Local bifurcation for steady periodic capillary water waves with constant vorticity

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Abstract. We study periodic capillary waves at the free surface of water in a flow with constant vorticity over a flat bed. Using bifurcation theory the local existence of waves of small amplitude is proved even in the presence of stagnation points in the flow. We also derive the dispersion relation.

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1. Introduction

It is by now well known that nonuniform currents give rise to water flows with vorticity [20, 21, 22]. There is an extensive research literature in the area of water waves with vorticity, see [1, 2, 3, 4, 6] for existence results, [13, 24] for matters of uniqueness and [9, 14, 19] for regularity results. We shall be interested in rotational capillary water waves, i.e., we consider the effects of surface tension in the presence of vorticity, neglecting gravity. This approximation is valid for short wave lengths (see the discussion in [18]). The existence of steady periodic capillary water waves with arbitrary vorticity distributions was proved in [25] under the assumption that there are no stagnation points and that the free surface of the fluid domain is a graph. In our work we will allow stagnation points and overhanging surfaces. We will base our approach on a method developed in [8] for gravity waves.

We now present the free-boundary value problem of steady periodic traveling capillary water waves with constant vorticity γ in a flow of finite depth. We consider two- dimensional waves propagating over water with a flat bed. The X variable will represent the direction of propagation, Y will be the height variable and $((V_1(X,Y,t), V_2(X,Y,t))$ denotes the velocity field. The water

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domain Ω in the XY-plane is bounded below by the impermeable flat bed

$$\mathcal{B} = \{(X, 0); X \in \mathbb{R}\},$$

and above by an a priori unknown curve

$$\mathcal{S}(t) = \{(u(t,s), v(t,s)); s \in \mathbb{R}\}, t \geq 0 \tag{1.1}$$

with

$$(u_s(t,s))^2 + (v_s(t,s))^2 > 0 \text{ for all } s \in \mathbb{R}, t \geq 0 \tag{1.2}$$

and

$$u(t, s+L) = u(t,s) + L, \ v(t, s+L) = v(t,s) \text{ for all } s \in \mathbb{R}, t \geq 0, \tag{1.3}$$

representing the free surface of the water (not necessary the graph of a function), which is L-periodic in the horizontal direction. The equations of motion are the equation of mass conservation

$$V_{1X} + V_{2Y} = 0 \tag{1.4}$$

and Euler's equation

$$\begin{cases} V_{1t} + V_1 V_{1X} + V_2 V_{1Y} = -P_X \\ V_{2t} + V_1 V_{2X} + V_2 V_{2Y} = -P_Y \end{cases} \tag{1.5}$$

where $P(X, Y, t)$ denotes pressure. The boundary conditions are of two types: dynamic and kinematic boundary conditions. The dynamic boundary condition expresses the stresses that the atmosphere exerts on the fluid surface and takes therefore the form

$$P(u(t,s), v(t,s), t) = P_0 - \sigma \frac{u_s(t,s)v_{ss}(t,s) - u_{ss}(t,s)v_s(t,s)}{((u_s(t,s))^2 + (v_s(t,s))^2)^{3/2}} \text{ on } \mathcal{S}, \tag{1.6}$$

P_0 being the constant atmospheric pressure, $\sigma > 0$ the coefficient of surface tension which is a force per unit length due to a pressure difference across a curved surface, cf. [16], [1] and $\frac{u_s(t,s)v_{ss}(t,s) - u_{ss}(t,s)v_s(t,s)}{((u_s(t,s))^2 + (v_s(t,s))^2)^{3/2}}$ representing the mean curvature of $\mathcal{S}$. The kinematic boundary conditions require that the free surface and the bed always consist of the same fluid particles. If $S_0(X, Y, t) = 0$ is the implicit equation of the free surface, the kinematic boundary condition can be expressed as

$$S_{0t} + S_{0X} V_1 + S_{0Y} V_2 = 0 \text{ on } \mathcal{S}, \tag{1.7}$$

cf. [10], while the kinematic boundary condition on the bed is

$$V_2 = 0 \text{ on } \mathcal{B}. \tag{1.8}$$

The assumption of steady periodic traveling waves at speed $c > 0$ means that we have a space-time dependence of the form $X - ct$ for the free surface, the pressure, and for the velocity field. Then after the change of variables

[1]Surface tension is perpendicular to any line drawn in the surface having the same magnitude for all directions of the line and the same value at all points on the surface, cf. [10].

$x = X - ct, y = Y$, the equation of the free surface becomes $S(x,y) = 0$ for some S, and (1.5) – (1.8) are transformed to the stationary problem

$$\begin{cases} (V_1(x,y) - c)V_{1x}(x,y) + V_2(x,y)V_{1y}(x,y) = -P_x(x,y) \\ (V_1(x,y) - c)V_{2x}(x,y) + V_2(x,y)V_{2y}(x,y) = -P_y(x,y) \end{cases} \text{in } \Omega \tag{1.9}$$

and

$$\begin{cases} S_x(V_1 - c) + S_y V_2 = 0 & \text{on} \quad \mathcal{S} \\ P = P_0 - \sigma \frac{u_s(t,s)v_{ss}(t,s) - u_{ss}(t,s)v_s(t,s)}{((u_s(t,s))^2 + (v_s(t,s))^2)^{3/2}} & \text{on} \quad \mathcal{S}, \\ V_2 = 0 & \text{on} \quad \mathcal{B}. \end{cases} \tag{1.10}$$

Note that if $S(x,y) = y - \eta(x)$ the first equation becomes $V_2 = \eta_x(V_1 - c)$ The equation of mass conservation (1.4) permits us to introduce the stream function ψ which satisfies:

$$\psi_x = -V_2 \;\text{ and }\; \psi_y = V_1 - c \tag{1.11}$$

being defined through the line integral

$$\psi(x,y) = -m + \int_{(0,0)}^{(x,y)} (-V_2(x,y))dx + (V_1(x,y) - c)dy,$$

for some constant m. The equation of mass conservation $V_{1x} + V_{2y} = 0$ ensures the path-independence of the above line integral provided the path is in the simply connected domain Ω, where Ω is the set of all $(x,y) \in \mathbb{R}^2$ above the line $y = 0$ bounded above by the curve $\{(x,y) \in \mathbb{R}^2 : S(x,y) = 0\}$.

Remark 1.1. 1. In the integral expression of ψ, since $\psi_x = 0$ on $y = 0$ we can start from any point $(x,0)$ instead of $(0,0)$.
2. When $\mathcal{S}$ is the graph of the form $y = \eta(x)$, then we recover the standard formula [3]

$$\psi(x,y) = -m + \int_0^y (V_1(x,z) - c)dz.$$

The function ψ is constant on $\mathcal{S}$ since $\frac{d}{dt}(\psi(X(t) - ct, Y(t))) = \psi_x(X(t) - ct, Y(t))(V_1 - c) + \psi_y(X(t) - ct, Y(t))V_2 = -V_2(V_1 - c) + (V_1 - c)V_2 = 0$ if $t \to (X(t), Y(t))$ is a water particle trajectory (confined to the free surface). We then set $\psi = 0$ on $\mathcal{S}$. Also, since $\psi_x(x,0) = -V_2(x,0) = 0$ from (1.8), we see that ψ is constant on the bed too. If $(u(t,s_0) - ct, v(t,s_0))$ is the wave trough, then

