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A regularity result for Calderón commutators

Calin Iulian Martin

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Abstract We present a regularity result for the Calderon commutator $[u, \mathcal{H}](v)$ where u, v are moduli of continuity and $\mathcal{H}$ is the Hilbert transform.

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Keywords Hilbert transform · moduli of continuity · commutators

1 Introduction

There are many instances where commutators are of interest. Some of them are the theory of pseudodifferential calculus and the theory of singular integral operators (see [5]).
We are motivated in this paper by the regularity result obtained by Constantin and Varvaruca (see Lemma B.1 in [2]) in the context of water waves. Namely, they proved that if u is $C^{1,\alpha}$ periodic function and v is a C^{α} periodic function then the commutator $[u, \mathcal{H}]v$ is again a $C^{1,\delta}$ (with $\delta < \alpha$) periodic function, where $\mathcal{H}$ is the Hilbert trasform (see below). We extend in this paper their result by replacing the Hölder regularity by the moduli of continuity. The main results are contained in Theorem 1 and Theorem 2.

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Calin Martin
Faculty of Mathematics
University of Vienna
Nordbergstrasse 15
1090 Vienna
Austria
E-mail: calin.martin@univie.ac.at

2 Commutator estimates

Let $\mathcal{H}$ denote the singular integral operator

$$\mathcal{H}(u)(t) = \frac{1}{2\pi} PV \int_{-\pi}^{\pi} \cot\left(\frac{t-s}{2}\right) u(s)\, ds, \tag{1}$$

with PV denoting a principal value integral, where the formula (1) makes sense for $u \in L_p[-\pi,\pi]$ or for $u \in C^\alpha$, in which case $\mathcal{H}$ gives rise to bounded operators

$$: L_p[-\pi,\pi] \to L_p[-\pi,\pi] \text{ for all } p \in (1,\infty),$$

and

$$: C^\alpha \to C^\alpha \text{ for all } \alpha \in (0,1),$$

cf., [1].

Definition 1 We say that a continuous, increasing function $\omega : (0,\pi] \to [0,\infty)$ is a modulus of continuity if

$$\lim_{t\to 0} \omega(t) = 0, \tag{2}$$

$$\frac{\omega(t)}{t} \text{ is decreasing on } (0,\infty), \tag{3}$$

and

$$\int_0^\pi \frac{\omega(s)}{s}\, ds < \infty. \tag{4}$$

Definition 2 Let ω be a modulus of continuity as in Definition 1. Then

(i) $C_\omega^{2\pi}$ denotes the space of continuous periodic (of period 2π) functions u on $[-\pi,\pi]$ such that $|u(x)-u(y)| \le C\omega(|x-y|)$, for some constant $C>0$.
(ii) By $C_{\omega_0}^{2\pi}$ we denote the space of continuous periodic (of period 2π) functions u on $[-\pi,\pi]$ such that $|u(x)-u(y)| \le C\omega_0(|x-y|)$, where $\omega_0(t) = \omega(t)|\log t| + \int_0^t \frac{\omega(s)}{s}\, ds$, for all $t>0$, for some constant $C>0$.

Definition 3 For $u, v \in C_\omega^{2\pi}$ we denote $\mathcal{C}(u,v) = [M_u, \mathcal{H}](v) = u\mathcal{H}(v) - \mathcal{H}(uv)$.

Remark 1 This type of commutators arise for example in the theory of traveling water waves, see for instance the recent papers [2], [3], [4].

