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RESEARCH ARTICLE

The Generalized Dock Problem

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We show existence and uniqueness for a linearized water wave problem in a two dimensional domain G with corner, formed by two semi-axes Γ_1 and Γ_2 which intersect under an angle $\alpha \in (0, \pi]$. The existence and uniqueness of the solution is proved by considering an auxiliary mixed problem with Dirichlet and Neumann boundary conditions. The latter guarantees the existence of the Dirichlet to Neumann map. The water wave boundary value problem is then shown to be equivalent to an equation like $v_{tt} + g\Lambda v = P_t$ with initial conditions, where t stands for time, g is the gravitational constant, P means pressure, and Λ is the Dirichlet to Neumann map. We then prove that Λ is a positive self-adjoint operator.

Keywords: water waves; corner domain; Dirichlet to Neumann map.

AMS Subject Classification: 35Q35; 76B15.

1. Introduction

This paper considers the wave motion in water with a free surface and subjected to gravitational and other forces. Namely, a problem resembling the classical dock problem will be studied. We first give a brief account on the general theory of surface waves and then continue with the statement of the dock problem and previous work in this direction. We now summarize the fundamental mathematical basis for our later endeavors by formulating a typical problem which arises in the hydrodynamics of surface waves. One of the first papers in this field belongs to Lord Rayleigh [1]. For a thorough treatment of the theory of water waves one could consult the books of Stoker [2] and Lamb [3]. See also the paper [4].
Let us consider the physical situation of an ocean beach. The water is assumed to be initially at rest occupying the region defined by the equation

$$-h(x_1, x_2) \leq y \leq 0,\ x^s(x_2, t) \leq x_1 < +\infty, -\infty < x_2 < +\infty,$$

where $x^s(x_2, t)$ is the horizontal coordinate of the water line on the shore. We assume that at time $t = 0$ a disturbance is created on the surface of the water, and one then wants to determine the subsequent motion of the water, namely the form of the free surface $\eta(x_1, x_2; t)$ and the velocity field components u, v, w as functions of the space variables x_1, x_2, y and the time t. We will also assume all flows to be incompressible and irrotational. The incompressibility of the flow gives the law of

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mass conservation

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial y} = 0, \tag{1}$$

where $U = (u_1, u_2, u_3)$ denotes the velocity field. Since the flow is assumed to be irrotational we have that

$$\operatorname{curl} U = 0. \tag{2}$$

The fact that $\operatorname{curl} U = 0$ implies the existence of a single-valued velocity potential $v(x_1, x_2, y, t)$ in any simple connected region, i.e.,

$$(u_1, u_2, u_3) = \nabla v = (v_{x_1}, v_{x_2}, v_y). \tag{3}$$

Equations (1) and (3) give that the velocity potential v satisfies the Laplace equation

$$\Delta v = 0. \tag{4}$$

From the irrotational character of the water flow (2) we obtain the Bernoulli law

$$v_t + \frac{1}{2}(u_1^2 + u_2^2 + u_3^2) + gy = P(x_1, x_2, y, t), \tag{5}$$

where g is the gravitational constant, and $P(x_1, x_2, y, t)$ plays the role of disturbances over the free surface. In our paper we will work under the assumption that the amplitude of the surface waves is small with respect to the wave length. This will allow us to neglect the nonlinear terms in (5). Therefore we obtain from (5) the linearized equation

$$v_t + gy = P(x_1, x_2, y, t). \tag{6}$$

Boundary conditions

In the problem under consideration it is assumed that the fluid has a boundary surface S which has the property that any particle which is once on the surface remains on it.

Assume that S is given by an equation $\xi(x_1, x_2, y, t) = 0$. Differentiation with respect to t gives that the condition

$$\frac{d\xi}{dt} = u_1\xi_{x_1} + u_2\xi_{x_2} + u_3\xi_y + \xi_t = 0 \tag{7}$$

holds on S. Using relations (3), (7) and the fact that the vector $(\xi_{x_1}, \xi_{x_2}, \xi_y)$ is normal to S we obtain that

$$\frac{\partial v}{\partial \nu} = -\frac{\xi_t}{\sqrt{\xi_{x_1}^2 + \xi_{x_2}^2 + \xi_y^2}}, \tag{8}$$

where $\frac{\partial}{\partial \nu}$ means differentiation in the direction of the normal to S.
An important special case is when the boundary S is independent of the time t (the bottom of the sea for eg.), situation which leads to the boundary condition

$$\frac{\partial v}{\partial \nu} = 0 \text{ on S} \tag{9}$$

Another important situation is when the boundary surface S is given by the equation

$$y = \eta(x_1, x_2, t), \tag{10}$$

and the surface is not prescribed apriori. In this case we have $\xi = y - \eta(x_1, x_2, t) = 0$ for any particle, and (7) together with the assumption of small amplitude of the surface waves lead to

$$-v_y + \eta_t = 0 \text{ on } \mathrm{y} = \eta(\mathrm{x}_1, \mathrm{x}_2, \mathrm{t}), \tag{11}$$

while the Bernoulli's law gives the condition

$$g\eta + v_t = P(x_1, x_2, y, t) \text{ on } \mathrm{y} = \eta(\mathrm{x}_1, \mathrm{x}_2, \mathrm{t}). \tag{12}$$

Previous work
In the case of the dock problem the upper surface of the water is constrained by the dock for all $x_2 < 0$ and is a free surface described by $y = \eta(x_1, x_2, t)$, subject to atmospheric presure for all $x_2 > 0$. The standing solution of the homogeneous ($P = 0$) two-dimensional dock problem has been given by Friedrichs and Lewy [5], (see also [6]) as a special case of periodic waves on sloping beaches which behave at infinity like an arbitrary progressing wave. The general three-dimensional case of periodic waves cresting on a beach sloping at any angle α was first considered by Peters [7] and Roseau [8]. The case of the three-dimensional dock problem in water of uniform depth was first solved by Heins [9] by means of the Wiener-Hopf technique, see also Holford [10]. Varley developed in [11] methods which he applied to a generalized dock problem. In the papers mentioned above the authors looked for very special solutions in the form of standing waves, i.e., solutions which are of the form $e^{iat}\Phi$, where Φ is some unknown function in the space variables, t denotes the time variable and a is the so-called shallowness parameter. They also considered progressing waves, which are solutions of the form $e^{i(at-\lambda x_1)}\Phi$ where λ is the wave length and t, a, Φ are as before. Rahimizadeh [12] proved for the first time the well-posedness of the full time dependent dock problem. In our work we prove existence and uniqueness for the full time-dependent problem and where the underlying space is a planar domain with corner.
For a recent account on the dock-problem see the paper [13].
Outline of the paper
Our paper deals with a problem in a two dimensional sector with a corner point. We will denote the space variables by (x_1, x_2). With this notation the equation of the free surface (10) now becomes

$$x_2 = \eta(x_1, t). \tag{13}$$

We shall denote by G the corner domain in $\mathbb{R}^2$ formed by the semi-axis $\Gamma_1 = \{x_1 > 0, x_2 = 0\}$ and $\Gamma_2 = \{y_1 = -x_1 \cos\alpha - x_2 \sin\alpha < 0, y_2 = x_1 \sin\alpha - x_2 \cos\alpha = 0\}$, where α represents the interior angle of G and $0 < \alpha \leq \pi$ (See Figure 1).

