\right) - \frac{\nu}{k^2}
\end{aligned}
\tag{44}
$$

From representation (92) it follows that

$$
\partial_{(\mu,w)}F(\lambda,(0,0))(\nu,f) = -\frac{2}{k^2}\sum_{n=1}^{\infty}(\lambda\gamma - g + \lambda^2 kn\coth(nkh) - \sigma k^2 n^2)a_n\cos(nx) - \frac{\nu}{k^2}.
\tag{45}
$$

if

$$
f = \sum_{n=1}^{\infty} a_n\cos(nx).
$$

Using now Lemma 12 it follows that the bounded linear operator $\partial_{(\mu,w)}F(\lambda,(0,0)) : X \to Y$ is invertible whenever

$$
\lambda\gamma - g + \lambda^2 kn\coth(nkh) - \sigma k^2 n^2 \neq 0 \text{ for any integer } n \geq 1.
\tag{46}
$$

Therefore all the candidates for the bifurcation points of (42) are to be found among the solutions of the equation

$$
\lambda\gamma - g + \lambda^2 kn\coth(nkh) - \sigma k^2 n^2 = 0,
\tag{47}
$$

for some integer $n \geq 1$. Since we are looking for solutions of (42) of minimal period 2π, we take $n = 1$ in (47). Let

$$\lambda_{\pm}^{*n} = -\frac{\gamma \tanh(nkh)}{2kn} \pm \sqrt{\frac{\gamma^2 \tanh^2(nkh)}{4k^2n^2} + (nk\sigma + \frac{g}{kn})\tanh(nkh)},$$

denote the two solutions of equation (47).

Remark 1. Note that

$$\lambda_{\pm}^{*1}(nk) = \lambda_{\pm}^{*n}(k), \tag{48}$$

for all integers $n \geq 1$ and all $k > 0$.

Lemma 3. *Let $\lambda(k) = \lambda_{+}^{*1}(k)$.*

(i) If $\frac{\sigma}{gh^2} > \frac{\gamma^2 h}{6g} + \frac{1}{3} - \frac{\gamma}{6g}\sqrt{\gamma^2h^2 + 4gh}$ then the function λ is strictly increasing.

(ii) If $\frac{\sigma}{gh^2} < \frac{\gamma^2 h}{6g} + \frac{1}{3} - \frac{\gamma}{6g}\sqrt{\gamma^2h^2 + 4gh}$ then the function λ has a maximum at $k = 0$ and a unique local extremum, namely a local minimum at $k = k_0 > 0$. Moreover, there is a strictly decreasing sequence $(k_n)_{n\geq 2}$ such that

$$\lambda(k) = \lambda(nk)$$

for $k > 0$ if and only if $k = k_n$.

Proof. Let us denote $F(x) = \frac{\gamma^2h^2}{4}\frac{\tanh^2(hk)}{(hk)^2}$ and $G(k) = \frac{\sigma}{h}hk\tanh(hk) + gh\frac{\tanh(hk)}{hk}$. We can then write

$$\lambda(k) = -\frac{\gamma h}{2}\frac{\tanh(hk)}{hk} + \sqrt{F(k) + G(k)}.$$

We have that $\frac{d}{dk}\frac{\tanh(hk)}{hk}|_{k=0} = 0$ and $F'(0) = G'(0) = 0$. This implies $\lambda'(0) = 0$ and

$$\begin{aligned}
\lambda''(0) &= -\frac{\gamma h}{2}\frac{d^2}{dk^2}\frac{\tanh(hk)}{hk}|_{\{k=0\}} + \frac{2(F+G)(F''+G'') - (F'+G')^2}{4(F+G)\sqrt{F+G}}|_{\{k=0\}} \\
&= \frac{\gamma}{3}h^3 + \frac{F''+G''}{2\sqrt{F+G}}|_{\{k=0\}} \\
&= \frac{\gamma}{3}h^3 + \frac{-\frac{\gamma^2}{3}h^4 + 2h\sigma - \frac{2}{3}gh^3}{\sqrt{\gamma^2h^2 + 4gh}}
\end{aligned} \tag{49}$$

One sees also easily that $\lim_{k\to\infty}\lambda(k)=\infty$.

Set now $f(k)=k\coth(hk)$. From the formula for $\lambda_+^{*1}(k)$ we obtain that

$$\lambda=-\frac{\gamma}{2f}+\sqrt{\frac{\gamma^2}{4f^2}+\frac{k^2\sigma+g}{f}}$$

which by squaring leads further to

$$\begin{aligned}\lambda^2 &= \tfrac{\gamma^2}{4f^2}+\tfrac{\gamma^2}{4f^2}+\tfrac{k^2\sigma+g}{f}-\tfrac{\gamma}{f}\sqrt{\tfrac{\gamma^2}{4f^2}+\tfrac{k^2\sigma+g}{f}}\\ &= \tfrac{\gamma^2}{2f^2}+\tfrac{k^2\sigma+g}{f}-\tfrac{\gamma}{f}\left(\lambda+\tfrac{\gamma}{2f}\right)\\ &= \tfrac{k^2\sigma+g-\lambda\gamma}{f}\end{aligned}$$

and therefore it follows that

$$\lambda^2 f=k^2\sigma+g-\lambda\gamma. \tag{50}$$

Implicit differentiation of (50) gives

$$\lambda'(\gamma+2\lambda f)=2k\sigma-\lambda^2 f' \tag{51}$$

Again by implicit differentiation of (51) we get

$$\lambda''(\gamma+2\lambda f)=2\sigma-2(\lambda')^2 f-4\lambda\lambda' f'-\lambda^2 f''. \tag{52}$$