Theorem 1 *Let $u, v \in C_\omega^{2\pi}$. Then $\mathcal{C} = \mathcal{C}(u,v) \in C_{\omega_0}^{2\pi}$.*

Proof The definition of $\mathcal{H}$ yields

$$\mathcal{C}(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cot\left(\frac{t-s}{2}\right) (u(t)-u(s))v(s)\, ds, \tag{5}$$

Since the operator defining $\mathcal{C}$ commutes with translations, it suffices to show that

$$|\mathcal{C}(t) - \mathcal{C}(0)| \le C\omega(|t|) \text{ for all } t \text{ close to } 0. \tag{6}$$

Above and in what follows below C denotes a positive constant.
Rearranging and taking into account the periodicity of the functions involved in (6) we have that

$$\mathcal{C}(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cot\left(\frac{s}{2}\right)(u(t) - u(t-s))v(t-s)\,ds \tag{7}$$

Hence,

$$2\pi[\mathcal{C}(t) - \mathcal{C}(0)] = I_1(t) + I_2(t),$$

where

$$I_1(t) = \int_{-2|t|}^{2|t|} \cot\left(\frac{s}{2}\right)\{(u(t) - u(t-s))v(t-s) - (u(0) - u(-s))v(-s)\}\;ds,$$

and

$$I_2(t) = \int_T \cot\left(\frac{s}{2}\right)\{(u(t) - u(t-s))v(t-s) - (u(0) - u(-s))v(-s)\}\;ds,$$

with $T = (-\pi, -2|t|) \cup (2|t|, \pi)$ and with t chosen such that $2|t| < \frac{1}{\pi}$. Using the inequality

$$\frac{1}{\sin^2 x} \le \psi(|x|)\frac{1}{x^2} \text{ for all } 0 < |x| < \pi, \tag{8}$$

with $\psi : (0,\pi) \to [0,\infty)$ bounded on $(0,a]$ for all $a \in (0,\pi)$, the boundedness of v and that $|u(t) - u(t-s)| \le \omega(|s|)$ yields

$$\left|\int_{-2|t|}^{2|t|} \cot\left(\frac{s}{2}\right)(u(t) - u(t-s))v(t-s)\,ds\right| \le C\int_0^{2|t|} \frac{\omega(s)}{s}\,ds \le C\omega_0(|t|). \tag{9}$$

An analogous argument gives that

$$\left|\int_{-2|t|}^{2|t|} \cot\left(\frac{s}{2}\right)(u(0) - u(-s))v(-s)\,ds\right| \le C\int_0^{2|t|} \frac{\omega(s)}{s}\,ds \le C\omega_0(|t|). \tag{10}$$

Therefore, (9) and (10) yield that $|I_1(t)| \le C\omega_0(|t|)$. In estimating I_2 we write

$$\begin{aligned} &\int_T \cot\left(\frac{s}{2}\right)[(u(t) - u(t-s))v(t-s) - (u(0) - u(-s))v(-s)]\;ds \\ &= \int_T \cot\left(\frac{s}{2}\right)\{[u(t) - u(0)]v(t-s) - [u(t-s) - u(-s)]v(-s)\}\;ds \\ &+ \int_T \cot\left(\frac{s}{2}\right)[u(0) - u(t-s)][v(t-s) - v(-s)]\,ds \end{aligned} \tag{11}$$

Using inequality (8), the boundedness of v and that $|u(t) - u(0)| \le \omega(|t|)$ we find that

$$\left|\int_T \cot\left(\frac{s}{2}\right)[u(t) - u(0)]v(t-s)\,ds\right| \le C\omega(|t|)\int_T \frac{ds}{|s|} \le C\omega(|t|)|\log|t||. \tag{12}$$

Similarly, one can show that

$$\left|\int_T \cot\left(\frac{s}{2}\right)[u(t-s)-u(-s)]v(-s)\,ds\right| \le C\omega(|t|)\int_T \frac{ds}{|s|} \le C\omega(|t|)|\log|t||. \tag{13}$$

In the same vein, using now the boundedness of u, inequality (8), and that $|v(t-s)-v(-s)| \le C\omega(|t|)$ we obtain

$$\left|\int_T \cot\left(\frac{s}{2}\right)[u(0)-u(t-s)][v(t-s)-v(-s)]\,ds\right| \le C\omega(|t|)|\log|t|| \tag{14}$$

The estimates (12), (13), and (14) yield that $|I_2(t)| \le C\omega(|t|)|\log|t|| \le C\omega_0(|t|)$.