$$m(t) = \int_0^{v(t,s_0)} [V_1(u(t,s_0) - ct, y) - c]\, dy, \tag{1.12}$$

an expression which is called the relative mass flux. We then have

$$
\begin{aligned}
& m_t = [V_1\,(u(t,s_0) - ct, v(t,s_0)) - c]\, v_t(t,s_0) \\
& + \int_0^{v(t,s_0)} V_{1x}\,(u(t,s_0) - ct, y)\,(u_t(t,s_0) - c)\,dy \\
& = [V_1\,(u(t,s_0) - ct, v(t,s_0)) - c]\, V_2(u(t,s_0) - ct, v(t,s_0)) \\
& - (u_t(t,s_0) - c) \int_0^{v(t,s_0)} V_{2y}\,(u(t,s_0) - ct, y)\,dy \\
& = [V_1\,(u(t,s_0) - ct, v(t,s_0)) - c]\, V_2(u(t,s_0) - ct, v(t,s_0)) \\
& - [V_1\,(u(t,s_0) - ct, v(t,s_0)) - c]\,[V_2(u(t,s_0) - ct, v(t,s_0)) - V_2(u(t,s_0) - ct, 0)] \\
& = 0,
\end{aligned}
$$

where the last equality is true since $V_2(u(t,s_0)-ct, 0) = 0$ by (1.8). Therefore it follows that m is independent of t. Since $\psi = 0$ on $\mathcal{S}$, we obtain from (1.11) and (1.12) that

$$m = \psi(u(t,s_0) - ct, v(t,s_0)) - \psi(u(t,s_0) - ct, 0) = -\psi(u(t,s_0) - ct, 0)$$

which shows that the stream function is constant on the flat bed, namely $\psi = -m$ on $\mathcal{B}$. Alltogether we can write (1.5) – (1.8) in terms of ψ as follows

$$\begin{cases} \psi_y \psi_{xy} - \psi_x \psi_{yy} = -P_x \\ -\psi_y \psi_{xx} + \psi_x \psi_{xy} = -P_y \end{cases} \quad \text{in } \Omega$$

and

$$\begin{cases} \psi = 0 & \text{on} \quad \mathcal{S} \\ P = P_0 - \sigma \frac{u_s(t,s) v_{ss}(t,s) - u_{ss}(t,s) v_s(t,s)}{((u_s(t,s))^2 + (v_s(t,s))^2)^{3/2}} & \text{on} \quad \mathcal{S}, \\ \psi = -m & \text{on} \quad \mathcal{B}. \end{cases} \tag{1.13}$$

If $\gamma = V_{2X} - V_{1Y}$ is the vorticity we obtain from (1.9) and properties of ψ Bernoulli's law, which says that

$$E = \frac{(V_1 - c)^2 + V_2^2}{2} + P + \gamma\psi$$

is constant throughout the fluid domain (this is a rephrasing of the Euler equations in the moving frame). On the free surface we have

$$E = \frac{(V_1 - c)^2 + V_2^2}{2} + P_0 - \sigma \frac{u_s(t,s) v_{ss}(t,s) - u_{ss}(t,s) v_s(t,s)}{((u_s(t,s))^2 + (v_s(t,s))^2)^{3/2}}.$$

Therefore setting $Q = 2(E - P_0)$ we obtain

$$\psi_x^2 + \psi_y^2 - 2\sigma \frac{u_s(t,s) v_{ss}(t,s) - u_{ss}(t,s) v_s(t,s)}{((u_s(t,s))^2 + (v_s(t,s))^2)^{3/2}} = Q$$

on the free surface. Thus the stream function ψ satisfies the following free boundary value problem:

$$\begin{array}{rcll} \Delta\psi & = & -\gamma & \text{in } \Omega, \\ \psi & = & -m & \text{on } \mathcal{B}, \\ \psi & = & 0 & \text{on } \mathcal{S}, \\ |\nabla\psi|^2 - 2\sigma\frac{u_s(t,s)v_{ss}(t,s)-u_{ss}(t,s)v_s(t,s)}{((u_s(t,s))^2+(v_s(t,s))^2)^{3/2}} & = & Q & \text{on } \mathcal{S}. \end{array} \tag{1.14}$$

Choosing a parametrization so that u and v are independent of t in the moving frame leads to the following free boundary value problem

$$\begin{array}{rcll} \Delta\psi & = & -\gamma & \text{in } \Omega, \\ \psi & = & -m & \text{on } \mathcal{B}, \\ \psi & = & 0 & \text{on } \mathcal{S}, \\ |\nabla\psi|^2 - 2\sigma\frac{u_s v_{ss}-u_{ss}v_s}{(u_s^2+v_s^2)^{3/2}} & = & Q & \text{on } \mathcal{S}. \end{array} \tag{1.15}$$

We will prove in the paper the local existence of waves of small amplitude to the problem (1.15). Moreover we derive the dispersion relation, i.e., a formula which gives the speed of the bifurcating laminar flow (see the proof of Theorem 3.3) in terms of the depth, the period and the vorticity. This dispersion relation is obtained even in the presence of stagnation points in the flow, feature that is not allowed in the paper [25]. We show that for flows with sufficiently small wave-length L there do not exist stagnation points while, if the vorticity is big enough we do have stagnation points.
Our investigation opens up possibilities for the detailed examination of the flow pattern, in the same vein to the ones pursued in [5, 7] for irrotational gravity water waves and in [15] for linear periodic capillary and capillary-gravity water waves.

2. Equivalence of the free boundary problem with a problem with a fixed domain

We give in this section a reformulation of the boundary value problem (1.15) as the quasilinear equation (2.5) for a periodic function of one variable. We give a few notations and state the necessary results; for the proofs we refer the reader to [8].
For an integer $p \geq 0$ and for $\alpha \in (0,1)$ we denote $C^{p,\alpha}$ the standard space of functions whose partial derivatives up to order p are Hölder continuous with exponent α over their domain of definition. $C^{p,\alpha}_{\text{loc}}$ will denote the set of functions of class $C^{p,\alpha}$ over any compact subset of their domain of definition. By $C^{p,\alpha}_{2\pi}$ we denote the space of functions of one real variable which are 2π periodic and of class $C^{p,\alpha}_{\text{loc}}$ in $\mathbb{R}$. By $C^{p,\alpha}_{2\pi,\text{o}}$ we denote the functions that are in $C^{p,\alpha}_{2\pi}$ and have zero mean over one period. For any $d > 0$ let

$$\mathcal{R}_d = \{(x,y) \in \mathbb{R}^2 : -d < y < 0\}.$$

For any $w \in C^{p,\alpha}_{2\pi}$ let $W \in C^{p,\alpha}(\overline{R}_d)$ be the unique solution of

$$\begin{array}{rcl} \Delta W & = & 0 \text{ in } \mathcal{R}_d, \\ W(x,-d) & = & 0, \ x \in \mathbb{R}, \\ W(x,0) & = & w(x), \ x \in \mathbb{R}. \end{array} \tag{2.1}$$

