EQUATORIAL WIND WAVES WITH CAPILLARY EFFECTS AND STAGNATION POINTS

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Abstract. We consider here waves that propagate in equatorial oceanic regions. By means of Crandall-Rabinowitz bifurcation theory we prove existence of steady periodic two-dimensional equatorial geophysical water waves with capillary effects and stagnation points. We also show that if the vorticity is big enough these flows possess stagnation points. Moreover, we prove that the free surface has a priori regularity. The dispersion relation, i.e. a formula giving the speed at the free surface of the bifurcating laminar flow (in terms of the constant vorticity, mean depth, wave number, Earth's rotation speed), is also provided.

1. Introduction

The equatorial region throughout the extent of the Pacific Ocean (about $13,000$ km) presents several peculiarities. One of them is the presence of the Equatorial Undercurrent (EUC), cf. [21], characterized by vertical variations near the surface (at great depths the water being motionless); while, owing to winds that blow westward, the surface water flow is directed westward, cf. [14], the flow reverses at a depth of several tens of meters according to [30]. Another feature of the waves in the equatorial region is the smallness of the variation in latitude of the EUC. The latter implies that the variations of the Coriolis parameter can be neglected and one can use the f-plane approximation (see the discussion in [4] for specific features in the equatorial region and [16] in a general context).

In the last decade there have appeared various papers involving rotational water waves on topics like existence [10], regularity of the free surface and of the stream lines [9, 15, 17, 18], symmetry [7, 8, 28], and the important issue of stability [11, 20] . Nevertheless, rigorous results which incorporate the Earth's rotation in the equations of motion are very recent [3, 4, 5, 6, 19, 26, 27] and concern gravitational water waves without stagnation points. We consider in this paper equatorial geophysical water waves where we also include the action of surface tension as a restoring force and we allow for stagnation points (whose existence we will in fact prove) in the flow.

Let us give in the sequel a general presentation of the free boundary value problem we will be working with.

We choose a rotating framework having the origin at a point on the Earth's surface, with the X-axis pointing horizontally to the east, the X-axis horizontally to the north and the Z-axis upward. We are also looking for two-dimensional flows which are independent of the Y-coordinate and possessing vanishing horizontal velocity along the Y-axis. The fluid domain Ω in the XZ-plane is bounded below by the impermeable flat bed

$$\mathcal{B} = \{(X, 0); X \in \mathbb{R}\},$$

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and above by an a priori unknown curve

$$\mathcal{S}(t) = \{(a(t,s), b(t,s)); s \in \mathbb{R}\}, t \geq 0 \tag{1.1}$$

with

$$(a_s(t,s))^2 + (b_s(t,s))^2 > 0 \text{ for all } s \in \mathbb{R}, t \geq 0 \tag{1.2}$$

and

$$a(t, s+L) = a(t,s) + L,\ b(t, s+L) = b(t,s) \text{ for all } s \in \mathbb{R}, \tag{1.3}$$

representing the free surface of the water (not necessary the graph of a function), which is L-periodic in the horizontal direction. According to [4] the governing equations in the f-plane approximation near the Equator are the Euler equations.

$$\begin{cases} u_t + uu_X + wu_Z + 2\omega w &= -P_X \\ w_t + uw_X + ww_Z - 2\omega u &= -P_Z - g, \end{cases} \tag{1.4}$$

where t is the time, $\omega = 73 \cdot 10^{-6}\, rad/s$ is the constant rotational speed of the Earth round the polar axis towards the east, g denotes the gravitational constant, P stands for the pressure, and $(u, 0, w)$ is the Y-independent two-dimensional velocity field. The Euler equations are supplemented by the equation of mass conservation

$$u_X + w_Z = 0, \tag{1.5}$$

which is a consequence of the assumption of constant density. The flows in our setting are assumed to possess a non-vanishing constant vorticity

$$\gamma = w_X - u_Z$$

which is preserved by the vorticity equation (see [29]). The assumption of non-vanishing vorticity is required by the presence of the non-uniform current (EUC). The constant vorticity is not only a mathematical convenience but can also be justified on physical grounds cf. the discussion in [4]. Moreover, we seek steady traveling waves, i.e., waves whose velocity field, pressure and the free surface present an (X, t) dependence of the form $(X - ct)$ with $c < 0$, where $|c|$ is the westward propagation speed of the surface wave. The boundary conditions associated to (1.4) and (1.5) are of two types: dynamic and kinematic boundary conditions. The dynamic boundary condition incorporates the stresses that the atmosphere exerts on the fluid surface and the effect of the surface tension-a force per unit length due to a pressure difference across a curved surface, cf. [22]. It therefore takes the form