Theorem 2 *Let u be differentiable with $u' \in C_\omega^{2\pi}$ and $v \in C_\omega^{2\pi}$. Then $\mathcal{C} = \mathcal{C}(u,v)$ is differentiable and $\mathcal{C}(u,v)' \in C_{\omega_0}^{2\pi}$*

Proof We adapt a method from [2] to the situation where the Hölder regularity is replaced by the one given by the moduli of continuity.
From the definition of $\mathcal{H}$ we have that

$$\mathcal{C}(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cot\left(\frac{t-s}{2}\right)(u(t)-u(s))v(s)\,ds, \tag{15}$$

for all $t \in \mathbb{R}$. We show first that $\mathcal{C}$ is differentiable on $\mathbb{R}$ with derivative given by formal differentiation in (15), namely:

$$\mathcal{C}'(t) = \frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{u'(t)\sin(t-s)-(u(t)-u(s))}{\sin^2\left(\frac{t-s}{2}\right)}v(s)\,ds. \tag{16}$$

We observe that the above integral is finite by noticing that it is equal to the integral

$$\frac{1}{4\pi}\int_{t-\pi}^{t+\pi} \frac{u'(t)\sin(t-s)-(u(t)-u(s))}{\sin^2\left(\frac{t-s}{2}\right)}v(s)\,ds, \tag{17}$$

about which we will show that it is finite. In doing so we write

$$u'(t)\sin(t-s)-(u(t)-u(s)) = u'(t)[\sin(t-s)-(t-s)]+u'(t)(t-s)-(u(t)-u(s)) \tag{18}$$

Using the inequalities

$$|x-\sin x| \le \frac{1}{6}|x|^3 \text{ for all } x \in \mathbb{R}, \tag{19}$$

and (8), we see that

$$\left|\frac{u'(t)[\sin(t-s)-(t-s)]}{\sin^2\left(\frac{t-s}{2}\right)}v(s)\right| \le C|u'(t)(t-s)| \le C|t-s|$$

which implies that

$$\int_{t-\pi}^{t+\pi} \left| \frac{u'(t)[\sin(t-s)-(t-s)]}{\sin^2\left(\frac{t-s}{2}\right)} v(s) \right| ds < \infty. \tag{20}$$

From the Mean Value Theorem and properties of the modulus of continuity ω we have that

$$\begin{aligned} |u'(t)(t-s)-(u(t)-u(s))| &= |u'(t)-u'(c)||t-s| \\ &\leq C|t-s|\omega(|t-c|) \leq C|t-s|\omega(|t-s|) \end{aligned} \tag{21}$$

for some $c = c(s,t)$ between s and t. Hence, using also (8) we see that

$$\left| \frac{u'(t)(t-s)-(u(t)-u(s))}{\sin^2\left(\frac{t-s}{2}\right)} \right| \leq C \frac{\omega(|t-s|)}{|t-s|} \tag{22}$$

By a change of variable we obtain that

$$\int_{t-\pi}^{t+\pi} \frac{\omega(|t-s|)}{|t-s|} ds = 2\int_0^{\pi} \frac{\omega(s)}{s} ds < \infty,$$

by requirement (4) in Definition 1. Therefore

$$\int_{t-\pi}^{t+\pi} \left| \frac{u'(t)(t-s)-(u(t)-u(s))}{\sin^2\left(\frac{t-s}{2}\right)} v(s) \right| ds < \infty. \tag{23}$$

From (20) and (23) we see that the integral in (17) is finite which implies, as argued before, that the integral in (16) is finite. We will now show that $\mathcal{C}$ is differentiable at $t = 0$. This will imply that $\mathcal{C}$ is differentiable everywhere due to the fact that $\mathcal{C}$ commutes with translations as it can be seen from (15). We see that

$$2\pi \frac{\mathcal{C}(t)-\mathcal{C}(0)}{t} = I_1(t) + I_2(t),$$

with

$$I_1(t) = \frac{1}{t}\int_{-2|t|}^{2|t|} \left[\cot\left(\frac{t-s}{2}\right)(u(t)-u(s)) - \cot\left(\frac{-s}{2}\right)(u(0)-u(s))\right] v(s)\, ds,$$