Let $k=k_0$ be a potential critical point of λ. It follows then from (51) that $\lambda^2(k_0)=\frac{2\sigma k_0}{f'(k_0)}$ and therefore

$$\lambda''(k_0)(\gamma+2\lambda f(k_0))=2\sigma-\lambda^2 f''(k_0)=2\sigma\frac{f'(k_0)-k_0 f''(k_0)}{f'(k_0)} \tag{53}$$

where $f'(k)-kf''(k)=\frac{\sinh^2(hk)\cosh(hk)+hk\sinh(hk)-2(hk)^2\cosh(hk)}{\sinh^3(hk)}$. Denoting $g(x)=\sinh^2(x)\cosh(x)+x\sinh(x)-2x^2\cosh(x)$ a computation reveals that $g^{(3)}(x)>0$ for all $x>0$, $g(0)=g'(0)=g''(0)=0$ which implies that $g(x)>0$ for all $x>0$. The latter together with $\gamma+2\lambda f(k_0)>0$ and $f'(k_0)>0$ imply via (53) that $\lambda''(k_0)>0$. Hence λ has at most one critical point for $k>0$, which is then a minimum point. Note now that the requirement in (i) is equivalent to asking $\lambda''(0)>0$ which in turn says that λ has a local minimum at $k=0$. The latter together with $\lim_{k\to\infty}\lambda(k)=\infty$ and the fact λ can not have a local maximum at $k > 0$ implies the conclusion. The condition in (ii) is equivalent to asking $\lambda''(0) < 0$ which in turn says that λ has a local maximum at $k = 0$. Since $\lim_{k\to\infty} \lambda(k) = \infty$ and because λ can not have a local maximum at $k > 0$ we have that λ has a unique local minimum at some $k_0 > 0$.

It remains to prove the existence of the sequence k_n with the asserted property. For $k \in (0, k_0)$ let $\tilde{k} > k_0$ be such that $\lambda(k) = \lambda(\tilde{k})$. Define now $F : (0, k_0) \to (1, \infty)$, $F(k) = \frac{\tilde{k}}{k}$. It is easy to see that F is strictly decreasing with $F(k) \to \infty$ as $k \to 0$ and $F(k) \to 1$ as $k \to k_0$. Set $k_n = F^{-1}(n)$. Since F^{-1} is decreasing it follows that k_n is well-defined and decreases to 0. Moreover, the only point k with $\lambda(k) = \lambda(nk)$ is $k = k_n$. $\square$

Lemma 4. *Let* $\tilde{\lambda}(k) = \lambda_-^{*1}(k)$.

(i) If $\frac{\sigma}{gh^2} \geq \frac{\gamma^2 h}{6g} + \frac{1}{3} + \frac{\gamma}{6g}\sqrt{\gamma^2 h^2 + 4gh}$ *then the function* $\tilde{\lambda}$ *is strictly decreasing.*

(ii) If $\frac{\sigma}{gh^2} < \frac{\gamma^2 h}{6g} + \frac{1}{3} + \frac{\gamma}{6g}\sqrt{\gamma^2 h^2 + 4gh}$ *then the function* $\tilde{\lambda}$ *has a minimum at* $k = 0$ *and a unique local extremum, namely a local maximum at* $k = \tilde{k}_0 > 0$. *Moreover, there is a strictly decreasing sequence* $(\tilde{k}_n)_{n\geq 2}$ *such that*

$$\tilde{\lambda}(k) = \tilde{\lambda}(nk)$$

for $k > 0$ *if and only if* $k = \tilde{k}_n$.

Proof. Except for the equality in (i) the proof is similar to the one in Lemma 3. Assume now that $\frac{\sigma}{gh^2} = \frac{\gamma^2 h}{6g} + \frac{1}{3} + \frac{\gamma}{6g}\sqrt{\gamma^2 h^2 + 4gh}$. With the notation from Lemma 3 we have that

$$\begin{aligned}\tilde{\lambda}''(0) &= \frac{\gamma}{3}h^3 - \frac{F'' + G''}{2\sqrt{F+G}}\Big|_{\{k=0\}} \\ &= \frac{\gamma}{3}h^3 - \frac{-\frac{\gamma^2}{3}h^4 + 2h\sigma - \frac{2}{3}gh^3}{\sqrt{\gamma^2 h^2 + 4gh}}\end{aligned} \tag{54}$$

Notice that $\frac{\sigma}{gh^2} = \frac{\gamma^2 h}{6g} + \frac{1}{3} + \frac{\gamma}{6g}\sqrt{\gamma^2 h^2 + 4gh}$ is equivalent to $\tilde{\lambda}''(0) = 0$. We claim that

$$\tilde{\lambda}^{(3)}(0) = 0 \text{ and } \tilde{\lambda}^{(4)}(0) < 0. \tag{55}$$

This implies that $\tilde{\lambda}''$ has a local maximum at $k = 0$ and since we are in the case $\tilde{\lambda}''(0) = 0$ it follows further that $\tilde{\lambda}''(k) < 0$ for $k > 0$ and close to 0. Therefore $\tilde{\lambda}'$ is strictly decreasing for $k > 0$. Since $\tilde{\lambda}'(0) = 0$ we have that $\tilde{\lambda}'(k) < 0$ for $k > 0$ close to 0. Hence $\tilde{\lambda}$ is decreasing for $k > 0$ close to 0 and since $\tilde{\lambda}$ does not have local minima and $\lim_{k\to\infty} \tilde{\lambda}(k) = -\infty$ we infer that $\tilde{\lambda}$ is strictly decreasing.
We now proceed with the claim (55).
We have first that

$$\begin{aligned}\tilde{\lambda}^{(3)} &= -\frac{\gamma h}{2}\frac{d^3}{dk^3}\frac{\tanh(hk)}{hk}\\ &- \frac{(F+G)^2(F^{(3)}+G^{(3)})\sqrt{F+G} - [(F+G)(F''+G'') - (F'+G')^2]E}{2(F+G)^3}\end{aligned} \tag{56}$$

where $E = \frac{3}{2}(F'+G')\sqrt{F+G}$. Using that $F'(0) = G'(0) = F^{(3)}(0) = G^{(3)}(0) = 0$ we see that $\tilde{\lambda}^{(3)}(0) = 0$.
From (56) and using that

$$0 = \tilde{\lambda}''(0) = \frac{\gamma}{3}h^3 - \frac{F''+G''}{2\sqrt{F+G}}|_{\{k=0\}}$$