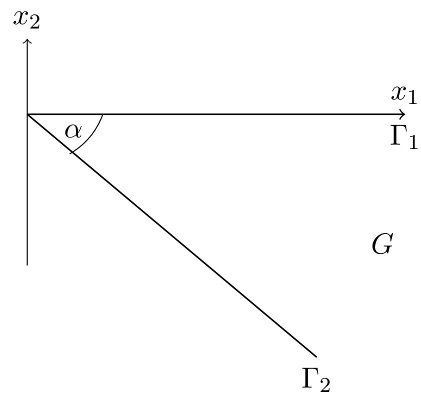

Figure 1.

The case $\alpha = \pi$ which corresponds to the full time dependent dock problem with initial conditions was treated using different methods in [12].
Let $v(x_1, x_2, t)$ denote the velocity potential function. From (4), (9) and since we work under the assumption that the amplitude of the surface waves is small with respect to the wave length we obtain using (11) and (12) the following linearized boundary value problem

$$\begin{cases} \Delta v(x_1, x_2, t) & = & 0, \quad (x_1, x_2) \in G \text{ and } t \geq 0 \\ g\eta(x_1, t) + v_t(x_1, 0, t) & = & P(x_1, 0, t), \quad (x_1, 0) \in \Gamma_1 \text{ and } t \geq 0 \\ \eta_t(x_1, t) - v_{x_2}(x_1, 0, t) & = & 0, \quad (x_1, 0) \in \Gamma_1 \text{ and } t \geq 0 \\ \frac{\partial v}{\partial \nu}(x_1, x_2, t) & = & 0, \quad (x_1, x_2) \in \Gamma_2 \text{ and } t \geq 0 \end{cases}, \tag{14}$$

subject to the initial conditions

$$\begin{cases} \eta(x_1, 0) & = \eta_0 \\ v(x_1, x_2, 0) & = v_0 \end{cases} \tag{15}$$

with given η_0 and v_0 in appropriate Sobolev spaces whose definitions, cf. for example [14], [15], we recall now.
As usual $H_s(\mathbb{R}^2)$ denotes the Sobolev space with the norm

$$||u||_s^2 = \int_{\mathbb{R}^2} (1 + |\xi|^2)^s |\tilde{u}(\xi)|^2 d\xi,$$

where $\tilde{u}$ is the Fourier transform of u. By $\overset{\circ}{H}_s(G)$ we denote the subspace of $H_s(\mathbb{R}^2)$ consisting of functions with support in $\overline{G}$. $H_s(G)$ is defined to be the space of all restrictions of functions in $H_s(\mathbb{R}^2)$ to the domain G with the norm

$$||f||_s^+ = \inf_l ||lf||_s, \tag{16}$$

where f is a distribution in G, lf is an arbitrary extension of f to $\mathbb{R}^2$ belonging to $H_s(\mathbb{R}^2)$, and the infimum is taken over all extensions of f.
On Γ_k, $k = 1, 2$, we define $H_s(\Gamma_k)$ to be the space of all restrictions of distributions in $H_s(\mathbb{R}^1)$ to Γ_k with the norm

$$[h]_s^+ = \inf_l [lh]_s, \tag{17}$$

where lh is an arbitrary extension of h to $\mathbb{R}^1$ and $[lh]_s$ is the norm in $H_s(\mathbb{R}^1)$.
The Sobolev space $\overset{\circ}{H}_s(\Gamma_k)$ is defined as the completion of $C_0^\infty(\Gamma_k)$ with respect to the norm (17).
We shall also need the following modifications of the Sobolev spaces. Let us denote with $\dot{H}_s(\mathbb{R}^2)$ the closure of $C_0^\infty(\mathbb{R}^2)$ with respect to the norm

$$\int_{\mathbb{R}^2} |\xi|^{2s}|\tilde{u}(\xi)|^2 d\xi, \quad |\xi| = \sqrt{\xi_1^2 + \xi_2^2} \tag{18}$$

Then, for the domain G we define $\dot{H}_s(G)$ to be the space of restrictions of distributions from $\dot{H}_s(\mathbb{R}^2)$.
We now state the main result of the paper.

Theorem 1.1: *For any* $T > 0$ *and for any* $P(x_1, x_2, t)$ *such that* $P(x_1, 0, t) \in C([0,T], L_2(\Gamma_1))$ *and* $P_t(x_1, 0, t) \in L_1([0,T], L_2(\Gamma_1))$ *there exist unique* $v(x_1, x_2, t) \in C([0,T], \dot{H}_1(G))$ *and* $\eta(x_1, t) \in C([0,T], L_2(\Gamma_1))$ *such that* $v(x_1, 0, t) \in C([0,T], H_{\frac{1}{2}}(\Gamma_1)), v_t(x_1, 0, t) \in C([0,T], L_2(\Gamma_1)), \eta_t \in C([0,T], H_{-\frac{1}{2}}(\Gamma_1))$ *which satisfy the boundary value problem* (14) *with initial conditions* (15) *for* $\eta_0 \in L_2(\Gamma_1)$ *and* $v_0 \in \dot{H}_1(G)$.

In order to prove Theorem 1.1 we consider in Section 2 an auxiliary boundary value problem for which we show existence and uniqueness. This will ensure that the Dirichlet to Neumann operator is well defined. Another fact which is established is the selfadjointness and the positivity of the Dirichlet to Neumann operator which is done in section 3. We conclude in Section 4 with the proof of Theorem 1.1.