$$I_2(t) = \frac{1}{t}\int_{T} \left[\cot\left(\frac{t-s}{2}\right)(u(t)-u(s)) - \cot\left(\frac{-s}{2}\right)(u(0)-u(s))\right] v(s)\, ds,$$

where $T = (-\pi, -2|t|) \cup (2|t|, \pi)$. First, we prove that $I_1(t) \to 0$ as $t \to 0$. In doing this we rewrite $I_1(t)$ as

$$\begin{aligned} I_1(t) = \frac{1}{t}\int_{-2|t|}^{2|t|} &\left[\left(\cot\left(\frac{t-s}{2}\right) - \frac{2}{t-s} + \frac{2}{t-s}\right)(u(t)-u(s)) \right. \\ &\left. - \left(\cot\left(\frac{-s}{2}\right) - \frac{2}{-s} + \frac{2}{-s}\right)(u(0)-u(s))\right] v(s) ds, \end{aligned}$$

Let

$$
\begin{aligned}
J_1(t) = \frac{1}{t} \int_{-2|t|}^{2|t|} \Bigg[&\left(\cot\left(\frac{t-s}{2}\right) - \frac{2}{t-s} \right) (u(t) - u(s)) \\
&- \left(\cot\left(\frac{-s}{2}\right) - \frac{2}{-s} \right) (u(0) - u(s)) \Bigg] v(s) ds.
\end{aligned} \tag{24}
$$

Using the fact that

$$
\left| \cot(x) - \frac{1}{x} \right| \leq \varphi(|x|)|x| \text{ for all } x, 0 < |x| < \pi, \tag{25}
$$

with $\varphi : (0, \pi) \to [0, \infty)$ bounded function on $(0, a]$ for all $a \in (0, \pi)$, we find that

$$
\begin{aligned}
|J_1(t)| &\leq \frac{C}{|t|} \int_{-2|t|}^{2|t|} \left(\varphi\left(\frac{|t-s|}{2}\right) |t-s| + \varphi\left(\frac{|s|}{2}\right) |s| \right) ds \\
&\leq \frac{C}{|t|} \int_{-2|t|}^{2|t|} (|t-s| + |s|) \, ds \\
&= C(|t - s_0(t)| + |s_0(t)|),
\end{aligned} \tag{26}
$$

with $s_0(t) \in (-2|t|, 2|t|)$ and where the last equality was obtained by applying the mean value theorem for integrals. Since $\lim_{t \to 0} s_0(t) = 0$ we have from (26) that $\lim_{t \to 0} J_1(t) = 0$. Let

$$
J_2(t) = \frac{2}{t} \int_{-2|t|}^{2|t|} \left\{ \frac{u(t) - u(s)}{t - s} - \frac{u(0) - u(s)}{0 - s} \right\} v(s) \, ds
$$

Using the mean value theorem for integrals we obtain that

$$
J_2(t) = \frac{8|t|}{t} \left\{ \frac{u(t) - u(s_1(t))}{t - s_1(t)} - \frac{u(0) - u(s_1(t))}{0 - s_1(t)} \right\} v(s_1(t)),
$$

where $s_1(t) \in (-2|t|, 2|t|)$. Using now the mean value theorem for differentiable functions reveals that

$$
J_2(t) = \frac{8|t|}{t} \{u'(c_1(t)) - u'(c_2(t))\} v(s_1(t)),
$$

with $c_1(t), c_2(t) \in (-2|t|, 2|t|)$. From the boundedness of v and the continuity of u' it follows that the right hand side of the above equality goes to 0 as $t \to 0$. Therefore we have that $\lim_{t \to 0} I_1(t) = 0$, since $I_1(t) = J_1(t) + J_2(t)$.
We are now going to use the Dominated Convergence Theorem to prove that

$$
I_2(t) \to \int_{-\pi}^{\pi} \frac{u'(0) \sin(-s) - (u(0) - u(s))}{2 \sin^2(\frac{s}{2})} v(s) \, ds, \tag{27}
$$