we have

$$\begin{aligned}\tilde{\lambda}^{(4)}(0) &= -\frac{\gamma h}{2}\frac{d^4}{dk^4}\frac{\tanh(hk)}{hk}|_{\{k=0\}}\\ &- \frac{2(F+G)^2(F^{(4)}+G^{(4)})\sqrt{F+G} - 2(F+G)(F''+G'')\frac{3}{2}\sqrt{F+G}(F''+G'')}{4(F+G)^3}|_{\{k=0\}}\\ &= -\frac{24}{15}\gamma h^5 - \frac{2(F+G)^2(F^{(4)}+G^{(4)})\sqrt{F+G} - 3(F+G)\sqrt{F+G}\frac{4}{9}(F+G)\gamma^2 h^6}{4(F+G)^3}|_{\{k=0\}}\\ &= -\frac{24}{15}\gamma h^5 - \frac{(F+G)^2\sqrt{F+G}[2(F^{(4)}+G^{(4)}) - \frac{4}{3}\gamma^2 h^6]}{4(F+G)^3}|_{\{k=0\}}\\ &= -\frac{24}{15}\gamma h^5 - \frac{1}{4\sqrt{F+G}}[2(F^{(4)}+G^{(4)}) - \frac{4}{3}\gamma^2 h^6]|_{\{k=0\}}\\ &= \frac{-96\gamma h^5\sqrt{F+G} - 15[2(F^{(4)}+G^{(4)}) - \frac{4}{3}\gamma^2 h^6]}{60\sqrt{F+G}}|_{\{k=0\}}\end{aligned} \tag{57}$$

Now, the expression at the numerator equals

$$\begin{aligned}
&-96\gamma h^5 \frac{3}{2}\frac{F''(0)+G''(0)}{\gamma h^3} - 15\left[2\left(\frac{34}{15}\gamma^2 h^6 + \frac{48}{15}gh^5 - 8\sigma h^3\right) - \frac{4}{3}\gamma^2 h^6\right]\\
&= -48\cdot 3h^2\left(2h\sigma - \frac{2}{3}gh^3 - \frac{1}{3}\gamma^2 h^4\right) - 68\gamma^2 h^6 - 96gh^5 + 240\sigma h^3 + 20\gamma^2 h^6\\
&= -48\sigma h^3 < 0
\end{aligned} \tag{58}$$

□

The conclusions of Lemmas 3 and 4 and Remark 1 yield the following necessary and sufficient conditions for the one-dimensionality of the kernel $\mathcal{N}(\partial_{(\mu,w)}F(\lambda^*,0))$

Lemma 5. *Let $\lambda^{*\,1}$ be a solution of (47) with $n=1$, i.e.,*

$$\lambda^{*\,1}_{\pm} = -\frac{\gamma\tanh(kh)}{2k} \pm \sqrt{\frac{\gamma^2\tanh^2(kh)}{4k^2} + (k\sigma + \frac{g}{k})\tanh(kh)}.$$

Then the kernel $\mathcal{N}(\partial_{(\mu,w)}F(\lambda^*,0)))$ is one-dimensional if and only if

$$\frac{\sigma}{gh^2} \geq \frac{\gamma^2 h}{6g} + \frac{1}{3} + \frac{\gamma}{6g}\sqrt{\gamma^2h^2+4gh} \tag{59}$$

or

$$\frac{\sigma}{gh^2} < \frac{\gamma^2 h}{6g} + \frac{1}{3} + \frac{\gamma}{6g}\sqrt{\gamma^2h^2+4gh} \text{ and } k \neq k_n \text{ and } k \neq \tilde{k}_n \text{ for all } n \geq 2 \tag{60}$$

Moreover, in these situations, $\mathcal{N}(\partial_{(\mu,w)}F(\lambda^*,0))$ is generated by $(0,w^*) \in X$, where $w^*(x) = \cos(x)$ for all $x \in \mathbb{R}$.

Remark 2. Due to the length of the paper we do not address here the question of double bifurcation which takes place in the situation when the kernel $\mathcal{N}(\partial_{(\mu,w)}F(\lambda^*,0))$ is two-dimensional.

Remark 3. If we set $\gamma = 0$ in (59) and (60) we rediscover the necessary and sufficient condition for the one-dimensionality of the kernel $\mathcal{N}(\partial_{(\mu,w)}F(\lambda^*,0))$ found in the irrotational case in the paper [27].

We now give a sufficient condition for the one dimensionality of the kernel $\mathcal{N}(\partial_{(\mu,w)}F(\lambda^*,0))$. This will prove to be very useful later on when we establish the existence of stagnation points.

Lemma 6. *Let $\lambda^{*\,1}$ be a solution of* (47) *with $n=1$, i.e.,*

$$\lambda_{\pm}^{*\,1} = -\frac{\gamma\tanh(kh)}{2k} \pm \sqrt{\frac{\gamma^2\tanh^2(kh)}{4k^2} + (k\sigma + \frac{g}{k})\tanh(kh)}.$$

Assume that $k^3 \geq 2\frac{\gamma^2}{\sigma}$, $k^2 \geq 4\frac{g}{\sigma}$ and $kh \geq \frac{1}{2}$. Then it follows from (45) *and Lemma 12 that the kernel $\mathcal{N}(\partial_{(\mu,w)}F(\lambda^*,0))$ is one-dimensional being generated by $(0,w^*) \in X$, where $w^*(x) = \cos(x)$ for all $x \in \mathbb{R}$.*

Proof. It suffices to show that if $n>1$ is an integer then the equations

$$\lambda\gamma + \lambda^2 kn\coth(nkh) - \sigma k^2 n^2 - g = 0,$$

and

$$\lambda\gamma + \lambda^2 k\coth(kh) - \sigma k^2 - g = 0,$$

do not have common solutions. Let

$$\lambda_{\pm}^{*\,n} = -\frac{\gamma\tanh(nkh)}{2kn} \pm \sqrt{\frac{\gamma^2\tanh^2(nkh)}{4k^2n^2} + (nk\sigma + \frac{g}{kn})\tanh(nkh)},$$

denote the two solutions of equation (47). Since

$$\lambda_{+}^{*\,n} > 0, \ \ \lambda_{+}^{*\,n} < 0,$$

for all $n \geq 1$ it suffices to show that $\lambda_{+}^{*\,n} \neq \lambda_{+}^{*\,1}$ and $\lambda_{-}^{*\,n} \neq \lambda_{-}^{*\,1}$ for all $n>1$. We consider only the case $\gamma>0$, since for $\gamma<0$ we can proceed in a similar way.