2. The elliptic problem in a corner

Consider the following auxiliary boundary value problem:

$$\begin{cases} \Delta v(x_1, x_2, t) = \frac{\partial^2 v}{\partial x_1^2}(x_1, x_2, t) + \frac{\partial^2 v}{\partial x_2^2}(x_1, x_2, t) = 0 \text{ for } (\mathrm{x}_1, \mathrm{x}_2) \in \mathrm{G}, \mathrm{t} \geq 0, \\ v|_{\Gamma_1} \quad = f \\ \frac{\partial v}{\partial \nu}|_{\Gamma_2} \quad = 0. \end{cases} \tag{19}$$

Theorem 2.1 will show that for f in a suitable Sobolev space we obtain a unique solution v for the boundary value problem (19). This allows us to define by

$$\Lambda f = \frac{\partial v}{\partial x_2}|_{\Gamma_1}$$

the so called Dirichlet to Neumann operator. From the system of boundary conditions on Γ_1,

$$\begin{cases} g\eta(x_1, t) + v_t(x_1, 0, t) = P(x_1, 0, t) \\ \eta_t(x_1, t) - v_{x_2}(x_1, 0, t) = 0 \end{cases}, \tag{20}$$

with initial conditions

$$\begin{cases} \eta(x_1, 0) \quad = \eta_0(x_1) \\ v(x_1, 0, 0) = v_0(x_1, 0) \end{cases} \tag{21}$$

we obtain by elimination of η between the above relations and using Λ the single equation in v:

$$v_{tt}(x_1,0,t) + g\Lambda v(x_1,0,t) = P_t(x_1,0,t), \tag{22}$$

with initial conditions

$$v(x_1,0,0) = v_0(x_1,0), \quad v_t(x_1,0,0) = P(x_1,0,0) - g\eta_0(x_1). \tag{23}$$

By denoting $\eta = \frac{P}{g} - \frac{v_t}{g}$ we see that $\eta_t = \frac{P_t}{g} - \frac{v_{tt}}{g}$ and from (22) we have that $\eta_t - v_{x_2} = 0$, and this shows that (20) and (22) are equivalent. Therefore we will show in Section 3 existence and uniqueness for (22) with initial conditions (23).
We first show existence and uniqueness for the boundary value problem (19). We will prove the following:

Theorem 2.1: *For any* $f \in \overset{\circ}{H}_{\frac{1}{2}}(\Gamma_1)$ *there exists a unique* $v(x_1,x_2) \in \dot{H}_1(G)$ *which solves the boundary value problem* (19).

This Theorem will be proved by showing that it is equivalent with another boundary value problem whose existence and uniqueness are proved in the paper [15]. We need first to fix some notations.
Notation:
For $t \in \mathbb{R}$ we denote

$$(t - i0)^{\frac{1}{2}-s} = \lim_{\varepsilon \to 0} e^{(\frac{1}{2}-s)\ln(t-i\varepsilon)}, \ \varepsilon > 0,$$

where we take the branch of $\ln(t - i\varepsilon)$ that is real for $t > 0$ and $\varepsilon = 0$. Now, let

$$\Lambda_-^{\frac{1}{2}-s} = \left(i\frac{\partial}{\partial_{x_1}} \cos\frac{\alpha}{2} + i\frac{\partial}{\partial x_2} \sin\frac{\alpha}{2} - i0 \right)^{\frac{1}{2}-s}$$

be a pseudodifferential operator in $\mathbb{R}^2$ with symbol

$$\Lambda_-^{\frac{1}{2}-s}(\xi_1,\xi_2) = \left(\xi_1 \cos\frac{\alpha}{2} + \xi_2 \sin\frac{\alpha}{2} - i0 \right)^{\frac{1}{2}-s},$$

cf. [15]. For the general theory of pseudodifferential operators see also [16] and [17].

Remark 1: The operator $\Lambda_-^{\frac{1}{2}-s}$ has the property that if u_- is a distribution with support in CG then the support of $\Lambda_-^{\frac{1}{2}-s} u_-$ is also in CG. An operator with such a property is called a "minus" operator with respect to the domain G. For proofs and details concerning "minus" operators see Lemma 20.2 in [14] and Lemma 2.2 in [18].

Remark 2: If A_- is a "minus" operator and u is a distribution in G we have that $p_G A_- lu$ is independent of the choice of the extension lu of u to $\mathbb{R}^2$ where p_G is the restriction operator to G.

Proof of Theorem 2.1**:**
Step 1
We first reduce to order zero boundary conditions and then apply a result from [15]. Note that due to Remark **2** it makes sense to consider the following boundary

value problem:

$$\begin{cases} \Delta u & = 0,\ (x_1,x_2)\in G \\ p_1^+\left(i\frac{\partial}{\partial x_1}-i0\right)^{s-m_1-\frac12} B_1\left(i\frac{\partial}{\partial_{x_1}}, i\frac{\partial}{\partial_{x_2}}\right)\Lambda_-^{\frac12-s} lu & = h_1(x_1),\ x_1>0 \\ p_2^-\left(-i\frac{\partial}{\partial_{x_1}}\cos\alpha - i\frac{\partial}{\partial_{x_2}}\sin\alpha + i0\right)^{s-m_2-\frac12} B_2\left(i\frac{\partial}{\partial_{x_1}}, i\frac{\partial}{\partial_{x_2}}\right)\Lambda_-^{\frac12-s} lu & = h_2(y_1),\ y_1<0 \end{cases} \tag{24}$$

where p_1^+, p_2^- are restrictions operators to Γ_1, Γ_2, respectively, $B_1(\xi_1,\xi_2), B_2(\xi_1,\xi_2)$ are homogeneous polynomials of degrees m_1, m_2 respectively, and the coordinates (y_1, y_2) are related to (x_1, x_2) through the equations

$$y_1 = -x_1\cos\alpha - x_2\sin\alpha,\ \ y_2 = x_1\sin\alpha - x_2\cos\alpha. \tag{25}$$