$$P(a(t,s), b(t,s), t) = P_0 - \sigma \frac{a_s(t,s)b_{ss}(t,s) - a_{ss}(t,s)b_s(t,s)}{((a_s(t,s))^2 + (b_s(t,s))^2)^{3/2}} \text{ on } \mathcal{S}, \tag{1.6}$$

where P_0 being the constant atmospheric pressure, $\sigma > 0$ is the coefficient of surface tension and $\frac{a_s(t,s)b_{ss}(t,s) - a_{ss}(t,s)b_s(t,s)}{((a_s(t,s))^2 + (b_s(t,s))^2)^{3/2}}$ represents the mean curvature of $\mathcal{S}$. The kinematic boundary conditions require that the free surface and the bed always consist of the same fluid particles (see the discussion in [2] for details). If $S(X, Z, t) = 0$ is the implicit equation of the free surface, the kinematic boundary conditions can be expressed as:

$$\begin{cases} S_t + S_X u + S_Z w &= 0 \text{ on } \mathcal{S}, \\ w = 0 \text{ on } \mathcal{B}. \end{cases} \tag{1.7}$$

The change of variables $x = X - ct, z = Z$ transforms (1.4)-(1.7) to the stationary problem

$$\begin{cases} (u-c)u_x + wu_z + 2\omega w &= -P_x \\ (u-c)w_x + ww_z - 2\omega u &= -P_z - g \\ \quad u_x + w_z = 0 & \\ \quad w_x - u_z &= \gamma \end{cases} \quad \text{in } \Omega \tag{1.8}$$

together with

$$\begin{cases} S_x(u-c) + S_z w = 0 & \text{on } \mathcal{S}, \\ P = P_0 - \sigma \frac{a_s b_{ss} - a_{ss} b_s}{((a_s)^2 + (b_s)^2)^{3/2}} & \text{on } \mathcal{S}, \\ w = 0 & \text{on } \mathcal{B}. \end{cases} \tag{1.9}$$

In order to reformulate (1.8) and (1.9) we introduce the stream function ψ by means of the following line integral

$$\psi(x,z) = -m + \int_{(0,0)}^{(x,z)} (-w(x,z))dx + (u(x,z) - c)dz,$$

for some constant m. The path-independence of the above line integral is ensured by the equation of mass conservation (1.5) provided the path is in the simply connected domain Ω, where Ω is the set of all $(x,z) \in \mathbb{R}^2$ above the line $z = 0$ bounded above by the curve $\{(x,z) \in \mathbb{R}^2 : S(x,z) = 0\}$. The definition of ψ implies immediately that

$$\psi_x = -w \ \text{ and } \ \psi_z = u - c \text{ in } \Omega. \tag{1.10}$$

Moreover, repeating the arguments we provided in [25] in a context that ignored geophysical effects, one can see that

$$\psi = 0 \text{ on } \mathcal{S}$$

and

$$\psi = -m \text{ on } \mathcal{B},$$

where (the constant) m is the relative mass flux defined through

$$m = \int_0^{b(t,s_0)} \left[u(a(t,s_0) - ct, z) - c\right] dz,$$

with $(a(t,s_0) - ct, b(t,s_0))$ being the wave trough. We are now able to reformulate (1.4)-(1.7) by means of ψ as the problem

$$\begin{cases} \psi_z\psi_{xz} - \psi_x\psi_{zz} - 2\omega\psi_x = -P_x \\ -\psi_z\psi_{xx} + \psi_x\psi_{xz} - 2\omega(\psi_z + c) = -P_z - g \\ \Delta\psi = -\gamma \end{cases} \quad \text{in } \Omega \tag{1.11}$$

with the boundary conditions

$$\begin{cases} P = P_0 - \sigma \frac{a_s(t,s) b_{ss}(t,s) - a_{ss}(t,s) b_s(t,s)}{((a_s(t,s))^2 + (b_s(t,s))^2)^{3/2}} & \text{on } \mathcal{S}, \\ \psi = 0 & \text{on } \mathcal{S}, \\ \psi = -m & \text{on } \mathcal{B}. \end{cases} \tag{1.12}$$

Properties of ψ and (1.8) ensure the Bernoulli law, which states that

$$E := \frac{\psi_x^2 + \psi_z^2}{2} + (g - 2\omega c)z - (2\omega - \gamma)\psi + P$$

is constant throughout the fluid domain. On the free surface $\mathcal{S}$ we have

$$E := \frac{\psi_x^2+\psi_z^2}{2} + (g-2\omega c)b + P_0 - \sigma\frac{a_s(t,s)b_{ss}(t,s)-a_{ss}(t,s)b_s(t,s)}{((a_s(t,s))^2+(b_s(t,s))^2)^{3/2}}.$$