We now assume ab absurdum that $\lambda_{-}^{*\,n} = \lambda_{-}^{*\,1}$ for some $n>1$ which leads to

$$\frac{\tanh(nkh)}{n} - \tanh(kh) = \sqrt{\tanh^2(kh) + \frac{4\sigma k^3}{\gamma^2}\tanh(kh) + \frac{4kg}{\gamma^2}\tanh(kh)} - \sqrt{\frac{\tanh^2(nkh)}{n^2} + \frac{4\sigma k^3}{\gamma^2}n\tanh(nkh) + \frac{4kg}{\gamma^2}\frac{1}{n}\tanh(nkh)} \tag{61}$$

To ease the notation we set $f(n) = \frac{\tanh(nkh)}{n}$ and $g(n) = n\tanh(nkh)$, $c = \frac{4\sigma k^3}{\gamma^2}$ and $d = \frac{4kg}{\gamma^2}$. Then (61) is equivalent to

$$f(n) - f(1) = \sqrt{f^2(1) + cg(1) + df(1)} - \sqrt{f^2(n) + cg(n) + df(n)},$$

which by squaring leads to

$$-2f(n)f(1) = c(g(n)+g(1))+d(f(n)+f(1))-2\sqrt{\left[f^2(n)+cg(n)+df(n)\right]\left[f^2(1)+cg(1)+df(1)\right]}.$$

By rearranging and squaring we arrive at

$$\begin{aligned}
&c^2\left[g(n)+g(1)\right]^2 + d^2\left[f(n)+f(1)\right]^2 + 2cd\left[g(n)+g(1)\right]\left[f(n)+f(1)\right] + 4f^2(n)f^2(1)\\
&+ 4f(n)f(1)\left[c(g(n)+g(1)) + d(f(n)+f(1))\right]\\
&= 4[f^2(n)f^2(1) + cf^2(n)g(1) + df^2(n)f(1) + cf^2(1)g(n) + c^2g(n)g(1) + cdg(n)f(1) + df(n)f^2(1)\\
&+ cdf(n)g(1) + d^2f(n)f(1)]
\end{aligned} \tag{62}$$

which after grouping and canceling terms is equivalent to

$$\begin{aligned}
&c^2\left[g(n)-g(1)\right]^2 + d^2\left[f(n)-f(1)\right]^2 + 2cd\left[g(n)-g(1)\right]\left[f(n)-f(1)\right]\\
&+ 4cf(n)\left[f(1)g(n)-f(n)g(1)\right] + 4cf(1)\left[f(n)g(1)-f(1)g(n)\right] = 0
\end{aligned} \tag{63}$$

which is simply written as

$$\begin{aligned}
c^2\left[g(n)-g(1)\right]^2 &+ d^2\left[f(n)-f(1)\right]^2 + 2cd\left[g(n)-g(1)\right]\left[f(n)-f(1)\right]\\
&+ 4c\left[f(n)-f(1)\right]\left[f(1)g(n)-f(n)g(1)\right] = 0.
\end{aligned} \tag{64}$$

We rearrange (64) as

$$\begin{aligned}
E(n) :=\ & \frac{c^2}{2}\left[g(n)-g(1)\right]^2 + 2cd\left[g(n)-g(1)\right]\left[f(n)-f(1)\right]\\
&+ \frac{c^2}{2}\left[g(n)-g(1)\right]^2 + 4c\left[f(n)-f(1)\right]\left[f(1)g(n)-f(n)g(1)\right]\\
&+ d^2\left[f(n)-f(1)\right]^2 = 0.
\end{aligned} \tag{65}$$

We aim to prove that $E(n) > 0$ for all $n \in \mathbb{N}$, $n \geq 2$. Since $k^3 \geq 2\frac{\gamma^2}{\sigma}$ it follows that $\frac{c^2}{2} \geq 4c$ and therefore

$$\begin{aligned}
&\frac{c^2}{2}\left[g(n)-g(1)\right]^2 + 4c\left[f(n)-f(1)\right]\left[f(1)g(n)-f(n)g(1)\right]\\
&\geq 4c\left\{\left[g(n)-g(1)\right]^2 + \left[f(n)-f(1)\right]\left[f(1)g(n)-f(n)g(1)\right]\right\}\\
&= 4c\left\{\left[n\tanh(nkh)-\tanh(kh)\right]^2 + \left[\tanh(nkh)-n\tanh(kh)\right](1-\frac{1}{n^2})\tanh(nkh)\tanh(kh))\right\}\\
&> 4c(n\tanh(nkh)-\tanh(kh)+\tanh(nkh)-n\tanh(kh))\\
&= 4c\left(n\left[\tanh(nkh)-\tanh(kh)\right]+\left[\tanh(nkh)-\tanh(kh)\right]\right) > 0,\ \text{ for all } n \in \mathbb{N}, n \geq 2,
\end{aligned} \tag{66}$$

where we have used the following inequalities proved in the Appendix of [30]

$$n\tanh(nkh)-\tanh(kh)\geq 1 \text{ for all } n\geq 2,\ kh\geq \frac{1}{2},$$

$$\tanh(nkh)-n\tanh(kh)<0 \text{ for all } n\geq 2,\ k>0, h>0,$$

$$(1-\frac{1}{n^2})\tanh(nkh)\tanh(kh)<1 \text{ for all } n\geq 2,$$

and the fact that the function $x\to\tanh(xkh)$ is strictly increasing for $kh>0$.
Now from $k^2\geq 4\frac{g}{\sigma}$ it follows that $\frac{c^2}{2}\geq 2cd$. Hence