We now invoke Theorem 2.1 from [15] which asserts that for any $(h_1, h_2) \in L_2(\Gamma_1)\times L_2(\Gamma_2)$ the boundary value problem (24) has a unique solution $u \in \dot{H}_{\frac12}(G)$ provided s satisfies the so-called "corner condition" (2.73) from [15]. In our case B_1 is the identity operator, $B_2 = \frac{\partial}{\partial\nu} = \nu_1\frac{\partial}{\partial x_1} + \nu_2\frac{\partial}{\partial x_2}$, and a calculation shows that $s = 1$ verifies the "corner condition" (2.73) from [15]. We only state what this conditions means in our case and show that $s = 1$ verifies it. For details we ask the reader to consult the proof in [15].
We set $\lambda_1 = -i, \lambda_2 = i$. Denote by

$$\mu_j = \frac{\sin\alpha - \lambda_j\cos\alpha}{-\cos\alpha - \lambda_j\sin\alpha},\ \ j = 1, 2,$$

from which follows that $\mu_1 = -i$ and $\mu_2 = i$. We will also need the numbers β_1, β_2 given by

$$i\beta_k = \ln(\cos\alpha + \lambda_k\sin\alpha) = \ln|\cos\alpha + \lambda_k\sin\alpha| + i\arg(\cos\alpha + \lambda_k\sin\alpha),\ \mathrm{k} = 1, 2. \tag{26}$$

from which we deduce that $\beta_1 = 2\pi - \alpha$ and $\beta_2 = \alpha$.
Denote also by $B_2^{(1)}(\eta_1, \eta_2)$ the symbol of B_2 in the (y_1, y_2) coordinates. It follows from (25) that it has the form

$$B_2^{(1)}(\eta_1,\eta_2) = B_2(-\eta_1\cos\alpha+\eta_2\sin\alpha, -\eta_1\sin\alpha-\eta_2\cos\alpha) = i\nu_1(-\eta_1\cos\alpha+\eta_2\sin\alpha)+i\nu_2(-\eta_1\sin\alpha-\eta_2\cos\alpha).$$

We can now formulate the corner condition, which is the following:

$$M_0\left(z - s + \frac12\right) \neq 0, \quad \text{for any } z = \frac12 + i\tau,\ \tau\in\mathbb{R} \tag{27}$$

where

$$M_0(z) = -b_2^{(0)} + e^{2\pi i z}e^{-i\beta_1 z}e^{i\beta_2 z},$$

$$b_2^{(0)} = (B_2^+)^{-1}e^{i\pi}B_2^-,$$

$$B_2^+ = B_2^{(1)}(1,\mu_1) = B_2^{(1)}(1,-i),$$

$$B_2^- = B_2^{(1)}(-1, -\mu_2) = B_2^{(1)}(-1, -i).$$

We will show that $s = 1$ verifies the condition (27), i.e., we will show that $M_0(i\tau) \neq 0$ for all $\tau \in \mathbb{R}$. We have the following

$$B_2^+ = \nu_1 \sin\alpha - \nu_2 \cos\alpha - i(\nu_2 \sin\alpha + \nu_1 \cos\alpha),$$

$$B_2^- = \nu_1 \sin\alpha - \nu_2 \cos\alpha + i(\nu_2 \sin\alpha + \nu_1 \cos\alpha),$$

equalities which show that $|b_2^{(0)}| = 1$. Since $\beta_1 = 2\pi - \alpha$ and $\beta_2 = \alpha$ we obtain that

$$M_0(i\tau) = -b_2^{(0)} + e^{-2\alpha\tau}. \tag{28}$$

Since $\alpha > 0$, we see from equation (28) that $M_0(i\tau) \neq 0$ for all $\tau \neq 0$. Therefore it only remains to show that $b_2^{(0)} \neq 1$.
Now

$$\begin{aligned} b_2^{(0)} &= -\frac{B_2-}{B_2^+} = -\frac{\nu_1 \sin\alpha - \nu_2 \cos\alpha + i(\nu_2 \sin\alpha + \nu_1 \cos\alpha)}{\nu_1 \sin\alpha - \nu_2 \cos\alpha - i(\nu_2 \sin\alpha + \nu_1 \cos\alpha)} \\ &= \frac{(\nu_1^2 - \nu_2^2)\cos(2\alpha) + 2\nu_1\nu_2 \sin(2\alpha) + \big(2\nu_1\nu_2 \cos(2\alpha) - (\nu_1^2 - \nu_2^2)\sin(2\alpha)\big)\, i}{\nu_1^2 + \nu_2^2} \\ &= -\cos^2(2\alpha) - \sin^2(2\alpha) = -1, \end{aligned} \tag{29}$$

since $(\nu_1, \nu_2) = (-\sin\alpha, \cos\alpha)$. Thus $s = 1$ verifies the corner condition (27). Therefore the boundary value problem

$$\begin{cases} \Delta u & = 0,\ (x_1, x_2) \in G \\ p_1^+ \left(i\frac{\partial}{\partial x_1} - i0\right)^{\frac{1}{2}} \Lambda_-^{-\frac{1}{2}} l u & = h(x_1),\ x_1 > 0 \\ p_2^- \left(-i\frac{\partial}{\partial_{x_1}} \cos\alpha - i\frac{\partial}{\partial_{x_2}} \sin\alpha + i0\right)^{-\frac{1}{2}} \frac{\partial}{\partial\nu} \Lambda_-^{-\frac{1}{2}} l u & = 0,\ y_1 < 0 \end{cases} \tag{30}$$

has a unique solution $u \in \dot{H}_{\frac{1}{2}}(G)$ for any $h \in L_2(\Gamma_1)$.
Step 2 (existence and uniqueness for the problem (19))
Let $f \in \overset{\circ}{H}_{\frac{1}{2}}(\Gamma_1)$. Put now

$$h = p_1^+ \left(i\frac{\partial}{\partial x_1} - i0\right)^{\frac{1}{2}} f.$$

This implies that $h \in L_2(\Gamma_1)$. Let u be the unique solution of boundary value problem (30). We then set

$$v = p_G \Lambda_-^{-\frac{1}{2}} l u, \tag{31}$$

where p_G and l are as before. Using Lemma 2.2 from [18] we obtain that $v \in \dot{H}_1(G)$. Due to the fact that “minus operators” commute with the differential operators we have that

$$\Delta v = 0, \text{ for } (\mathrm{x}_1, \mathrm{x}_2) \in \mathrm{G} \tag{32}$$

From (31) and the second equation of (30) we see that

$$p_1^+ \left(i\frac{\partial}{\partial x_1} - i0 \right)^{\frac{1}{2}} v = h = p_1^+ \left(i\frac{\partial}{\partial x_1} - i0 \right)^{\frac{1}{2}} f.$$