The function $(x,y) \to W(x,y)$ is 2π-periodic in x throughout $\mathcal{R}_d$. For $p \in \mathbb{Z}, p \geq 1$, and $\alpha \in (0,1)$ we define the *periodic Dirichlet-Neumann operator for a strip* $\mathcal{G}_d$ by

$$\mathcal{G}_d(w)(x) = W_y(x,0), \ x \in \mathbb{R}.$$

The operator $\mathcal{G}_d : C^{p,\alpha}_{2\pi} \to C^{p-1,\alpha}_{2\pi}$ is a bounded linear operator. If the function w takes the constant value c then

$$\mathcal{G}_d(c) = \frac{c}{d}. \tag{2.2}$$

Let Z be the unique (up to a constant) harmonic function in $\mathcal{R}_d$, such that $Z + iW$ is holomorphic in $\mathcal{R}_d$. If $w \in C^{p,\alpha}_{2\pi,\mathrm{o}}$ it follows from the discussion in Section 2 of [8] that the function $(x,y) \to Z(x,y)$ is 2π-periodic in x throughout $\mathcal{R}_d$. We specify the constant in the definition of Z by asking that $x \to Z(x,0)$ has zero mean over one period. We define $\mathcal{C}_d(w)$ by

$$\mathcal{C}_d(w)(x) = Z(x,0), \ x \in \mathbb{R}.$$

The obtained mapping $\mathcal{C}_d : C^{p,\alpha}_{2\pi,\mathrm{o}} \to C^{p,\alpha}_{2\pi,\mathrm{o}}$ is a bounded linear operator and is called the *periodic Hilbert transform for a strip*. If $w \in C^{p,\alpha}_{2\pi,\mathrm{o}}$ for $p \geq 1$ we have

$$\mathcal{G}_d(w) = (\mathcal{C}_d(w))' = \mathcal{C}_d(w'). \tag{2.3}$$

It also follows (see [8]) that for $p \geq 1$,

$$\mathcal{G}_d(w) = \frac{[w]}{d} + \mathcal{C}_d(w'), \tag{2.4}$$

where $[w]$ denotes the average of w over one period.

Definition 2.1. We say that a solution (Ω, ψ) of the water wave equation (1.15) is of class $C^{2,\alpha}$ if the free surface satifies (1.1),(1.2) and (1.3), with $u, v \in C^{2,\alpha}$ and $\psi \in C^\infty(\Omega) \cap C^{2,\alpha}(\overline{\Omega})$.

We are going to prove that the free boundary problem (1.15) is equivalent to the problem of finding a positive number h and a function $v \in C^{2,\alpha}_{2\pi}$ subject to the following

$$\begin{array}{l} \left\{ \frac{m}{kh} + \gamma \left(\mathcal{G}_{kh}(\frac{v^2}{2}) - v\mathcal{G}_{kh}(v) \right) \right\}^2 = \left(Q + 2\sigma \frac{\mathcal{G}_{kh}(v)v'' - \mathcal{G}_{kh}(v')v'}{(v'^2 + \mathcal{G}_{kh}(v)^2)^{3/2}} \right) \left(v'^2 + \mathcal{G}_{kh}(v)^2 \right) \\ [v] = h \\ v(x) > 0 \text{ for all } x \in \mathbb{R}, \\ \text{the mapping } x \to \left(\frac{x}{k} + \mathcal{C}_{kh}(v-h)(x), v(x) \right) \text{ is injective on } \mathbb{R}, \\ v'(x)^2 + \mathcal{G}_{kh}(v)(x)^2 \neq 0 \text{ for all } x \in \mathbb{R}, \end{array} \tag{2.5}$$

where $k = \frac{2\pi}{L}$. Before we state the result we explain the meaning of the constant h that appears in (2.5).

Definition 2.2.
- We say that $\Omega \subset \mathbb{R}^2$ is an *L-periodic strip like domain* if it is contained in the upper half (X, Y)-plane and if its boudary consists of the real axis $\mathcal{B}$ and a parametric curve $\mathcal{S}$ defined by (1.1) and (1.3).
- For any such domain, the *conformal mean depth* is defined to be the unique positive number h such that there exists an onto conformal mapping $\tilde{U} + i\tilde{V} : \mathcal{R}_h \to \Omega$ which admits an extension between the closures of these domains, with onto mappings

$$\{(x, 0) : x \in \mathbb{R}\} \to \mathcal{S},$$

and

$$\{(x, -h) : x \in \mathbb{R}\} \to \mathcal{B},$$

and such that

$$\begin{array}{rcl} \tilde{U}(x+L, y) & = & \tilde{U}(x, y) + L, \\ \tilde{V}(x+L, y) & = & \tilde{V}(x, y), \end{array} \quad (x, y) \in \mathcal{R}_h \tag{2.6}$$

The existence and uniqueness of such an h was proved in Appendix A of the paper [8].

Theorem 2.3. *If (Ω, ψ) of class $C^{2,\alpha}$ is a solution of (1.15) then there exists a positive number h, a function $v \in C^{2,\alpha}_{2\pi}$ and a constant $a \in \mathbb{R}$ such that (2.5) holds and moreover*

$$\mathcal{S} = \left\{ \left(a + \frac{x}{k} + \mathcal{C}_{kh}(v - h)(x), v(x) \right) : x \in \mathbb{R} \right\}, \tag{2.7}$$

for some constant $a \in \mathbb{R}$. Conversely, let $h > 0$ and $v \in C^{2,\alpha}_{2\pi}$ be such that (2.5) holds. Assume also that $\mathcal{S}$ is defined by (2.7), let Ω be the domain whose boundary consists of $\mathcal{S}$ and of the real axis $\mathcal{B}$ and let $a \in \mathbb{R}$ be arbitrary. Then there exists a function ψ in Ω such that (Ω, ψ) is a solution of (1.15) of class $C^{2,\alpha}$.

Proof. We first prove the necessity. Let (Ω, ψ) be a solution of class $C^{2,\alpha}$ of (1.15). Then we denote by h the conformal mean depth of Ω and by $\tilde{U} + i\tilde{V}$ the conformal mapping associated to Ω. If we consider the mapping $U + iV : \mathcal{R}_{kh} \to \Omega$ given by

$$\begin{array}{l} U(x, y) = \tilde{U}(\frac{x}{k}, \frac{y}{k}), \\ V(x, y) = \tilde{V}(\frac{x}{k}, \frac{y}{k}), \end{array} \quad (x, y) \in \mathcal{R}_{kh}, \tag{2.8}$$

where $k = \frac{2\pi}{L}$ then following the proof of Theorem 2.2 in [8] we see that $U, V \in C^{2,\alpha}(\overline{\mathcal{R}_h})$ and $U + iV$ is a conformal mapping from $\mathcal{R}_{kh}$ onto Ω which extends homeomorphically to the closures of these domains, with onto mappings

$$\{(x, 0) : x \in \mathbb{R}\} \to \mathcal{S},$$

and

$$\{(x, -kh) : x \in \mathbb{R}\} \to \mathcal{B}.$$

Moreover,

$$U_x^2(x, 0) + V_x^2(x, 0) \neq 0 \text{ for all } x \in \mathbb{R}. \tag{2.9}$$

Let us set

$$v(x) = V(x,0) \text{ for all } x \in \mathbb{R}, \quad u(x) = U(x,0) \text{ for all } x \in \mathbb{R} \tag{2.10}$$

We then have that

$$u = \mathcal{C}_{kh}(v),$$

and from (2.3) it is immediate that

$$u' = \mathcal{G}_{kh}(v) \text{ and } u'' = \mathcal{G}_{kh}(v') \tag{2.11}$$

It also follows [8] that $v \in C^{2,\alpha}_{2\pi}$ and

$$\begin{array}{l} [v] = h \\ v(x) > 0 \text{ for all } x \in \mathbb{R}, \\ \text{the mapping } x \to \left(\frac{x}{k} + \mathcal{C}_{kh}(v-h)(x), v(x)\right) \text{ is injective on } \mathbb{R}, \\ \mathcal{S} = \left\{\left(a + \frac{x}{k} + \mathcal{C}_{kh}(v-h)(x), v(x)\right) : x \in \mathbb{R}\right\}, \end{array} \tag{2.12}$$

for some $a \in \mathbb{R}$, whose presence in the formula for $\mathcal{S}$ is due to the invariance of problem (1.15) to horizontal translations. From (2.9) and the Cauchy-Riemann equations it follows that

$$v'(x)^2 + \mathcal{G}_{kh}(v)(x)^2 \neq 0 \text{ for all } x \in \mathbb{R}. \tag{2.13}$$