$$\begin{aligned}
\frac{c^2}{2}\left[g(n)-g(1)\right]^2 &+2cd\left[g(n)-g(1)\right]\left[f(n)-f(1)\right]\\
&\geq 2cd\left[g(n)-g(1)\right]\left[g(n)-g(1)+f(n)-f(1)\right]\\
&=\left[g(n)-g(1)\right]\left[n\tanh(nkh)-\tanh(kh)+\frac{1}{n}\tanh(nkh)-\tanh(kh)\right]\\
&=\left[g(n)-g(1)\right]\left[n\tanh(nkh)-2\tanh(kh)+\frac{1}{n}\tanh(nkh)\right]>0
\end{aligned}\tag{67}$$

for all $n\in\mathbb{N}, n\geq 2$. From (66) and (67) we have that $E(n)>0$ for all $n\in\mathbb{N}, n\geq 2$ and therefore (61) can not hold. This proves that $\lambda_-^{*n}\neq\lambda_-^{*1}$ for all $n\in\mathbb{N}, n\geq 2$. It is easy to see that the same argument as above can be applied to prove that $\lambda_+^{*n}\neq\lambda_+^{*1}$ for all $n\in\mathbb{N}, n\geq 2$. □

We now proceed with checking the remaining conditions from the Crandall-Rabinowitz Theorem.
It is easy to see that $\mathcal{R}(\partial_{(\mu,w)}F(\lambda^*,0))$ is the closed subspace of Y consisting of all functions $f\in Y$ which satisfy

$$\int_{-\pi}^{\pi} f(x)\cos(x)dx=0,$$

and therefore $Y/\mathcal{R}(\partial_{(\mu,w)}F(\lambda^*,0))$ is the one dimensional subspace of Y generated by the function $w^*(x)=\cos(x)$.
Using (44) we compute

$$\begin{aligned}
&\partial^2_{\lambda,(\mu,w)}F(\lambda^*,(0,0))(1,(0,w^*))\\
&=\lim_{t\to 0}\frac{\partial_{(\mu,w)}F(\lambda^*+t,(0,0))(0,w^*)-\partial_{(\mu,w)}F(\lambda^*,(0,0))(0,w^*)}{t}
\end{aligned}\tag{68}$$

Since

$$\partial_{(\mu,w)}F(\lambda^*+t,(0,0))(0,w^*) = -\frac{2}{k^2}\left[(\lambda^*+t)\gamma w^* + (\lambda^*+t)^2 k\mathcal{C}_{kh}(w^{*\prime}) + \sigma k^2 w^{*\prime\prime} - gw^*\right] \tag{69}$$

and

$$\partial_{(\mu,w)}F(\lambda^*,(0,0))(0,w^*) = -\frac{2}{k^2}\left[\lambda^*\gamma w^* + \lambda^{*2}k\mathcal{C}_{kh}(w^{*\prime}) + \sigma k^2 w^{*\prime\prime} - gw^*\right] \tag{70}$$

we obtain that

$$\partial^2_{\lambda,(\mu,w)}F(\lambda^*,(0,0))(1,(0,w^*)) = -\frac{2}{k^2}\gamma w^* - \frac{4}{k^2}\lambda^* k\mathcal{C}_{kh}(w^{*\prime}) \tag{71}$$

For $w^* = \cos(x)$ we have from (92) that $\mathcal{C}_{kh}(w^*) = \coth(kh)\sin(x)$ and $\mathcal{C}_{kh}(w^{*\prime}) = (\mathcal{C}_{kh}(w^*))' = \coth(kh)\cos(x) = \coth(kh)w^*$, and therefore

$$\begin{aligned}\partial^2_{\lambda,(\mu,w)}F(\lambda^*,(0,0))(1,(0,w^*)) =& \frac{2}{k^2}(-\gamma - 2\lambda^* k\coth(kh))w^* \\ &\notin \mathcal{R}(\partial_{(\mu,w)}F(\lambda^*,(0,0))),\end{aligned} \tag{72}$$

since by (47) we have

$$-\gamma - 2\lambda^* k\coth(kh) = -\lambda^*\left(k\coth(kh) + \frac{\sigma k^2 + g}{(\lambda^*)^2}\right) \neq 0.$$

From the local bifurcation theorem we obtain the bifurcation values

$$\lambda_\pm = -\frac{\gamma\tanh(kh)}{2k} \pm \sqrt{\frac{\gamma^2\tanh^2(kh)}{4k^2} + \frac{k^2\sigma + g}{k}\tanh(kh)}. \tag{73}$$

Please note that if $\sigma = 0$ i.e., in the case of gravity waves the formula for the bifurcation values (73) is the same as formula 5.13 in [11], while for $g = 0$ i.e., for pure capillary waves we regain the corresponding formula from [30].
From (41) we obtain the corresponding values for the relative mass flux m as

$$m_\pm = \frac{\gamma h^2}{2} - \frac{\gamma h\tanh(kh)}{2k} \pm h\sqrt{\frac{\gamma^2\tanh^2(kh)}{4k^2} + \frac{k^2\sigma + g}{k}\tanh(kh)} \tag{74}$$

In the case of gravity waves ($g = 0$) we obtain from (74) the formula 5.14 from [11] and in the case of pure capillary waves we regain from (74) the corresponding formula from [30]. We can now formulate the results concerning the existence of small amplitude periodic capillary-gravity water waves of constant vorticity.