It then follows that

$$p_1^+ \left(i\frac{\partial}{\partial x_1} - i0 \right)^{\frac{1}{2}} (v - f) = 0$$

which means that

$$\left(i\frac{\partial}{\partial x_1} - i0 \right)^{\frac{1}{2}} (v - f) = v_-,$$

where v_- has its support in $\mathbb{R}^1 \setminus \Gamma_1$. But then

$$v - f = \left(i\frac{\partial}{\partial x_1} - i0 \right)^{-\frac{1}{2}} (v_-)$$

and since $\left(i\frac{\partial}{\partial x_1} - i0 \right)^{-\frac{1}{2}}$ is a "minus operator" we obtain that the support of $v - f$ is contained in $\mathbb{R}^1 \setminus \Gamma_1$. This just means that

$$v|_{\Gamma_1} = f. \tag{33}$$

Using the second boundary condition in (30) and the fact that

$$p_2^- \left(-i\frac{\partial}{\partial_{x_1}} \cos\alpha - i\frac{\partial}{\partial_{x_2}} \sin\alpha + i0 \right)^{\frac{1}{2}}$$

is a "minus operator", we obtain that

$$\frac{\partial v}{\partial \nu}|_{\Gamma_2} = 0. \tag{34}$$

The relations (32), (33), and (34) prove the existence of a solution to the boundary value problem (19). In order to prove the uniqueness we will show that the boundary value problem

$$\begin{cases} \Delta v(x_1, x_2, t) &= 0 \text{ for } (\mathrm{x}_1, \mathrm{x}_2) \in \mathrm{G}, \mathrm{t} \geq 0, \\ v|_{\Gamma_1} &= 0, \\ \frac{\partial v}{\partial \nu}|_{\Gamma_2} &= 0. \end{cases} \tag{35}$$

has only the trivial solution in $\dot{H}_1(G)$. Let $v \in \dot{H}_1(G)$ be a solution of the boundary value problem (35). Denote by lv the extension by zero of v to $\mathbb{R}^2$ and put $u =$

$p_G \Lambda_-^{\frac{1}{2}} lv$. Then $u \in \overset{\circ}{H}_{\frac{1}{2}}(G)$ and u satisfies the following boundary value problem

$$\begin{cases} \Delta u & = 0, \ (x_1, x_2) \in G \\ p_1^+ \left(i\frac{\partial}{\partial x_1} - i0 \right)^{\frac{1}{2}} \Lambda_-^{-\frac{1}{2}} lu & = 0, \ x_1 > 0 \\ p_2^- \left(-i\frac{\partial}{\partial_{x_1}} \cos\alpha - i\frac{\partial}{\partial_{x_2}} \sin\alpha + i0 \right)^{-\frac{1}{2}} \frac{\partial}{\partial \nu} \Lambda_-^{-\frac{1}{2}} lu & = 0, \ y_1 < 0 \end{cases} \tag{36}$$

Since by the conclusion in Step 1 the solution to the above boundary value problem is unique, it follows that $u = 0$. Using that $u = p_G \Lambda_-^{\frac{1}{2}} lv$ and that $\Lambda_-^{\frac{1}{2}}$ is a "minus operator" it follows that $v = 0$.

Remark 3: Theorem 2.1 gives rise to an operator

$$\Lambda : \overset{\circ}{H}_{\frac{1}{2}}(\Gamma_1) \to H_{-\frac{1}{2}}(\Gamma_1)$$

defined by

$$\Lambda f := \frac{\partial v}{\partial \nu}\Big|_{\Gamma_1}, \tag{37}$$

where $v \in \dot{H}_1(G)$ is the unique solution to the boundary value problem (19). Λ is called the Dirichlet to Neumann operator.

3. Selfadjointness of the Dirichlet to Neumann operator

We are now ready to show existence and uniqueness for the equation

$$v_{tt}(x_1, 0, t) + g\Lambda v(x_1, 0, t) = P_t(x_1, 0, t), \tag{38}$$

with initial conditions

$$v(x_1, 0, 0) = v_0(x_1, 0), \quad v_t(x_1, 0, 0) = P(x_1, 0, 0) - g\eta_0(x_1). \tag{39}$$

In order to do this we will show that the operator Λ is a positive and self-adjoint operator.

Theorem 3.1: *We have that $(\Lambda f, g) = (f, \Lambda g)$ for every $f, g \in \overset{\circ}{H}_{\frac{1}{2}}(\Gamma_1)$, where $(\cdot, \cdot)$ denotes the pairing between $\overset{\circ}{H}_{\frac{1}{2}}(\Gamma_1)$ and $H_{-\frac{1}{2}}(\Gamma_1)$.*

Proof: Let $f, g \in C_0^\infty(\Gamma_1)$. Let $v \in \dot{H}_1(G)$ be the unique solution of the boundary value problem

$$\begin{cases} \Delta v & = 0 \text{ in G}, \\ v|_{\Gamma_1} & = f, \\ \frac{\partial v}{\partial \nu}|_{\Gamma_2} & = 0 \end{cases} \tag{40}$$

cf. Theorem 2.1.
Let also $u \in \dot{H}_1(G)$ be the unique solution of the boundary value problem

$$\begin{cases} \Delta u & = 0 \text{ in G}, \\ u|_{\Gamma_1} & = g, \\ \frac{\partial u}{\partial \nu}|_{\Gamma_2} & = 0 \end{cases} \tag{41}$$

cf. Theorem 2.1.
Let $\varepsilon > 0$, $N > 0$ be arbitrary positive numbers. We will now apply the first Green formula for the domain

$$G_{\varepsilon N} := \{(r,\theta) : \varepsilon \leq r \leq N,\ 0 \leq \theta \leq \alpha\},$$

and for the functions u and v. Let us also denote $C_\varepsilon := \{(\varepsilon,\theta) : 0 \leq \theta \leq \alpha\}$ and $C_N := \{(N,\theta) : 0 \leq \theta \leq \alpha\}$. We then have

$$\int\int_{G_{\varepsilon N}} v\Delta u\, dx_1 dx_2 = -\int\int_{G_{\varepsilon N}} \nabla u \nabla v\, dx_1 dx_2 + \int_{C_\varepsilon} v\frac{\partial u}{\partial \nu} d\sigma + \int_{C_N} v\frac{\partial u}{\partial \nu} d\sigma + \int_\varepsilon^N v\frac{\partial u}{\partial \nu} dx_1 \tag{42}$$