for $t \to 0$. Notice that

$$
I_2(t) = \int_{-\pi}^{\pi} F(t, s) \chi_{(-\pi, -2|t|) \cup (2|t|, \pi)} \, ds,
$$

where

$$F(t,s) = \left\{ \frac{u(t)-u(0)}{t} \cot\left(\frac{t-s}{2}\right) + (u(0)-u(s)) \frac{\cot(\frac{t-s}{2}) - \cot(-\frac{s}{2})}{t} \right\} v(s)$$

where χ_T denotes the characteristic function of the set T.
Note that $F(t,s)\chi_{(-\pi,-2|t|)\cup(2|t|,\pi)}$ converges pointwise to $\frac{u'(0)\sin(-s)-(u(0)-u(s))}{2\sin^2(\frac{s}{2})} v(s)$ as $t \to 0$. Also notice that for each $s \in (-\pi,0)\cup(0,\pi)$ and for each t with $0 < |t| < \frac{|s|}{2}$ we obtain from the mean value theorem that

$$\begin{aligned} &\frac{1}{t}\left[\cot\left(\frac{t-s}{2}\right)(u(t)-u(s)) - \cot\left(\frac{-s}{2}\right)(u(0)-u(s))\right] \\ &\quad = u'(q)\cot\left(\frac{q-s}{2}\right) - \frac{1}{2\sin^2\left(\frac{q-s}{2}\right)}(u(q)-u(s)) \\ &\quad = \frac{u'(q)\sin(q-s) - (u(q)-u(s))}{2\sin^2\left(\frac{q-s}{2}\right)} \end{aligned} \tag{28}$$

with $q = q(t,s)$ between 0 and t. We use (19) and (8) and obtain

$$\begin{aligned} &\left|\frac{u'(q)\sin(q-s) - (u(q)-u(s))}{2\sin^2\left(\frac{q-s}{2}\right)}\right| \\ &\le C\left|\frac{u'(q)(q-s) - (u(q)-u(s))}{(q-s)^2}\right| + C|q-s| \\ &= C\left|\frac{u'(q)-u'(q_0)}{q-s}\right| + C|q-s| \\ &\le C\frac{\omega(|q-q_0|)}{|q-s|} + C|q-s| \\ &\le C\frac{\omega(|q-s|)}{|q-s|} + C|q-s|, \end{aligned} \tag{29}$$

where we used that q_0 is between q and s, $0 < |t| < |s|/2$ and $u' \in C_\omega$. For $s > 0$ and since $0 < q < t < \frac{s}{2} < s$ we have that $|q-s| = s-q > s-t > \frac{|s|}{2}$. Analogously, for $s < 0$ and since $s < \frac{s}{2} < t < q < 0$ we have that $|q-s| = q-s > t-s > -\frac{s}{2} = \frac{|s|}{2}$. Therefore $\frac{\omega(|q-s|)}{|q-s|} \le \frac{\omega(|s|/2)}{|s|/2}$. The conditions of the Dominated Convergence Theorem are thus checked since the function $s \to \frac{\omega(|s|)}{|s|}$ is integrable and v is bounded on $(-\pi,\pi)$. Hence (27) holds and, together with $\lim_{t\to 0} I_1(t) = 0$, it follows that $\mathcal{C}$ is differentiable at $t=0$ with $\mathcal{C}'(0)$ given by formula (16) at $t=0$. As already explained, this implies that $\mathcal{C}$ is differentiable on $\mathbb{R}$.
We now prove that $\mathcal{C}' \in C^{2\pi}_{\omega_0}$. It suffices to show that

$$|\mathcal{C}'(t) - \mathcal{C}'(0)| \le \omega_0(|t|) \text{ for all } t \text{ close to } 0.$$