Now let $\xi : \mathcal{R}_{kh} \to \mathbb{R}$ be defined by

$$\xi(x,y) = \psi(U(x,y), V(x,y)),\ (x,y) \in \mathcal{R}_{kh}. \tag{2.14}$$

The harmonicity in Ω of the function $(x,y) \to \psi(x,y) + \frac{\gamma}{2}y^2$ and the invariance of harmonic functions under conformal mappings imply that

$$\xi + \frac{\gamma}{2}V^2 \text{ is harmonic in } \mathcal{R}_{kh}. \tag{2.15}$$

The chain rule and the Cauchy-Riemann equations imply that

$$\xi_x^2 + \xi_y^2 = (\psi_x^2(U,V) + \psi_y^2(U,V))(V_x^2 + V_y^2) \text{ in } \overline{\mathcal{R}_{kh}}.$$

From the last equation in (1.15) and (2.11) it follows that

$$\xi_x^2 + \xi_y^2 = \left(Q + 2\sigma \frac{\mathcal{G}_{kh}(v)v'' - \mathcal{G}_{kh}(v')v'}{(v'^2 + \mathcal{G}_{kh}(v)^2)^{3/2}}\right)\left(v'^2 + \mathcal{G}_{kh}(v)^2\right) \tag{2.16}$$

Define $\zeta : \mathcal{R}_{kh} \to \mathbb{R}$ through

$$\zeta = \xi + m + \frac{\gamma}{2}V^2. \tag{2.17}$$

Using the boundary conditions from (1.15) we obtain the following

$$\begin{array}{rcl} \Delta\zeta & = & 0 \text{ in } \mathcal{R}_{kh}, \\ \zeta(x,-kh) & = & 0 \text{ for all } x \in \mathbb{R}, \\ \zeta(x,0) & = & m + \frac{\gamma}{2}v^2(x) \text{ for all } x \in \mathbb{R}, \\ (\zeta_y - \gamma V V_y)^2 & = & \left(Q + 2\sigma \frac{\mathcal{G}_{kh}(v)v'' - \mathcal{G}_{kh}(v')v'}{(v'^2 + \mathcal{G}_{kh}(v)^2)^{3/2}}\right)\left(v'^2 + \mathcal{G}_{kh}(v)^2\right) \text{ at } (x,0) \text{ for all } x \in \mathbb{R}. \end{array} \tag{2.18}$$

The system (2.18) can be reformulated by using the Dirichlet-Neumann operator and (2.2) as

$$\left\{ \frac{m}{kh} + \gamma\left(\mathcal{G}_{kh}(\frac{v^2}{2}) - v\mathcal{G}_{kh}(v)\right)\right\}^2 = \left(Q + 2\sigma\frac{\mathcal{G}_{kh}(v)v'' - \mathcal{G}_{kh}(v')v'}{(v'^2 + \mathcal{G}_{kh}(v)^2)^{3/2}}\right)(v'^2 + \mathcal{G}_{kh}(v)^2) \tag{2.19}$$

For the sufficiency suppose that the positive number h and the function $v \in C^{2,\alpha}_{2\pi}$ satisfy (2.5). Let V be the harmonic function on $\mathcal{R}_{kh}$ which satisfies

$$V(x, -kh) = 0$$

and

$$V(x, 0) = v(x) \text{ for all } x \in \mathbb{R},$$

and let $U : \mathcal{R}_{kh} \to \mathcal{R}$ be such that $U + iV$ is holomorphic. An application of Lemma 2.1 from [8] yields that $U + iV \in C^{2,\alpha}(\overline{\mathcal{R}_{kh}})$. From $[v] = h$ we obtain

$$\begin{cases} U(x + 2\pi, y) = U(x, y) + \frac{2\pi}{k}, \\ V(x + 2\pi, y) = V(x, y), \end{cases} \quad (x, y) \in \mathcal{R}_{kh}. \tag{2.20}$$

The injectivity of the mapping $x \to \big(\frac{x}{k} + \mathcal{C}_{kh}(v - h)(x), v(x)\big)$ gives that the curve (2.7) is non-self-intersecting and from $v(x) > 0$ we have that (2.7) is contained in the upper half-plane. If Ω denotes the domain whose boundary consists of $\mathcal{S}$ and $\mathcal{B}$, it follows from Theorem 3.4 in [23] that $U + iV$ is a conformal mapping from $\mathcal{R}_{kh}$ onto Ω, which extends to a homeomorphism between the closures of these domains, with onto mappings

$$\{(x, 0) : x \in \mathbb{R}\} \to \mathcal{S},$$

and

$$\{(x, -kh) : x \in \mathbb{R}\} \to \mathcal{B}.$$

Together with (2.20) this implies that Ω is a L-periodic strip-like domain, with $L = 2\pi/k$. The conformal mean depth of Ω is h as it can be seen from the properties of the mapping $\tilde{U} + i\tilde{V} : \mathcal{R}_h \to \Omega$, where $\tilde{U}, \tilde{V}$ are given by (2.8). Let ζ be defined as the unique solution of the first three equations of (2.18). Then $\zeta \in C^{2,\alpha}(\overline{\mathcal{R}_{kh}}) \cap C^\infty(\mathcal{R}_{kh})$. Now, let ξ be defined by (2.17) and ψ by (2.14). We obtain that ψ satisfies the first three equations in (1.15). From the first equation in (2.5) we also have that the last equation from (1.15) holds. □

3. Local bifurcation

This section is devoted to proving the existence of solutions to (2.5). The relation $[v] = h$ makes natural to set

$$v = w + h \tag{3.1}$$

Equation (3.1) implies immediately that $[w] = 0$. We then use (2.4) to find that

$$\mathcal{G}_{kh}(w + h) = \mathcal{G}_{kh}(w) + \mathcal{G}_{kh}(h) = \frac{[w]}{kh} + \mathcal{C}_{kh}(w') + \frac{h}{kh} = \frac{1}{k} + \mathcal{C}_{kh}(w')$$

and

$$\mathcal{G}_{kh}(v') = \mathcal{G}_{kh}(w') = \frac{[w']}{kh} + \mathcal{C}_{kh}(w'') = \mathcal{C}_{kh}(w''),$$

since w is periodic. Therefore we can rewrite (2.5) as

$$\begin{cases} \left\{ \frac{m}{kh} + \gamma \left(\frac{[w^2]}{2kh} - \frac{w}{k} - \frac{h}{2k} + \mathcal{C}_{kh}(ww') - w\mathcal{C}_{kh}(w') \right) \right\}^2 \\ = \left\{ Q + 2\sigma \frac{\frac{w''}{k} + w''\mathcal{C}_{kh}(w') - w'\mathcal{C}_{kh}(w'')}{\left(w'^2 + \left(\frac{1}{k} + \mathcal{C}_{kh}(w') \right)^2 \right)^{3/2}} \right\} \left\{ w'^2 + \left(\frac{1}{k} + \mathcal{C}_{kh}(w') \right)^2 \right\} \\ [w] = 0 \\ w(x) > -h \text{ for all } x \in \mathbb{R}, \\ \text{the mapping } x \to \left(\frac{x}{k} + \mathcal{C}_{kh}(w)(x), w(x) + h \right) \text{ is injective on } \mathbb{R}, \\ w'(x)^2 + \left(\frac{1}{k} + \mathcal{C}_{kh}(w')(x) \right)^2 \neq 0 \text{ for all } x \in \mathbb{R}, \end{cases} \tag{3.2}$$