Since $\Delta u = 0$ the last equation becomes

$$\int\int_{G_{\varepsilon N}} \nabla u \nabla v\, dx_1 dx_2 = \int_{C_\varepsilon} v\frac{\partial u}{\partial \nu} d\sigma + \int_{C_N} v\frac{\partial u}{\partial \nu} d\sigma + \int_\varepsilon^N v\frac{\partial u}{\partial \nu} dx_1 \tag{43}$$

for every $\varepsilon > 0$ and every $N > 0$. We are going to prove that

$$\lim_{N\to\infty} \int_{C_N} v\frac{\partial u}{\partial \nu} d\sigma = 0. \tag{44}$$

and that

$$\lim_{\varepsilon\to 0} \int_{C_\varepsilon} v\frac{\partial u}{\partial \nu} d\sigma = 0. \tag{45}$$

We now pass to polar coordinates (r,θ) and perform the standard procedure of separation of variables. We obtain that the general solutions to the equation $\Delta v = 0$ are of the form

$$v(r,\theta) = (A\cos(\sqrt{\lambda}\theta) + B\sin(\sqrt{\lambda}\theta))(Cr^{\sqrt{\lambda}} + Dr^{-\sqrt{\lambda}})$$

We now exploit the boundary condition on Γ_1 and Γ_2.
From

$$\frac{\partial v}{\partial x_1} = \cos\theta\frac{\partial v}{\partial r} - \frac{\sin\theta}{r}\frac{\partial v}{\partial \theta} \tag{46}$$

and

$$\frac{\partial v}{\partial x_2} = \sin\theta\frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\frac{\partial v}{\partial \theta} \tag{47}$$

we obtain that

$$\frac{\partial v}{\partial \nu}\Big|_{\Gamma_2} = -\sin\alpha\frac{\partial v}{\partial x_1}\Big|_{\theta=\alpha} + \cos\alpha\frac{\partial v}{\partial x_2}\Big|_{\theta=\alpha} = \frac{\sin^2\alpha}{r}\frac{\partial v}{\partial \theta}\Big|_{\theta=\alpha} + \frac{\cos^2\alpha}{r}\frac{\partial v}{\partial \theta}\Big|_{\theta=\alpha} = \frac{1}{r}\frac{\partial v}{\partial \theta}\Big|_{\theta=\alpha}. \tag{48}$$

Since $f \in C_0^\infty(\Gamma_1)$ there exist $\varepsilon_0 > 0$ and $N_0 > 0$ such that $v\big|_{\Gamma_1} = 0$ for $x_1 < \varepsilon_0$ and $v\big|_{\Gamma_1} = 0$ for $x_1 > N_0$. From the condition $\frac{\partial v}{\partial \nu}\big|_{\Gamma_2} = 0$ we obtain utilizing (48) the separated solutions

$$v_n(r,\theta) = B_n \sin\left(n+\frac{1}{2}\right)\frac{\pi}{\alpha}\theta\left(C_n r^{(n+\frac{1}{2})\frac{\pi}{\alpha}} + D_n r^{-(n+\frac{1}{2})\frac{\pi}{\alpha}}\right). \tag{49}$$

Since $v \in \dot{H}_1(G)$ it follows that there exists an integer $n_0 > 0$ such that v has the representation

$$v(r,\theta) = \sum_{n=-\infty}^{n_0} A_n r^{(n+\frac{1}{2})\frac{\pi}{\alpha}} \sin\left(n+\frac{1}{2}\right)\frac{\pi}{\alpha}\theta, \text{ for } \mathrm{r} \geq \mathrm{N_0} \tag{50}$$

and n_0 is to be determined from the condition

$$\int\int_G |\nabla u(x_1,x_2)|^2\, dx_1 dx_2 < \infty.$$

Using relations (46) and (47) the last condition is written in polar coordinates as

$$\int_0^\alpha \int_{N_0}^\infty \left(\left(\frac{\partial v}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial v}{\partial \theta}\right)^2\right) r\, dr\, d\theta < \infty. \tag{51}$$

From the representation (50) of v we obtain, using (51), the condition

$$\int_{N_0}^\infty r^{2[(n_0+\frac{1}{2})\frac{\pi}{\alpha}-1]}\left(1+O\left(\frac{1}{r}\right)\right) r\, dr < \infty,$$

which is satisfied if and only if $2\left(n_0+\frac{1}{2}\right)\frac{\pi}{\alpha} - 2 + 1 < -1$, which is equivalent to $n_0 < -\frac{1}{2}$. Since n_0 is an integer we have that (51) is satisfied if and only if $n_0 \leq -1$. In order to prove that

$$\lim_{N\to\infty} \int_{C_N} v\frac{\partial u}{\partial \nu} d\sigma = 0. \tag{52}$$

we note first that

$$\int_{C_N} v\frac{\partial u}{\partial \nu} d\sigma = \int_0^\alpha v(N,\theta)\frac{\partial u}{\partial r}(N,\theta) N\, d\theta.$$

Since

$$\begin{aligned} v(r,\theta) &= \textstyle\sum_{n=-\infty}^{-1} A_n r^{(n+\frac{1}{2})\frac{\pi}{\alpha}} \sin\left(n+\frac{1}{2}\right)\frac{\pi}{\alpha}\theta \\ &= r^{-\frac{\pi}{2\alpha}}\left(A_{-1}\sin\left(\frac{-\pi}{2\alpha}\theta\right) + O\left(r^{\frac{-\pi}{\alpha}}\right)\right) \end{aligned} \tag{53}$$

and since a formula like (53) is true for u it suffices to show that

$$\lim_{N\to\infty} \int_0^\alpha N^{-\frac{\pi}{2\alpha}} N^{-\frac{\pi}{2\alpha}-1} N\, d\theta = 0.$$

The last equality is obviously true and therefore the equality (52) is proved.

Our next task is to prove that

$$\lim_{\varepsilon\to 0}\int_{C_\varepsilon} v\frac{\partial u}{\partial \nu}d\sigma = 0. \tag{54}$$

First of all note that

$$\int_{C_\varepsilon} v\frac{\partial u}{\partial \nu}d\sigma = \int_0^\alpha v(\varepsilon,\theta)\frac{\partial u}{\partial r}(\varepsilon,\theta)\varepsilon\, d\theta \tag{55}$$

Using again that $v \in \dot{H}_1(G)$ it follows that there exists an $\varepsilon_0 > 0$ such that v has the representation

$$v(r,\theta) = \sum_{n=n^0}^{\infty} A_n r^{(n+\frac{1}{2})\frac{\pi}{\alpha}} \sin\left(n+\frac{1}{2}\right)\frac{\pi}{\alpha}\theta, \text{ for r} \leq \varepsilon_0 \tag{56}$$

where n^0 is some fixed integer which is to be determined from the condition

$$\int\int |\nabla u(x_1,x_2)|^2\, dx_1 dx_2 < \infty,$$

which in polar coordinates is written as

$$\int_0^\alpha \int_0^{\varepsilon_0} \left(\left(\frac{\partial v}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial v}{\partial \theta}\right)^2\right) r\, dr\, d\theta < \infty. \tag{57}$$