Theorem 7. *For any $h > 0$, $k > 0$, $\sigma > 0$, $\gamma \in \mathbb{R}$ satisfying condition* (59) *or* (60) *and $m \in \mathbb{R}$ there exists laminar flows with a flat free surface in water of depth h, of constant vorticity γ and relative mass flux m. The laminar flows of flux $m_\pm$ are exactely those with horizontal speeds at the flat free surface equal to $\lambda_\pm$ given by* (73)*. The values of $m_\pm$ of the flux given by* (74) *trigger the appearance of periodic steady waves of small amplitude, with period $\frac{2\pi}{k}$ and conformal mean depth h, which have a smooth profile with one crest and one trough per period, monotone between consecutive crests and troughs and symmetric about any crest line.*

Proof. See the proof of Theorem 8 below. □

Theorem 8. *For any $h > 0$, $k > 0$, $\gamma \in \mathbb{R}$ and $m \in \mathbb{R}$ satisfying $k^3 \geq 2\frac{\gamma^2}{\sigma}$, $k^2 \geq 4\frac{g}{\sigma}$ and $kh \geq \frac{1}{2}$ there exists laminar flows with a flat free surface in water of depth h, of constant vorticity γ and relative mass flux m. The laminar flows of flux $m_\pm$ are exactely those with horizontal speeds at the flat free surface equal to $\lambda_\pm$ given by* (73)*. The values of $m_\pm$ of the flux given by* (74) *trigger the appearance of periodic steady waves of small amplitude, with period $\frac{2\pi}{k}$ and conformal mean depth h, which have a smooth profile with one crest and one trough per period, monotone between consecutive crests and troughs and symmetric about any crest line.*

Proof. Using the argument from the proof of Theorem 3.2 in [30] we see that $w = 0$ gives rise to laminar flows in the fluid domain bounded below by the rigid bed $\mathcal{B}$ and above by the flat free surface $Y = h$. These laminar flow solutions are given by

$$\psi(X,Y) = -\frac{\gamma}{2}Y^2 + \left(\frac{m}{h} + \frac{\gamma h}{2}\right)Y - m,\ X \in \mathbb{R},\ 0 \leq Y \leq h,$$

while the velocity field is

$$(\psi_Y, -\psi_X) = \left(-\gamma Y + \frac{m}{h} + \frac{\gamma h}{2}, 0\right),\ X \in \mathbb{R},\ 0 \leq Y \leq h. \tag{75}$$

Note that using (41) we can rewrite (75) as

$$(\psi_Y, -\psi_X) = (\lambda_\pm + \gamma(h - Y), 0),\ X \in \mathbb{R},\ 0 \leq Y \leq h, \tag{76}$$

where $\lambda_\pm$ is given by (73). Observe that $\psi_{Y|Y=h} = -\gamma h + \frac{m}{h} + \frac{\gamma h}{2} = \frac{m}{h} - \frac{\gamma h}{2}$ which shows that for laminar flows the horizontal velocity at the free surface coincides with λ given in (41). The formula (73), of the speed $\lambda_\pm$ at the free surface in terms of the depth h, period $2\pi/k$ and vorticity γ, is called the *dispersion relation*.

Concerning the existence of waves of small amplitude with the properties mentioned in the statement of the theorem we apply the Crandall-Rabinowitz theorem which asserts the existence of the local bifurcation curve

$$\{(\lambda(s), (0 + o(s), s\cos(x) + o(s))) : |s| < \varepsilon\} \subset \mathbb{R} \times X$$

consisting of solutions of (42) with $\lambda_\pm$ given by (73).

Choosing ε sufficiently small and using Lemma 12 we can ensure that

$$w(x) > -h \text{ for all } x \in \mathbb{R},$$

and

$$\frac{1}{k} + \mathcal{C}_{kh}(w')(x) > 0 \text{ for all } x \in \mathbb{R}. \tag{77}$$

The inequality (77) implies that the corresponding non-flat free surface $\mathcal{S}$ given by (22) with $v = w + h$ is the graph of a smooth function, symmetric with respect to the points obtained for the values $x = n\pi, n \in \mathbb{Z}$. From

$$w(x; s) = s\cos(x) + o(s) \text{ in } C_{2\pi}^{p+1,\alpha},$$

we have that

$$sw'(x; s) < 0 \text{ for all } x \in (0, \pi),\ 0 < |s| < \varepsilon,$$

for $\varepsilon > 0$ sufficiently small and $p \geq 1$. Using the eveness of $x \to w(x; s)$ we conclude the proof of the assertion about the free surface $\mathcal{S}$, i.e., $\mathcal{S}$ has one crest and one trough per minimal period and is monotone between consecutive crests and troughs. $\square$

Remark 4. Please note that our formula for λ_--the horizontal velocity at the free surface for laminar flows- coincides with the one given in [42]. However the formula in [42] was obtained under stronger assumptions namely that the flow

does not contain stagnation points and that the free surface is always a graph. While for the small amplitude waves whose existence we prove, at least close to the bifurcation point, the free surface is always a graph, we will see that stagnation is possible.
In the case of periodic traveling gravity waves in a flow with constant vorticity the formula giving λ was rigourously obtained by Constantin and Varvaruca in [11] although by formal arguments appeared already in the papers [1, 37].

Remark 5. Concerning the hypothesis that the free surface is a graph (discarded in the present paper and assumed in [42] we want to note that there are examples of capillary irrotational waves whose free surface is not a graph. In water of finite depth the explicit solutions are known as Kinnersley's waves [29], and in the case of infinite depth as Crapper's waves [16].

We are now concerned with the question as whether such flows contain stagnation points. Note first that it follows from (76) that the necessary and sufficient condition for the existence of stagnation points is

$$\lambda_{\pm} + \gamma(h - Y) = 0$$

for at least one Y in $[0, h]$. The latter condition is satisfied if and only if

$$\lambda_{\pm}(\lambda_{\pm} + \gamma h) \leq 0. \tag{78}$$

If $\gamma > 0$ it follows that $\lambda_+(\lambda_+ + \gamma h) > 0$ so from (78) we have that the flow corresponding to λ_+ does not contain stagnation points. The flow corresponding to λ_- contains stagnation points if and only if $\lambda_- + \gamma h \geq 0$, which is equivalent to

$$\tanh(kh) \leq \frac{\gamma^2 h^2 k}{\gamma^2 h + k^2 \sigma + g}. \tag{79}$$

The case $\gamma < 0$ can be treated similarly. Namely, if $\gamma < 0$ we see that $\lambda_-(\lambda_- + \gamma h) > 0$ and therefore from (78) we have that the flow corresponding to λ_- does not contain stagnation points. The flow corresponding to λ_+ contains stagnation points if and only if $\lambda_+ + \gamma h \leq 0$, which also is equivalent to (79).