Through changing of variables, rearranging and using the periodicity we obtain the formula

$$\mathcal{C}'(t) = \frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{u'(t)\sin(s) - (u(t) - u(t-s))}{\sin^2\left(\frac{s}{2}\right)} v(t-s)\,ds, \tag{30}$$

for all $t \in \mathbb{R}$. Then

$$4\pi(\mathcal{C}'(t) - \mathcal{C}'(0)) = T_1(t) - T_2(t) + T_3(t), \tag{31}$$

with

$$\begin{aligned} T_1(t) &= \int_{-2|t|}^{2|t|} \frac{u'(t)\sin(s) - (u(t) - u(t-s))}{\sin^2\left(\frac{s}{2}\right)} v(t-s)\,ds, \\ T_2(t) &= \int_{-2|t|}^{2|t|} \frac{u'(0)\sin(s) - (u(0) - u(-s))}{\sin^2\left(\frac{s}{2}\right)} v(-s)\,ds, \\ T_3(t) &= \int_{(-\pi,-2|t|)\cup(2|t|,\pi)} \left\{ \frac{u'(t)\sin(s) - (u(t) - u(t-s))}{\sin^2\left(\frac{s}{2}\right)} v(t-s) \right. \\ &\quad \left. - \frac{u'(0)\sin(s) - (u(0) - u(-s))}{\sin^2\left(\frac{s}{2}\right)} v(-s) \right\} ds. \end{aligned} \tag{32}$$

We write

$$\begin{aligned} u'(t)\sin(s) - (u(t) - u(t-s)) &= u'(t)(\sin(s) - s) + su'(t) - (u(t) - u(t-s)) \\ &= u'(t)(\sin(s) - s) + \int_{-s}^{0} [u'(t) - u'(t+\tau)]\,d\tau \end{aligned} \tag{33}$$

Therefore

$$\begin{aligned} |T_1(t)| &\le \int_{-2|t|}^{2|t|} |u'(t)||v(t-s)| \frac{|\sin(s) - s|}{\sin^2\left(\frac{s}{2}\right)}\,ds \\ &\quad + \int_{-2|t|}^{2|t|} |v(t-s)| \frac{\left|\int_{-s}^{0}[u'(t) - u'(t+\tau)]\,d\tau\right|}{\sin^2\left(\frac{s}{2}\right)}\,ds. \end{aligned} \tag{34}$$

Using that $|u'(t) - u'(t+\tau)| \le \omega(|\tau|)$ and that ω is increasing it follows via the Mean Value Theorem for integrals that

$$\left| \int_{-s}^{0} [u'(t) - u'(t+\tau)]\,d\tau \right| \le |s|\omega(|s|). \tag{35}$$

From the boundedness of u' and v, inequalities (34), (35), (19), (8) we have that

$$\begin{aligned} |T_1(t)| &\le C\left(\int_{-2|t|}^{2|t|} |s|\,ds + \int_{-2|t|}^{2|t|} \frac{\omega(|s|)}{|s|}\,ds \right) \\ &\le C\left(|t|^2 + \int_{0}^{2|t|} \frac{\omega(|s|)}{|s|}\,ds \right) \le C\omega_0(t). \end{aligned} \tag{36}$$

Analogously one can prove that $|T_2(t)| \leq C\omega_0(t)$. We now write $T_3(t) = R_1(t) + R_2(t)$ where

$$R_1(t) = \int_T \left\{ \frac{[u'(t) - u'(0)]\sin(s)}{\sin^2\left(\frac{s}{2}\right)} v(t-s) \right.$$
$$\left. + \frac{[u(0) - u(-s)] - [u(t) - u(t-s)]}{\sin^2\left(\frac{s}{2}\right)} v(t-s) \right\} ds$$

$$R_2(t) = \int_T \frac{u'(0)\sin(s) - [u(0) - u(-s)]}{\sin^2\left(\frac{s}{2}\right)} [v(t-s) - v(-s)]\, ds, \qquad (37)$$

where $T = (-\pi, -2|t|) \cup (2|t|, \pi)$. Concerning R_1 we have using (8) that

$$|R_1(t)| \leq C \int_T |v(t-s)| \left\{ \frac{|u'(t) - u'(0)|}{|s|} + \frac{|\int_{-s}^{0}[u'(\tau) - u'(t+\tau)]\, d\tau|}{|s|^2} \right\} ds \qquad (38)$$