From the representation (56) of v and using (57) we obtain the condition

$$\int_0^{\varepsilon_0} r^{2\left[\left(n^0+\frac{1}{2}\right)\frac{\pi}{\alpha}-1\right]} \left(1+O\left(r\right)\right) r\, dr < \infty,$$

which is satisfied if and only if $2\left(n^0+\frac{1}{2}\right)\frac{\pi}{\alpha} - 2 + 1 > -1$, which is equivalent to $n^0 > -\frac{1}{2}$. Since n^0 is an integer we have that (57) is satisfied if and only if $n^0 \geq 0$. Therefore,

$$\begin{aligned} v(r,\theta) &= \textstyle\sum_{n=0}^{\infty} A_n r^{(n+\frac{1}{2})\frac{\pi}{\alpha}} \sin\left(n+\frac{1}{2}\right)\frac{\pi}{\alpha}\theta \\ &= r^{\frac{\pi}{2\alpha}}\left(A_0 \sin\left(\frac{\pi}{2\alpha}\theta\right) + O\left(r^{\frac{\pi}{\alpha}}\right)\right), \end{aligned} \tag{58}$$

and a formula like (58) is also valid for u. In order to prove (54) we use (55), (58) and therefore it suffices to show that

$$\lim_{\varepsilon\to 0}\int_0^\alpha \varepsilon^{\frac{\pi}{2\alpha}}\varepsilon^{\frac{\pi}{2\alpha}-1}\varepsilon\, d\theta = 0,$$

which is true.
Passing to the limit with $\varepsilon \to 0$ and $N \to \infty$ in the formula (43) and using (54) and (52) we obtain that

$$\int\int_G \nabla u \nabla v\, dx_1 dx_2 = \int_{\Gamma_1} v\frac{\partial u}{\partial \nu}dx_1. \tag{59}$$

Analogously we obtain that

$$\int\int_G \nabla v \nabla u \, dx_1 dx_2 = \int_{\Gamma_1} u \frac{\partial v}{\partial \nu} dx_1. \tag{60}$$

From (59) and (60) we then see that $(f, \Lambda g) = (g, \Lambda f)$ for every $f, g \in C_0^\infty(\Gamma_1)$, and then we take the closure to obtain the desired result. □

Corollary 3.2: *For every non-zero* $f \in \mathring{H}_{\frac{1}{2}}(\Gamma_1)$ *we have that* $(\Lambda f, f) > 0$.

Proof: Taking $v = u$ and using $u\big|_{\Gamma_1} = f$, $\frac{\partial u}{\partial \nu}\big|_{\Gamma_1} = \Lambda f$ and (60) we obtain that

$$(\Lambda f, f) = \int\int_G |\nabla u|^2 \, dx_1 dx_2, \tag{61}$$

which proves the claim. □

Lemma 3.3: *The Dirichlet to Neumann operator is invertible and*

$$\Lambda^{-1} : H_{-\frac{1}{2}}(\Gamma_1) \to \mathring{H}_{\frac{1}{2}}(\Gamma_1)$$

is bounded.

Proof: Let $f \in \mathring{H}_{\frac{1}{2}}(\Gamma_1)$ and let $u \in \dot{H}_1(G)$ be the unique solution to the boundary value problem (19). From Proposition 3.1 of [15] we have that

$$\int\int_G |\nabla u|^2 \, dx_1 dx_2 \geq C([u(x_1, 0)]^+_{\frac{1}{2}})^2, \tag{62}$$

where $C > 0$ is a constant, and $[\cdot]^+_{\frac{1}{2}}$ denotes the norm in $H_{\frac{1}{2}}(\Gamma_1)$ cf., (17). From (62) and (61) we obtain that

$$(\Lambda f, f) \geq C([f]^+_{\frac{1}{2}})^2. \tag{63}$$

Since $(\Lambda f, f) \leq [\Lambda f]^+_{-\frac{1}{2}} [f]^+_{\frac{1}{2}}$ we have using (63) that

$$[\Lambda f]^+_{-\frac{1}{2}} \geq C[f]^+_{\frac{1}{2}}, \text{ for all } f \in \mathring{H}_{\frac{1}{2}}(\Gamma_1). \tag{64}$$

Inequality (64) implies that the range of Λ is closed. Since Λ is self-adjoint and $\ker \Lambda^* = 0$ it follows that Λ is invertible with

$$\Lambda^{-1} : H_{-\frac{1}{2}}(\Gamma_1) \to \mathring{H}_{\frac{1}{2}}(\Gamma_1).$$

bounded. □

4. The hyperbolic evolution equation on Γ_1 and the conclusion of the proof of the main theorem

Remark 1: Lemma 3.3 allows us to show existence and uniqueness for our initial problem:

$$v_{tt}(x_1, 0, t) + g\Lambda v(x_1, 0, t) = P_t(x_1, 0, t), \tag{65}$$

with initial conditions

$$v(x_1, 0, 0) = v_0(x_1, 0), \quad v_t(x_1, 0, 0) = P(x_1, 0, 0) - g\eta_0(x_1). \tag{66}$$

For simplicity, we denote $v_1(x_1) := P(x_1, 0, 0) - g\eta_0(x_1)$ and obtain the following initial problem:

$$v_{tt}(x_1, t) + g\Lambda v(x_1, t) = P_t(x_1, t), \quad v(x_1, 0) = v_0, \quad v_t(x_1, 0) = v_1. \tag{67}$$

Remark 2: In order to simplify the notation, we will write for a moment $v(x_1, t), v_t(x_1, t), v_{tt}(x_1, t)$ instead of $v(x_1, 0, t), v_t(x_1, 0, t), v_{tt}(x_1, 0, t)$ when we are on $\Gamma_1(x_2 = 0)$.

Lemma 4.1: *The solution of the homogeneous problem*

$$v_{tt}(x_1, t) + g\Lambda v(x_1, t) = 0, \quad v(x_1, 0) = v_0, \quad v_t(x_1, 0) = v_1, \tag{68}$$

is given by the formula

$$v(x_1, t) = \cos(t\underline{\Lambda}^{\frac{1}{2}})v_0 + \underline{\Lambda}^{-\frac{1}{2}} \sin(t\underline{\Lambda}^{\frac{1}{2}})v_1, \tag{69}$$

where $\underline{\Lambda} = g\Lambda$.