Note that in the case of gravity water waves ($\sigma = 0$) the condition (79) becomes $\tanh(kh) \leq \frac{\gamma^2 h^2 k}{\gamma^2 h + g}$, which is the same as the condition for existence of stagnation points found in [11]. Also if we set $g = 0$ in (79) we regain the condition of existence of stagnation points in the case of capillary water waves which we found in [30].

Remark 6. Notice that (79) does not hold true for $k \to \infty$ since the limit of the left-hand side is 1 as $k \to \infty$ while the limit of the right hand side is 0 as $k \to \infty$. It follows that a flow with $L \to 0$ does not have stagnation points since in that case $k = \frac{2\pi}{L} \to \infty$ and we saw that this violates (79).

Lemma 9. *If the vorticity γ is such that*

$$\gamma^2 \geq \frac{4\sigma h + \sqrt{16\sigma^2 h^2 + 16h^4 \sigma g}}{2h^4} \tag{80}$$

there are values $k_1 \leq k_2$ with the property that (79) *holds true whenever $k \in [k_1, k_2]$.*

Proof. Note that

$$\gamma_{\pm} = \pm\sqrt{\frac{4\sigma h + \sqrt{16\sigma^2 h^2 + 16h^4 \sigma g}}{2h^4}}$$

are the only real solutions of the equation

$$h^4\gamma^4 - 4\sigma h \gamma^2 - 4\sigma g = 0,$$

which besides $\gamma_{\pm}$ has another two complex conjugate solutions. It follows that

$$h^4\gamma^4 - 4\sigma h \gamma^2 - 4\sigma g \geq 0, \tag{81}$$

for all $\gamma \in (-\infty, \gamma_-] \cup [\gamma_+, \infty)$. But (81) ensures that the equation

$$\sigma k^2 - \gamma^2 h^2 k + \gamma^2 h + g = 0$$

has two (not necessarily distinct) solutions

$$k_1 := \frac{\gamma^2 h^2 - \sqrt{\gamma^4 h^4 - 4\sigma\gamma^2 h - 4\sigma g}}{2\sigma}$$

$$k_2 := \frac{\gamma^2 h^2 + \sqrt{\gamma^4 h^4 - 4\sigma\gamma^2 h - 4\sigma g}}{2\sigma}$$

We then have that

$$\sigma k^2 - \gamma^2 h^2 k + \gamma^2 h + g \leq 0,$$

for all $k \in [k_1, k_2]$ which implies that

$$\frac{\gamma^2 h^2 k}{\gamma^2 h + k^2\sigma + g} \geq 1.$$

Therefore (79) holds true for all $k \in [k_1, k_2]$. $\square$

Remark 7. Observe that if we set $g = 0$ in (80) we discover the sufficient condition for the existence of stagnation points for capillary water waves which we found already in [30].

Remark 8. Let γ and h be such that

$$\gamma^2 \geq \frac{4\sigma h + \sqrt{16\sigma^2 h^2 + 64 h^4 \sigma g}}{2h^4}. \tag{82}$$

Let k_1 and k_2 be the values from Lemma 9. If $k \in \left[\frac{k_1+k_2}{2}, k_2\right]$ then the sufficient conditions for the existence of laminar flows from Theorem 8 are satisfied. In addition, these flows posses stagnation points as the proof of Lemma 9 shows.

Proof. We have to show that for all such k we have that $k^3 \geq \frac{\gamma^2}{\sigma}$, $k^2 \geq 4\frac{g}{\sigma}$ and $kh \geq \frac{1}{2}$. We only need to show that the last two inequalities hold true for $k = \frac{k_1+k_2}{2} = \frac{\gamma^2 h^2}{2\sigma}$ and then the rest follows. Note first that (82) implies that

$$\gamma^2 \geq 4\frac{\sigma}{h^3}$$

and

$$\gamma^2 \geq \frac{\sqrt{64 h^4 \sigma g}}{2h^4} = 4\frac{\sqrt{\sigma g}}{h^2}.$$

If $k = \frac{k_1+k_2}{2} = \frac{\gamma^2 h^2}{2\sigma}$ then we have

$$k^3 = \left(\frac{\gamma^2 h^2}{2\sigma}\right)^3 = \frac{\gamma^6 h^6}{8\sigma^3} = \frac{\gamma^2}{\sigma} \cdot \frac{\gamma^4 h^6}{8\sigma^2} \geq \frac{\gamma^2}{\sigma} \cdot \frac{16\sigma^2}{8\sigma^2} = 2\frac{\gamma^2}{\sigma},$$

$$k^2 = \left(\frac{\gamma^2 h^2}{2\sigma}\right)^2 = \frac{\gamma^4 h^4}{4\sigma^2} \geq \frac{16\sigma g}{4\sigma^2} = 4\frac{g}{\sigma},$$

$$kh = \frac{\gamma^2 h^3}{2\sigma} \geq 2.$$

$\square$

4. Regularity

We are now ready to prove a regularity result for the free surface. We would like to mention that for flows without stagnation points recent results proving *a priori* regularity for the free-surface and streamlines of capillary and capillary-gravity waves with vorticity were obtained in [23], [24]-(for a recent survey of regularity results for flows with vorticity see [18]).