Setting $Q := 2(E-P_0)$ we obtain, after choosing a parametrization of the fluid domain for which a and b are independent of t in the moving frame, the following free boundary value problem for ψ:

$$\left\{\begin{array}{rcll} \Delta\psi &=& -\gamma & \text{in } \Omega,\\ \psi &=& -m & \text{on } \mathcal{B},\\ \psi &=& 0 & \text{on } \mathcal{S},\\ |\nabla\psi|^2 - 2\sigma\frac{a_s b_{ss}-a_{ss}b_s}{(a_s^2+b_s^2)^{3/2}} + 2(g-2\omega c)b &=& Q & \text{on } \mathcal{S}. \end{array}\right. \tag{1.13}$$

Let us now summarize the content of the paper and comment on the mathematical difficulties raised by the study of problem (1.13). First of all an equivalent formulation of the free boundary problem (1.13) is required so that we end up with a problem over a fixed domain. Allowing for stagnation points and overhanging profiles hinders the use of the Dubreil-Jacotin transformation, a tool very much utilized in previous investigations concerning water waves. We choose instead a more recent approach [1] employed in the case of irrotational water waves and extended to the case of any constant vorticity γ in [12] in the context of gravity waves. This new approach reformulates the problem as a quasi differential equation for a periodic function of one real variable, which gives the elevation of the free surface when the fluid domain is sought to be the conformal image of a half-plane. We present this new reformulation in Section 2 adapting it to our scenario of equatorial capillary-gravity water waves with stagnation points and overhanging profiles. We then use in Section 3 bifurcation tools in the spirit of Crandall-Rabinowitz to prove the existence of solutions to (1.13). Section 4 is concerned with establishing the existence of stagnation points. We show in Section 5 that the surface tension has a smoothing effect on the free surface. Namely, we prove that the free surface is C^∞ regular.

2. Reformulation as a quasilinear equation in a fixed domain

We start by a few notations and definitions of the function spaces we will be working with. For $p\in\mathbb{N}$ and $\alpha\in(0,1)$ we set $C^{p,\alpha}$ to be the space of functions whose partial derivatives up to order p are Hölder continuous with exponent α over their domain of definition. By $C^{p,\alpha}_{\text{loc}}$ we mean the functions of class $C^{p,\alpha}$ over any compact subset of their domain of definition. By $C^{p,\alpha}_{2\pi}$ we denote the space of functions of one real variable which are 2π periodic and of class $C^{p,\alpha}_{\text{loc}}$ in $\mathbb{R}$. Finally $C^{p,\alpha}_{2\pi,o}$ will denote the space of functions that are in $C^{p,\alpha}_{2\pi}$ and have zero mean over one period.
For $d>0$ let $S_d=\{(x,z)\in\mathbb{R}^2 : -d<z<0\}$. For any $\mathfrak{b}\in C^{p,\alpha}_{2\pi}$ let $\mathfrak{B}\in C^{p,\alpha}(\overline{S}_d)$ be the unique solution of

$$\begin{array}{rcl} \Delta\mathfrak{B} &=& 0 \text{ in } S_d,\\ \mathfrak{B}(x,-d) &=& 0,\ x\in\mathbb{R},\\ \mathfrak{B}(x,0) &=& \mathfrak{b}(x),\ x\in\mathbb{R}. \end{array} \tag{2.1}$$

For $p\in\mathbb{N}$, $\alpha\in(0,1)$ we define the *periodic Dirichlet-Neumann operator associated to the strip S_d* by

$$\mathcal{G}_d(\mathfrak{b})(x) = \mathfrak{B}_z(x,0),\ x\in\mathbb{R}.$$

Remark 2.1. (i) The operator $\mathcal{G}_d : C^{p,\alpha}_{2\pi} \to C^{p-1,\alpha}_{2\pi}$ is a bounded linear operator for all $p \in \mathbb{N}, p \geq 1$ and all $\alpha \in (0,1)$.

(ii) If w is the constant function taking the value s we have that

$$\mathcal{G}_d(s) = \frac{s}{d}. \tag{2.2}$$

Let $\mathfrak{A}$ be the unique (up to a constant) harmonic function in S_d, such that $\mathfrak{A} + i\mathfrak{B}$ is holomorphic in S_d. If $\mathfrak{b} \in C^{p,\alpha}_{2\pi,o}$ it follows from the discussion in Section 2 of [12] that the function $(x,z) \to \mathfrak{A}(x,z)$ is 2π-periodic in x throughout S_d. We specify the constant in the definition of $\mathfrak{A}$ by asking that $x \to \mathfrak{A}(x,0)$ has zero mean over one period. We define $\mathcal{C}_d(\mathfrak{b})$ by

$$\mathcal{C}_d(\mathfrak{b})(x) = \mathfrak{A}(x,0),\ x \in \mathbb{R}.$$

The map $\mathcal{C}_d$ is called the periodic Hilbert transform for a strip and is a bounded linear operator from $C^{p,\alpha}_{2\pi,o}$ into itself. The following properties for $\mathfrak{b} \in C^{p,\alpha}_{2\pi,o}$ (with $p \geq 1$), cf. [12], will prove to be useful throughout the paper.