Since $|u'(\tau) - u'(t+\tau)| \leq \omega(|t|)$ we obtain from (38) and from the boundedness of v that

$$|R_1(t)| \leq C\omega(|t|) \int_T \frac{ds}{|s|} \leq C\omega(|t|)|\log|t||,$$

for all t with $2|t| < \frac{1}{\pi}$. Using now (8) we have

$$|R_2(t)| \leq C \int_T |v(t-s) - v(-s)| \left\{ \frac{|u'(0)|}{|s|} + \frac{|\int_{-s}^{0} u'(\tau)\, d\tau|}{|s|^2} \right\} ds. \qquad (39)$$

Since $v \in C_\omega$ we have that $|v(t-s) - v(-s)| \leq \omega(|t|)$. The latter inequality, the boundedness of u' and (39) yield that

$$|R_1(t)| \leq C\omega(|t|) \int_T \frac{ds}{|s|} \leq C\omega(|t|)|\log|t||,$$

for all t with $2|t| < \frac{1}{\pi}$.

Remark 2 Note that for $\alpha \in (0,1)$, the function $\omega_\alpha(t) = t^\alpha$ satisfies all the properties required by Definition 1. Thus the Hölder regularity results in [2] are particular cases of ours. To see that our result is more general, let us impose the following condition

$$\omega(t) \geq t^\alpha \text{ for all } t \text{ close to } 0, \text{ and all } \alpha \in (0,1), \qquad (40)$$

in addition to those in Definition 1. The reason for this is that whenever a modulus of continuity is bounded above near $t = 0$ by some ω_α, one may replace it by ω_α. New classes of functions are opened up by means of (40). Indeed, an example of ω satisfying (40), as well as the requirements of Definition 1, is the function

$$\omega(t) = \frac{1}{(2 + |\log(\frac{t}{\pi})|)^{1+\varepsilon}},$$

where $\varepsilon \in (0,1)$. Indeed, it is easy to see that $\omega(t)$ is increasing on $(0,\pi]$ and $\lim_{t\to 0}\omega(t) = 0$. Note that the property (3) is equivalent to showing that the function $f(t) := t(2+|\log(t/\pi)|)^{1+\varepsilon}$ is increasing. Computing the derivative of the latter function we find that

$$f'(t) = \begin{cases} (2-\log(\frac{t}{\pi}))^{\varepsilon}(1-\log(\frac{t}{\pi})-\varepsilon),\ t<\pi \\ (2+\log(\frac{t}{\pi}))^{\varepsilon}(3+\log(\frac{t}{\pi})+\varepsilon),\ t>\pi \end{cases} \tag{41}$$

Therefore we have that $f'(t) > 0$ for all $t > 0, t \neq \pi$. Hence f is increasing on $(0,\pi)\cup(\pi,\infty)$ and since f is continuous it follows that f is increasing on $(0,\infty)$ which implies that the function $t \to \frac{\omega(t)}{t}$ is decreasing on $(0,\infty)$.
We also have $\int_0^{\pi}\frac{\omega(s)}{s}\,ds = \int_0^{\infty}\frac{e^{-u}}{e^{-u}(2+|u|)^{1+\varepsilon}}\,du = \int_0^{\infty}\frac{du}{(2+u)^{1+\varepsilon}} < \infty$. This proves the property (4).
We now check (40). Note that $\frac{1}{(2+|\log(t/\pi)|)^{1+\varepsilon}} \geq t^{\alpha}$ is equivalent to $t^{\alpha}(2+|\log(t/\pi)|)^{1+\varepsilon} \leq 1$ for $t>0$ and sufficiently close to 0. From

$$t^{\alpha}(2+|\log(t/\pi)|)^{1+\varepsilon} \leq t^{\alpha}(2+|\log(t/\pi)|)^{2}$$

for t sufficiently small and using that $\lim_{t\to 0} t^{\alpha}(2+|\log(t/\pi)|)^2 = 0$ we obtain that (40) holds true.

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