Theorem 10. *Let $h > 0$ and $v \in C^{2,\alpha}_{2\pi}$ be a solution of* (21). *Then $v \in C^{\infty}_{2\pi}$.*

Proof. From (17) and (19) we find that

$$\mathcal{G}_{kh}(v)v'' - \mathcal{G}_{kh}(v')v' = \left(\frac{1}{k} + \mathcal{C}_{kh}(v')\right) v'' - v'\mathcal{C}_{kh}(v'') \tag{83}$$

From (19) and the second equation of (21) we have that

$$\mathcal{G}_{kh}(\frac{v^2}{2}) - v\mathcal{G}_{kh}(v) = \frac{[v^2]}{2kh} + \mathcal{C}_{kh}(vv') - \frac{v}{k} - v\mathcal{C}_{kh}(v') = \frac{[v^2]}{2kh} - \frac{v}{k} - \mathcal{Q}_{kh}(v), \tag{84}$$

where $\mathcal{Q}_{kh}(v) = v\mathcal{C}_{kh}(v') - \mathcal{C}_{kh}(vv')$. From Lemma 14 we have that $\mathcal{Q}_{kh}(v) \in C^{2,\alpha/3}_{2\pi}$ since $v \in C^{2,\alpha}_{2\pi}$. The latter fact together with the formulas (83), (84), (21) and using $v'^{\,2} + \mathcal{G}_{kh}(v)^2 \in C^{1,\alpha/3}_{2\pi}$ yield

$$\left(\frac{1}{k} + \mathcal{C}_{kh}(v')\right) v'' - v'\mathcal{C}_{kh}(v'') \in C^{1,\alpha/3}_{2\pi} \tag{85}$$

Now from Lemma 14 with $f = -v' \in C^{1,\alpha}_{2\pi}$ and $g = \mathcal{C}_{kh}(v'') \in C^{0,\alpha}_{2\pi}$ it follows that

$$-v'\mathcal{C}_{kh}(\mathcal{C}_{kh}(v'')) - \mathcal{C}_{kh}(-v'\mathcal{C}_{kh}(v'')) \in C^{1,\alpha/3}_{2\pi}, \tag{86}$$

and taking into account that $\mathcal{C}_{kh}^{-1} = -\mathcal{C}_{kh}$ (see Lemma 12) we obtain from above that

$$v'v'' - \mathcal{C}_{kh}(-v'\mathcal{C}_{kh}(v'')) \in C^{1,\alpha/3}_{2\pi}. \tag{87}$$

By applying $\mathcal{C}_{kh}$ to (85) and using (87) we get

$$\frac{1}{k}\mathcal{C}_{kh}(v'') + C_{kh}(v''\mathcal{C}_{kh}(v')) + v'v'' \in C^{1,\alpha/3}_{2\pi} \tag{88}$$

Setting $f = \mathcal{C}_{kh}(v') \in C^{1,\alpha}_{2\pi}$ and $g = v'' \in C^{0,\alpha}_{2\pi}$ we get by applying Lemma 14

$$\mathcal{C}_{kh}(v')\mathcal{C}_{kh}(v'') - \mathcal{C}_{kh}(v''\mathcal{C}_{kh}(v')) \in C^{1,\alpha/3}_{2\pi} \tag{89}$$

Adding up (88) and (89) yields

$$\left(\frac{1}{k}+\mathcal{C}_{kh}(v')\right)\mathcal{C}_{kh}(v'')+v'v''\in C^{1,\alpha/3}_{2\pi} \tag{90}$$

We now multiply (85) by $\frac{1}{k}+\mathcal{C}_{kh}(v')\in C^{1,\alpha}_{2\pi}$ and (90) by $v'\in C^{1,\alpha}_{2\pi}$ and by adding up the resulting expressions we obtain

$$\left(\left(\frac{1}{k}+\mathcal{C}_{kh}(v')\right)^2+v'^2\right)v''\in C^{1,\alpha/3}_{2\pi} \tag{91}$$

Since the expression in the bracket on the left-hand side of (91) is strictly positive and belongs to $C^{1,\alpha}_{2\pi}$ we obtain that $v''\in C^{1,\alpha/3}_{2\pi}$. Therefore $v\in C^{3,\alpha/3}_{2\pi}$. An iteration of this method shows that $v\in C^{\infty}_{2\pi}$. □

5. Appendix

This section contains a more precise description of the operator $\mathcal{C}_d$ as was obtained in [11]. We only formulate the results.

Denote by $L^2_{2\pi}$ the space of 2π-periodic locally square integrable functions of one real variable. By $L^2_{2\pi,\mathrm{o}}$ we denote the subspace of $L^2_{2\pi}$ whose elements have zero mean over one period.

Lemma 11. *If*

$$w=\sum_{n=1}^{\infty}a_n\cos(nx)+\sum_{n=1}^{\infty}b_n\sin(nx),$$

is the Fourier series expansion of $w\in L^2_{2\pi,o}$ then

$$\mathcal{C}_d(w)=\sum_{n=1}^{\infty}a_n\coth(nd)\sin(nx)-\sum_{n=1}^{\infty}b_n\coth(nd)\cos(nx) \tag{92}$$

Lemma 12. *For any $d>0$, $p\geq 0$ integer and $\alpha\in(0,1)$, $\mathcal{C}_d: C^{p,\alpha}_{2\pi,o}\to C^{p,\alpha}_{2\pi,o}$ is a bounded linear operator. Moreover, $\mathcal{C}_d^{-1}=-\mathcal{C}_d: C^{p,\alpha}_{2\pi,o}\to C^{p,\alpha}_{2\pi,o}$ is also a bounded linear operator.*

Lemma 13. *Let $w\in C^{p,\alpha}_{2\pi}$ with $p\geq 1$ an integer and $\alpha\in(0,1)$. Let $\mathcal{Q}_d$ denote the mapping*

$$w\to\mathcal{Q}_d(w)=w\mathcal{C}_d(w')-\mathcal{C}_d(ww'),$$

We then have that $\mathcal{Q}_d(w)\in C^{p,\delta}_{2\pi}$ for any $\delta\in(0,\alpha)$.

Lemma 14. *Let $p \geq 1$ be an integer, $\alpha \in (0,1)$ and $d > 0$. If $f \in C^{p,\alpha}_{2\pi}$ and $g \in C^{p-1,\alpha}_{2\pi}$ then*

$$f\mathcal{C}_d(g) - \mathcal{C}_d(fg) \in C^{p,\delta}_{2\pi} \text{ for all } \delta \in (0,\alpha).$$

Proof. The proof follows the line of the proofs of Lemma 3.2 and of Lemma B1 from the paper [11]. □

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