We will regard m and Q as parameters and will prove the existence of solutions $w \in C^{2,\alpha}_{2\pi}$ to the problem (3.2) for all $\gamma \in \mathbb{R}$, $k > 0$, and $h > 0$ fixed and such that $k^3 \geq \frac{\gamma^2}{\sigma}$ and $kh \geq \frac{1}{2}$.
We observe that $w = 0 \in C^{2,\alpha}_{2\pi,\mathrm{o}}$ is a solution of (3.2) if and only if

$$Q = \left(\frac{m}{h} - \frac{\gamma h}{2} \right)^2.$$

This suggests setting

$$\begin{aligned} \lambda &= \frac{m}{h} - \frac{\gamma h}{2}, \\ \mu &= Q - \left(\frac{m}{h} - \frac{\gamma h}{2} \right)^2 \end{aligned} \tag{3.3}$$

Note that the mapping $(m, Q) \to (\lambda, \mu)$ is a bijection from $\mathbb{R}^2$ onto itself. Using (3.3) we see that the equation (3.2) can be rewritten as

$$\begin{aligned} &\left\{ \frac{\lambda}{k} + \gamma \left(\frac{[w^2]}{2kh} - \frac{w}{k} + \mathcal{C}_{kh}(ww') - w\mathcal{C}_{kh}(w') \right) \right\}^2 \\ &= \left\{ \lambda^2 + \mu + 2\sigma \frac{\frac{w''}{k} + w''\mathcal{C}_{kh}(w') - w'\mathcal{C}_{kh}(w'')}{\left(w'^2 + \left(\frac{1}{k} + \mathcal{C}_{kh}(w') \right)^2 \right)^{3/2}} \right\} \left\{ w'^2 + \left(\frac{1}{k} + \mathcal{C}_{kh}(w') \right)^2 \right\} \end{aligned} \tag{3.4}$$

with $w \in C^{2,\alpha}_{2\pi,\mathrm{o}}$, $\mu \in \mathbb{R}$ and $\lambda \in \mathbb{R}$. It is clear that $w = 0 \in C^{2,\alpha}_{2\pi,\mathrm{o}}$ and $\mu = 0$ is a solution of (3.4) for all $\lambda \in \mathbb{R}$. We now apply the Crandall-Rabinowitz theorem [11] on bifurcation from a simple eingevalue in order to prove existence of non-trivial solutions to equation (3.4).

Theorem 3.1. *Let X and Y be Banach spaces, I be an open interval in $\mathbb{R}$ containing λ^*, and $F : I \times X \to Y$ be a continous map satisfying the following properties:*

1. *$F(\lambda, 0) = 0$ for all $\lambda \in I$;*
2. *$\partial_\lambda F$, $\partial_u F$ and $\partial^2_{\lambda,u} F$ exist and are continuous;*
3. *$\mathcal{N}(\partial_u F(\lambda^*, 0))$ and $Y/\mathcal{R}(\partial_u F(\lambda^*, 0))$ are one-dimensional, with the null space generated by u^*;*
4. *$\partial^2_{\lambda,u} F(\lambda^*, 0)(1, u^*) \notin \mathcal{R}(\partial_u F(\lambda^*, 0))$*

Then there exists a continuous local bifurcation curve $\{(\lambda(s), u(s) : |s| < \varepsilon\}$ *with* $\varepsilon > 0$ *sufficiently small such that* $(\lambda(0), u(0)) = (\lambda^*, 0)$ *and there exists a neighbourhood* $\mathcal{O}$ *of* $(\lambda^*, 0) \in I \times X$ *such that*

$$\{(\lambda, u) \in \mathcal{O} : u \neq 0, F(\lambda, u) = 0\} = \{(\lambda(s), u(s)) : 0 < |s| < \varepsilon\}.$$

Moreover, we have

$$u(s) = su^* + o(s) \text{ in } X, \ |s| < \varepsilon.$$

If $\partial_u^2 F$ *is also continuous, then the curve is of class* C^1*.*

In order to apply the local bifurcation theorem 3.1 to (3.4) we set

$$X = \mathbb{R} \times C^{p+1,\alpha}_{2\pi,\mathrm{o},e}, \ Y = C^{p,\alpha}_{2\pi,e},$$

where for any integer $p \geq 0$ we denote:

$$C^{p,\alpha}_{2\pi,e} = \{f \in C^{p,\alpha}_{2\pi} : f(x) = f(-x) \text{ for all } x \in \mathbb{R}\},$$
$$C^{p,\alpha}_{2\pi,\mathrm{o},e} = \{f \in C^{p,\alpha}_{2\pi,\mathrm{o}} : f(x) = f(-x) \text{ for all } x \in \mathbb{R}\}.$$

Equation (3.4) can be written as $F(\lambda, (\mu, w)) = 0$ where $F : \mathbb{R} \times X \to Y$ is given by

$$\begin{aligned} F(\lambda, (\mu, w)) = {} & \gamma^2 \left(\mathcal{C}_{kh}(ww') - w\mathcal{C}_{kh}(w') - \frac{w}{k} + \frac{[w^2]}{2kh} \right)^2 \\ & + \frac{2\lambda\gamma}{k} \left(\mathcal{C}_{kh}(ww') - w\mathcal{C}_{kh}(w') - \frac{w}{k} + \frac{[w^2]}{2kh} \right) \\ & - \left(\mu + 2\sigma \frac{\frac{w''}{k} + w''\mathcal{C}_{kh}(w') - w'\mathcal{C}_{kh}(w'')}{\left(w'^2 + \left(\frac{1}{k} + \mathcal{C}_{kh}(w')\right)^2 \right)^{3/2}} \right) \left(w'^2 + \left(\frac{1}{k} + \mathcal{C}_{kh}(w') \right)^2 \right) \\ & - \lambda^2 \left(w'^2 + \frac{2}{k}\mathcal{C}_{kh}(w') + (\mathcal{C}_{kh}(w'))^2 \right). \end{aligned} \tag{3.5}$$

Since $F(\lambda, (0,0)) = 0$ the first condition from the local bifurcation theorem 3.1 is verified. We now compute

$$\partial_{(\mu,w)} F(\lambda, (0,0))(\nu, f) = \lim_{t \to 0} \frac{F(\lambda, t(\nu, f)) - F(\lambda, (0,0))}{t}.$$

Using Lemma 4.2 it turns out that

$$\begin{aligned} \partial_{(\mu,w)} F(\lambda, (0,0))(\nu, f) &= -\frac{2\lambda\gamma}{k^2} f - \frac{\nu}{k^2} - 2\sigma f'' - 2\frac{\lambda^2}{k}\mathcal{C}_{kh}(f') \\ &= -\frac{2}{k^2}\left(\lambda\gamma f + \lambda^2 k \mathcal{C}_{kh}(f') + \sigma k^2 f''\right) - \frac{\nu}{k^2} \end{aligned} \tag{3.6}$$

From representation (4.1) it follows that

$$\partial_{(\mu,w)} F(\lambda, (0,0))(\nu, f) = -\frac{2}{k^2} \sum_{n=1}^{\infty} (\lambda\gamma + \lambda^2 kn \coth(nkh) - \sigma k^2 n^2) a_n \cos(nx) - \frac{\nu}{k^2}. \tag{3.7}$$

if

$$f = \sum_{n=1}^{\infty} a_n \cos(nx).$$

Using now Lemma 4.2 it follows that the bounded linear operator $\partial_{(\mu,w)} F(\lambda, (0,0)) : X \to Y$ is invertible whenever

$$\lambda\gamma + \lambda^2 kn \coth(nkh) - \sigma k^2 n^2 \neq 0 \text{ for any integer } n \geq 1. \tag{3.8}$$

Therefore all the candidates for the bifurcation points of (3.4) are to be found among the solutions of the equation

$$\lambda\gamma + \lambda^2 kn \coth(nkh) - \sigma k^2 n^2 = 0, \tag{3.9}$$

for some integer $n \geq 1$. Since we are looking for solutions of (3.4) of minimal period 2π, we take $n = 1$ in (3.9).