Proof: See for instance [19], pp. 309. □

For each $s \in \mathbb{R}$ let now $u(x_1, t; s)$ be the solution of

$$u_{tt} + \underline{\Lambda}u = 0, \;\; u(x_1, 0; s) = 0, \;\; u_t(x_1, 0; s) = P_t(x_1, 0; s).$$

From (69) it follows that

$$u(x_1, t, s) = \underline{\Lambda}^{-\frac{1}{2}} \sin(t\underline{\Lambda}^{\frac{1}{2}})P_t(x_1, 0; s). \tag{70}$$

We then have the following

Lemma 4.2: *The function defined by* $v(x_1, t) = \int_0^t u(x_1, t - s; s)\, ds$ *satisfies the boundary value problem*

$$v_{tt} + g\Lambda v = P_t(x_1, t) \;\; v(x_1, 0) = 0, \quad v_t(x_1, 0) = 0. \tag{71}$$

Proof: Clearly $v(x_1, 0) = 0$. We also have

$$v_t(x_1, t) = u(x_1, 0; t) + \int_0^t u_t(x_1, t - s; s)\, ds = \int_0^t u_t(x_1, t - s; s)\, ds,$$

which implies that $v_t(x_1, 0) = 0$. Finally, differentiating once more in t we obtain

$$\begin{aligned} v_{tt}(x_1, t) &= u_t(x_1, 0; t) + \int_0^t u_{tt}(x_1, t - s; s)\, ds \\ &= P_t(x_1, t) + \int_0^t -g\Lambda u(x_1, t - s; s)\, ds = P_t(x_1, t) - g\Lambda v(x_1, t), \end{aligned} \tag{72}$$

which proves (71). □

Corollary 4.3: *The solution of the problem* (67) *is given by the following formula*

$$v(x_1,t) = \int_0^t \underline{\Lambda}^{-\frac{1}{2}} \sin((t-s)\underline{\Lambda}^{\frac{1}{2}}) P_t(x_1,0;s)\,ds + \cos(t\underline{\Lambda}^{\frac{1}{2}})v_0 + \underline{\Lambda}^{-\frac{1}{2}}\sin(t\underline{\Lambda}^{\frac{1}{2}})v_1.$$

Proof: Adding up the solutions to the problems (68) and (71) and taking into account formula (70) we obtain the assertion. □

Remark **1** and Lemma 4.2 allow us to conclude the proof of the main Theorem 1.1. We restate it here for convenience.

Theorem 4.4: *For any* $T > 0$ *and for any* $P(x_1,x_2,t)$ *such that* $P(x_1,0,t) \in C([0,T],L_2(\Gamma_1))$ *and* $P_t(x_1,0,t) \in L_1([0,T],L_2(\Gamma_1))$ *there exist unique* $v(x_1,x_2,t) \in C([0,T],\dot{H}_1(G))$ *and* $\eta(x_1,t) \in C([0,T],L_2(\Gamma_1))$ *such that* $v(x_1,0,t) \in C([0,T],H_{\frac{1}{2}}(\Gamma_1)), v_t(x_1,0,t) \in C([0,T],L_2(\Gamma_1)), \eta_t \in C([0,T],H_{-\frac{1}{2}}(\Gamma_1))$ *which satisfy the boundary value problem*

$$\begin{cases} \Delta v(x_1,x_2,t) &= 0, & (x_1,x_2)\in G \text{ and } t\geq 0 \\ g\eta(x_1,t)+v_t(x_1,0,t) &= P(x_1,0,t), & (x_1,0)\in\Gamma_1 \text{ and } t\geq 0 \\ \eta_t(x_1,t)-v_{x_2}(x_1,0,t) &= 0, & (x_1,0)\in\Gamma_1 \text{ and } t\geq 0 \\ \frac{\partial v}{\partial \nu}(x_1,x_2,t) &= 0, & (x_1,x_2)\in\Gamma_2 \text{ and } t\geq 0 \end{cases} \tag{73}$$

with the initial conditions

$$\begin{cases} \eta(x_1,0) &= \eta_0(x_1) \\ v(x_1,x_2,0) &= v_0(x_1,x_2) \end{cases}, \tag{74}$$

where $\eta_0 \in L_2(\Gamma_1)$ *and* $v_0 \in \dot{H}_1(G)$.

Proof: We first prove the assertion about v. From Corollary 4.3 we have that

$$\underline{\Lambda}^{\frac{1}{2}} v(x_1,0,t) = \int_0^t \sin((t-s)\underline{\Lambda}^{\frac{1}{2}})P_t(x_1,0;s)\,ds + \cos(t\underline{\Lambda}^{\frac{1}{2}})\underline{\Lambda}^{\frac{1}{2}}v_0 + \sin(t\underline{\Lambda}^{\frac{1}{2}})v_1.$$

Therefore we have that

$$\max_{0\leq t\leq T} [\underline{\Lambda}^{\frac{1}{2}}v]_0^+ \leq C\left(\int_0^T [P_t]_0^+\,dt + [v_0]_{\frac{1}{2}}^+ + [v_1]_0^+\right),$$

where C is a constant. Since $[\underline{\Lambda}^{\frac{1}{2}}v]_0^+ = [v(x_1,0,t)]_{\frac{1}{2}}^+$ it follows that $v \in C([0,T],H_{\frac{1}{2}}(\Gamma_1))$.
Using again Corollary 4.3 we have that

$$v_t(x_1,0,t) = \int_0^t \left(\cos(t-s)\underline{\Lambda}^{\frac{1}{2}}\right) P_t(x_1,0;s)\,ds + \underline{\Lambda}^{\frac{1}{2}}\sin(t\underline{\Lambda}^{\frac{1}{2}})v_0 + \cos(t\underline{\Lambda}^{\frac{1}{2}})v_1,$$

from which we obtain that

$$\max_{0\leq t\leq T} [v_t]_0^+ \leq \tilde{C}\left(\int_0^T [P_t]_0^+\,ds + [v_0]_{\frac{1}{2}}^+ + [v_1]_0^+\right),$$

where $\tilde{C}$ is a constant. This shows that $v_t(x_1,0,t) \in C([0,T],L_2(\Gamma_1))$. The assertions about η follow from the conditions on Γ_1 in (73). □

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