Lemma 3.2. *Let λ^{*1} be a solution of* (3.9) *with $n = 1$, i.e.,*

$$\lambda^{*1}_{\pm} = -\frac{\gamma \tanh(kh)}{2k} \pm \sqrt{\frac{\gamma^2 \tanh^2(kh)}{4k^2} + k\sigma \tanh(kh)}.$$

Assume that $k^3 \geq \frac{\gamma^2}{\sigma}$ and $kh \geq \frac{1}{2}$. Then it follows from (3.7) *and Lemma* 4.2 *that the kernel $\mathcal{N}(\partial_{(\mu,w)} F(\lambda^*, 0)))$ is one-dimensional, being generated by $(0, w^*) \in X$, where $w^*(x) = \cos(x)$ for all $x \in \mathbb{R}$.*

Proof. It suffices to show that if $n > 1$ is an integer then the equations

$$\lambda\gamma + \lambda^2 kn \coth(nkh) - \sigma k^2 n^2 = 0,$$

and

$$\lambda\gamma + \lambda^2 k \coth(kh) - \sigma k^2 = 0,$$

do not have common solutions. Let

$$\lambda^{*n}_{\pm} = -\frac{\gamma \tanh(nkh)}{2kn} \pm \sqrt{\frac{\gamma^2 \tanh^2(nkh)}{4k^2 n^2} + nk\sigma \tanh(nkh)},$$

denote the two solutions of equation (3.9). Since

$$\lambda^{*n}_{+} > 0, \quad \lambda^{*n}_{+} < 0,$$

for all $n \geq 1$ it suffices to show that $\lambda^{*n}_{+} \neq \lambda^{*1}_{+}$ and $\lambda^{*n}_{-} \neq \lambda^{*1}_{-}$ for all $n > 1$. We consider only the case $\gamma > 0$, since for $\gamma < 0$ we can proceed in a similar way. We assume ab absurdum that $\lambda^{*n}_{-} = \lambda^{*1}_{-}$ for some $n > 1$ which leads to

$$\begin{aligned} &\frac{\tanh(nkh)}{n} - \tanh(kh) = \\ &\sqrt{\tanh^2(kh) + \frac{4\sigma k^3}{\gamma^2} \tanh(kh)} - \sqrt{\frac{\tanh^2(nkh)}{n^2} + \frac{4\sigma k^3}{\gamma^2} n \tanh(nkh)} \end{aligned} \tag{3.10}$$

To ease the notation we set $f(n) = \frac{\tanh(nkh)}{n}$ and $g(n) = n\tanh(nkh)$. By squaring (3.10) we obtain

$$-2f(n)f(1) = \frac{4\sigma k^3}{\gamma^2}\left(g(n)+g(1)\right) - 2\sqrt{\left[f^2(n)+\frac{4\sigma k^3}{\gamma^2}g(n)\right]\left[f^2(1)+\frac{4\sigma k^3}{\gamma^2}g(1)\right]}$$

which is equivalent to

$$2\sqrt{\left[f^2(n)+\frac{4\sigma k^3}{\gamma^2}g(n)\right]\left[f^2(1)+\frac{4\sigma k^3}{\gamma^2}g(1)\right]} = 2f(n)f(1) + \frac{4\sigma k^3}{\gamma^2}\left(g(n)+g(1)\right).$$

Further by squaring we obtain

$$\begin{aligned} 4\frac{4\sigma k^3}{\gamma^2}\left[f^2(n)g(1)+f^2(1)g(n)\right] &= \left(\frac{4\sigma k^3}{\gamma^2}\right)^2\left[g(n)-g(1)\right]^2 \\ &\quad + 4\frac{4\sigma k^3}{\gamma^2}f(n)f(1)\left[g(n)+g(1)\right] \end{aligned}$$

which is equivalent to

$$\left(\frac{4\sigma k^3}{\gamma^2}\right)^2\left[g(n)-g(1)\right]^2 + 4\frac{4\sigma k^3}{\gamma^2}\left[f(n)-f(1)\right]\left[f(1)g(n)-f(n)g(1)\right] = 0 \tag{3.11}$$

Since $k^3 \geq \frac{\gamma^2}{\sigma}$ we have that the left hand side of (3.11) is greater than

$$\begin{aligned} 4\frac{4\sigma k^3}{\gamma^2}\Big(\left[n\tanh(nkh)-\tanh(kh)\right]^2 + \left[\tanh(nkh)-n\tanh(kh)\right](1-\frac{1}{n^2})\times \\ \times\tanh(nkh)\tanh(kh)\Big) \end{aligned} \tag{3.12}$$

Using the following inequalities (whose proofs are given in Lemma 4.3 and Lemma 4.4)

$$n\tanh(nkh)-\tanh(kh) \geq 1 \text{ for all } n \geq 2,\ kh \geq \frac{1}{2},$$

$$\tanh(nkh)-n\tanh(kh) < 0 \text{ for all } n \geq 2,\ k>0, h>0,$$

$$(1-\frac{1}{n^2})\tanh(nkh)\tanh(kh) < 1 \text{ for all } n \geq 2,$$

we have that the expression in (3.12) is greater than

$$\begin{aligned} &4\frac{4\sigma k^3}{\gamma^2}\left(n\tanh(nkh)-\tanh(kh)+\tanh(nkh)-n\tanh(kh)\right) \\ &= \frac{4\sigma k^3}{\gamma^2}\left(n\left[\tanh(nkh)-\tanh(kh)\right]+\left[\tanh(nkh)-\tanh(kh)\right]\right) > 0, \end{aligned} \tag{3.13}$$

for all $n \in \mathbb{N}, n \geq 2$. This shows that the expression on the left hand side of (3.11) is strictly greater than zero and therefore (3.10) can not hold. But this implies that $\lambda_-^{*n} \neq \lambda_-^{*1}$ for all $n \in \mathbb{N}, n > 1$. It is easy to see that the same argument can be applied to show that $\lambda_+^{*n} \neq \lambda_+^{*1}$ for all $n \in \mathbb{N}, n > 1$. ☐

We also obtain that $\mathcal{R}(\partial_{(\mu,w)}F(\lambda^*,0)))$ is the closed subspace of Y consisting of all functions $f \in Y$ which satisfy

$$\int_{-\pi}^{\pi} f(x)\cos(x)dx = 0,$$

and therefore $Y/\mathcal{R}(\partial_{(\mu,w)}F(\lambda^*,0)))$ is the one dimensional subspace of Y generated by the function $w^*(x) = \cos(x)$. Using (3.6) we compute

$$\begin{aligned}
&\partial^2_{\lambda,(\mu,w)}F(\lambda^*,(0,0))(1,(0,w^*)) \\
&= \lim_{t\to 0} \frac{\partial_{(\mu,w)}F(\lambda^*+t,(0,0))(0,w^*) - \partial_{(\mu,w)}F(\lambda^*,(0,0))(0,w^*)}{t} \\
&= -\frac{2}{k^2}\gamma w^* - \frac{4}{k^2}\lambda^* k \mathcal{C}_{kh}(w^{*\prime})
\end{aligned} \tag{3.14}$$

For $w^* = \cos(x)$ we have that $\mathcal{C}_{kh}(w^*) = \coth(kh)\sin(x)$ and $\mathcal{C}_{kh}(w^{*\prime}) = (\mathcal{C}_{kh}(w^*))' = \coth(kh)\cos(x) = \coth(kh)w^*$, and therefore

$$\begin{aligned}
\partial^2_{\lambda,(\mu,w)}F(\lambda^*,(0,0))(1,(0,w^*)) =& \frac{2}{k^2}(-\gamma - 2\lambda^* k\coth(kh))w^* \\
&\notin \mathcal{R}(\partial_{(\mu,w)}F(\lambda^*,(0,0))),
\end{aligned} \tag{3.15}$$

since by (3.9) we have

$$-\gamma - 2\lambda^* k\coth(kh) = -\lambda^*\left(k\coth(kh) + \frac{\sigma k^2}{(\lambda^*)^2}\right) \neq 0.$$

From the considerations above we obtain the bifurcation values

$$\lambda_\pm = -\frac{\gamma\tanh(kh)}{2k} \pm \sqrt{\frac{\gamma^2\tanh^2(kh)}{4k^2} + k\sigma\tanh(kh)}. \tag{3.16}$$

From (3.3) we obtain the corresponding values for m as

$$m_\pm = \frac{\gamma h^2}{2} - \frac{\gamma h\tanh(kh)}{2k} \pm h\sqrt{\frac{\gamma^2\tanh^2(kh)}{4k^2} + k\sigma\tanh(kh)} \tag{3.17}$$

Theorem 3.3. *For any $h > 0$, $k > 0$, $\gamma \in \mathbb{R}$ and $m \in \mathbb{R}$ satisfying $k^3 > \frac{\gamma^2}{\sigma}$, $kh \geq \frac{1}{2}$ there exists laminar flows with a flat free surface in water of depth h, of constant vorticity γ and relative mass flux m. The laminar flows of flux $m_\pm$ are exactely those with horizontal speeds at the flat free surface equal to $\lambda_\pm$ given by* (3.16). *The values of $m_\pm$ of the flux given by* (3.17) *trigger the appearance of periodic steady waves of small amplitude, with period $\frac{2\pi}{k}$ and conformal mean depth h, which have a smooth profile with one crest and one trough per period, monotone between consecutive crests and troughs and symmetric about any crest line.*

Proof. We already saw that $w = 0$ satisfies (3.2) provided $Q = \left(\frac{m}{h} - \frac{\gamma h}{2}\right)^2$. We are going to see that these solutions correspond to laminar flows in the fluid domain bounded below by the rigid bed $\mathcal{B}$ and above by the flat free surface $Y = h$. This can be shown as follows. First, since $w = 0$ we have from (3.1) that $v = h$ which implies that the free surface $\mathcal{S}$ corresponding to this flow is flat. Since $\mathcal{S}$ is flat and the stream function ψ is 0 on $\mathcal{S}$ by (1.15) we conclude that $\psi_X = 0$ on $\mathcal{S}$. Similarly $\psi_X = 0$ on the flat bed $\mathcal{B}$. Moreover from $\Delta\psi = -\gamma$ in Ω we have that $\Delta\psi_X = 0$ in Ω. Therefore from the maximum principle it follows that $\psi_X = 0$ in Ω. Now from the first equation in (1.15) it follows that the laminar flow solutions have the form $\psi((X,Y) = -\frac{\gamma}{2}Y^2 + Ay + B$ and from the second and third equation of (1.15) we finally have that the laminar flow solutions are given by

$$\psi(X,Y) = -\frac{\gamma}{2}Y^2 + \left(\frac{m}{h} + \frac{\gamma h}{2}\right) Y - m,\ X \in \mathbb{R},\ 0 \leq Y \leq h,$$

while the velocity field is

$$(\psi_Y, -\psi_X) = \left(-\gamma Y + \frac{m}{h} + \frac{\gamma h}{2}, 0\right),\ X \in \mathbb{R},\ 0 \leq Y \leq h. \tag{3.18}$$

Note that using (3.3) we can rewrite (3.18) as

$$(\psi_Y, -\psi_X) = (\lambda_\pm + \gamma(h - Y), 0),\ X \in \mathbb{R},\ 0 \leq Y \leq h, \tag{3.19}$$

where $\lambda_\pm$ is given by (3.16). Observe that $\psi_{Y|Y=h} = -\gamma h + \frac{m}{h} + \frac{\gamma h}{2} = \frac{m}{h} - \frac{\gamma h}{2}$ which shows that for laminar flows the horizontal velocity at the free surface coincides with λ given in (3.3). The formula (3.16), of the speed $\lambda_\pm$ at the free surface in terms of the depth h, period $2\pi/k$ and vorticity γ, is called the *dispersion relation.*

The Crandall-Rabinowitz theorem provides the existence of the local bifurcation curve

$$\{(\lambda(s), (0 + o(s), s\cos(x) + o(s))) : |s| < \varepsilon\} \subset \mathbb{R} \times X$$

consisting of solutions of (3.4) with $\lambda_\pm$ given by (3.16). We choose ε sufficiently small and use Lemma 4.2 to ensure that

$$w(x) > -h \text{ for all } x \in \mathbb{R},$$

and

$$\frac{1}{k} + \mathcal{C}_{kh}(w')(x) > 0 \text{ for all } x \in \mathbb{R}.$$

The latter inequality implies that the corresponding non-flat free surface $\mathcal{S}$ given by (2.7) with $v = w + h$ is the graph of a smooth function, symmetric with respect to the points obtained for the values $x = n\pi, n \in \mathbb{Z}$. Choosing p such that $p \geq 1$ and using

$$w(x; s) = s\cos(x) + o(s) \text{ in } C_{2\pi}^{p+1,\alpha},$$

we have that

$$sw'(x; s) < 0 \text{ for all } x \in (0, \pi),\ 0 < |s| < \varepsilon,$$

for $\varepsilon > 0$ sufficiently small. Since $x \to w(x; s)$ is even we have alltogether that $\mathcal{S}$ has one crest and one trough per minimal period and is monotone between consecutive crests and troughs. $\square$

Remark 3.4. Please note that our formula for λ_--the horizontal velocity at the free surface for laminar flows- coincides with the one given in [25]. However the formula in [25] was obtained under stronger assumptions namely that the flow does not contain stagnation points and that the free surface is always a graph. While for the small amplitude waves whose existence we prove, at least close to the bifurcation point, the free surface is always a graph, we will see that stagnation is possible.

Remark 3.5. Concerning the hypothesis that the free surface is a graph (discarded in the present paper and assumed in [25] we want to note that there are examples of capillary irrotational waves whose free surface is not a graph. In water of finite depth the explicit solutions are known as Kinnersley's waves [17], and in the case of infinite depth as Crapper's waves [12].

We now want to analyze whether stagnation points can occur in such laminar flows. It follows from (3.19) that the necessary and sufficient condition for the existence of stagnation points is

$$\lambda_{\pm} + \gamma(h - Y) = 0$$

for at least one Y in $[0, h]$. The latter condition is satisfied if and only if

$$\lambda_{\pm}(\lambda_{\pm} + \gamma h) \leq 0. \tag{3.20}$$

If $\gamma > 0$ it follows that $\lambda_+(\lambda_+ + \gamma h) > 0$ so from (3.20) we have that the flow corresponding to λ_+ does not contain stagnation points. The flow corresponding to λ_- contains stagnation points if and only if $\lambda_- + \gamma h \geq 0$, which is equivalent to

$$(\gamma^2 h + k^2\sigma)\frac{\tanh(kh)}{kh} \leq \gamma^2 h. \tag{3.21}$$

The case $\gamma < 0$ can be treated similarly. Namely, if $\gamma < 0$ we see that $\lambda_-(\lambda_- + \gamma h) > 0$ and therefore from (3.20) we have that the flow corresponding to λ_- does not contain stagnation points. The flow corresponding to λ_+ contains stagnation points if and only if $\lambda_+ + \gamma h \leq 0$, which also is equivalent to (3.21).

Remark 3.6. Notice that (3.21) does not hold true for $k \to \infty$ since the limit of the left-hand side is $+\infty$ as $k \to \infty$. It follows that a flow with $L \to 0$ does not have stagnation points since in that case $k = \frac{2\pi}{L} \to \infty$ and we saw that this violates (3.21).

Lemma 3.7. *If the vorticity γ is such that $\gamma^2 \geq 4\frac{\sigma}{h^3}$ there are values*

$$k_1 := \frac{\gamma^2 h^2 - \sqrt{\gamma^4 h^4 - 4\sigma\gamma^2 h}}{2\sigma}$$

$$k_2 := \frac{\gamma^2 h^2 + \sqrt{\gamma^4 h^4 - 4\sigma\gamma^2 h}}{2\sigma}$$

with the property that (3.21) *holds true whenever $k \in [k_1, k_2]$. To see this note that* (3.21) *is equivalent to*

$$\tanh(kh) \leq \frac{\gamma^2 h^2 k}{\gamma^2 h + k^2\sigma} \tag{3.22}$$

Now note that $\tanh(kh) < 1$ *for all* k, h. *We will prove that there are* k_1, k_2 *such that* $k \in [k_1, k_2]$ *implies* $\frac{\gamma^2 h^2 k}{\gamma^2 h + k^2\sigma} \geq 1$. *Then* (3.22) *will follow. Due to* $\gamma^2 \geq 4\frac{\sigma}{h^3}$ *we obtain that the equation* $\sigma k^2 - \gamma^2 h^2 k + \gamma^2 h = 0$ *has two solutions* k_1 *and* k_2 *since its discriminant equals* $\gamma^4 h^4 - 4\sigma\gamma^2 h = \gamma^2 h(\gamma^2 h^3 - 4\sigma) \geq 0$. *This implies that* $\sigma k^2 - \gamma^2 h^2 k + \gamma^2 h \leq 0$ *for all* $k \in [k_1, k_2]$. *The latter inequality is equivalent to* $\frac{\gamma^2 h^2 k}{\gamma^2 h + k^2 \sigma} \geq 1$ *and we are done.*

Remark 3.8. Let γ and h be such that $\gamma^2 \geq 4\frac{\sigma}{h^3}$. Let k_1 and k_2 be the values from Lemma 3.7. If $k \in \left[\frac{k_1+k_2}{2}, k_2\right]$ then the sufficient conditions for the existence of laminar flows from Theorem 3.3 are satisfied. In addition these flows posses stagnation points as the proof of Lemma 3.7 shows.

Proof. Indeed, we only need to show that for all such k we have that $k^3 \geq \frac{\gamma^2}{\sigma}$ and $kh \geq \frac{1}{2}$. We show that the last two inequalities hold true for $k = \frac{k_1+k_2}{2} = \frac{\gamma^2 h^2}{2\sigma}$. Indeed, we have

$$\left(\frac{\gamma^2 h^2}{2\sigma}\right)^3 = \frac{\gamma^6 h^6}{8\sigma^3} = \frac{\gamma^2}{\sigma} \cdot \frac{\gamma^4 h^6}{8\sigma^2} \geq \frac{\gamma^2}{\sigma} \cdot \frac{16\sigma^2}{8\sigma^2} = 2\frac{\gamma^2}{\sigma}.$$

Also $\frac{k_1+k_2}{2} h = \frac{\gamma^2 h^3}{2\sigma} \geq 2$ and we are done. □

4. Appendix

This section contains some auxiliary results.
Denote by $L^2_{2\pi}$ the space of 2π-periodic locally square integrable functions of one real variable. By $L^2_{2\pi,\mathrm{o}}$ we denote the subspace of $L^2_{2\pi}$ whose elements have zero mean over one period.

Lemma 4.1 ([8])**.** *If*

$$w = \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx),$$

is the Fourier series expansion of $w \in L^2_{2\pi,o}$ *then*

$$\mathcal{C}_d(w) = \sum_{n=1}^{\infty} a_n \coth(nd) \sin(nx) - \sum_{n=1}^{\infty} b_n \coth(nd) \cos(nx) \qquad (4.1)$$

Lemma 4.2 ([8])**.** *For any* $d > 0$, $p \geq 0$ *integer and* $\alpha \in (0, 1)$, $\mathcal{C}_d : C^{p,\alpha}_{2\pi,\mathrm{o}} \to C^{p,\alpha}_{2\pi,\mathrm{o}}$ *is a bounded linear operator. Moreover,* $\mathcal{C}_d^{-1} : C^{p,\alpha}_{2\pi,\mathrm{o}} \to C^{p,\alpha}_{2\pi,\mathrm{o}}$ *is also a bounded linear operator.*

Lemma 4.3. *For all integers* $n \geq 2$ *and all* $k > 0, h > 0$ *with* $kh \geq \frac{1}{2}$ *we have that*

$$n \tanh(nkh) - \tanh(kh) \geq 1.$$

Proof. Since the function $x \to x\tanh(xkh)$ is strictly increasing for $kh > 0$ we only need to prove the inequality for $n = 2$. If we denote $\tanh(kh) = a$ then

$$2\tanh(2kh) - \tanh(kh) \geq 1$$

is equivalent to

$$\frac{-a^3 - a^2 + 3a - 1}{1 + a^2} \geq 0$$

which in turn is equivalent to

$$(a-1)(a^2 + 2a - 1) \leq 0 \tag{4.2}$$

Since $a = \tanh(kh) \in (0,1)$ the inequality in (4.2) is equivalent to $a = \tanh(kh) \geq \sqrt{2} - 1$. A routine calculation shows that $\tanh(\frac{1}{2}) > \sqrt{2} - 1$ and then it follows that $\tanh(kh) \geq \sqrt{2} - 1$ for all k, h with $kh \geq \frac{1}{2}$ since the function $y \to \tanh(h)$ is increasing. □

Lemma 4.4. *For all integers $n \geq 2$ and all $k > 0, h > 0$ we have that*

$$\tanh(nkh) - n\tanh(kh) < 0.$$

Proof. Note first that

$$\tanh(2kh) = \frac{2\tanh(kh)}{1 + \tanh^2(kh)} < 2\tanh(kh),$$

which shows that $\tanh(2kh) - 2\tanh(kh) < 0$, for all $k > 0, h > 0$. The asserted inequality is true if we show that the function $x \to \tanh(xkh) - x\tanh(kh) =: q(x)$ is decreasing. Indeed

$$q'(x) = \frac{kh}{\cosh^2(xkh)} - \tanh(kh) = \frac{kh\cosh(kh) - \sinh(kh)\cosh^2(xkh)}{\cosh^2(xkh)\cosh(kh)} < 0$$

for all $x \geq 2, k > 0, h > 0$, since

$$1 < \cosh(kh) < \cosh^2(kh) < \cosh^2(xkh) \text{ for all } x \geq 2, k > 0, h > 0$$

and

$$kh < \sinh(kh) \text{ for all } k > 0, h > 0.$$

□

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