$$\begin{aligned} \mathcal{G}_d(\mathfrak{b}) &= (\mathcal{C}_d(\mathfrak{b}))' = \mathcal{C}_d(\mathfrak{b}') \\ \mathcal{G}_d(\mathfrak{b}) &= \tfrac{[\mathfrak{b}]}{d} + \mathcal{C}_d(\mathfrak{b}'), \end{aligned} \tag{2.3}$$

where $[\mathfrak{b}]$ denotes the average of $\mathfrak{b}$ over one period. We present now a couple of properties of the operator $\mathcal{C}_d$. Mostly, we only formulate the results as given in[12].

Properties of the Dirichlet-Neumann map and of the Hilbert transform. Denote by $L^2_{2\pi}$ the space of 2π-periodic locally square integrable functions of one real variable. By $L^2_{2\pi,\mathrm{o}}$ we denote the subspace of $L^2_{2\pi}$ whose elements have zero mean over one period.

Lemma 2.2. *If*

$$\mathfrak{b} = \sum_{n=1}^{\infty} c_n \cos(nx) + \sum_{n=1}^{\infty} d_n \sin(nx), \tag{2.4}$$

is the Fourier series expansion of $\mathfrak{b} \in L^2_{2\pi,o}$ *then*

$$\mathcal{C}_d(\mathfrak{b}) = \sum_{n=1}^{\infty} c_n \coth(nd)\sin(nx) - \sum_{n=1}^{\infty} d_n \coth(nd)\cos(nx) \tag{2.5}$$

Setting $d = \infty$ in (2.5) we obtain for $\mathfrak{b} \in L^2_{2\pi,o}$ the familiar periodic Hilbert transform of $\mathfrak{b}$ defined through the relation

$$\mathcal{C}(\mathfrak{b}) = \sum_{n=1}^{\infty} c_n \sin(nx) - \sum_{n=1}^{\infty} d_n \cos(nx),$$

where the coefficients c_n and d_n are those from the Fourier expansion (2.4).

Remark 2.3. It is immediate that $\mathcal{C}(\mathcal{C}(\mathfrak{b})) = -\mathfrak{b}$ for all $\mathfrak{b} \in L^2_{2\pi,o}$.

Lemma 2.4. *For any* $d > 0$, $p \geq 0$ *integer and* $\alpha \in (0,1)$, $\mathcal{C}_d = \mathcal{C} + \mathcal{S}_d : C^{p,\alpha}_{2\pi,o} \to C^{p,\alpha}_{2\pi,o}$ *is a bounded linear operator and moreover,* $\mathcal{C}_d^{-1} = -\mathcal{C} + \tilde{\mathcal{S}}_d : C^{p,\alpha}_{2\pi,o} \to C^{p,\alpha}_{2\pi,o}$ *is also a bounded linear operator, whereby* $\mathcal{S}_d$ *and* $\tilde{\mathcal{S}}_d$ *are smoothing operators, i.e. operators mapping* $C^{p,\alpha}_{2\pi}$ *to* C^{∞}.

Proof. The assertion about $\mathcal{C}_d$ is proven in [12] where it is also shown that $\mathcal{C}_d = \mathcal{C} + \mathcal{S}_d$. We set now

$$\tilde{\mathcal{S}}_d := (\mathcal{C} + \mathcal{S}_d)^{-1}\mathcal{S}_d\mathcal{C}. \tag{2.6}$$

Since $\mathcal{S}_d$ is a smoothing operator, the same is true about $\tilde{\mathcal{S}}_d$ as one can easily see from the mapping properties of $\mathcal{S}_d$. It remains to prove that $-\mathcal{C} + \tilde{\mathcal{S}}_d$ is the inverse of $\mathcal{C}_d$. Denoting with I the identity operator and making use of the fact that $\mathcal{C}^{-1} = -\mathcal{C}$ we compute

$$(\mathcal{C} + \mathcal{S}_d)(-\mathcal{C} + (\mathcal{C} + \mathcal{S}_d)^{-1}\mathcal{S}_d\mathcal{C}) = I - \mathcal{S}_d\mathcal{C} + \mathcal{S}_d\mathcal{C} = I,$$

$$(-\mathcal{C} + (\mathcal{C} + \mathcal{S}_d)^{-1}\mathcal{S}_d\mathcal{C})(\mathcal{C} + \mathcal{S}_d) = I - \mathcal{C}\mathcal{S}_d + (\mathcal{C} + \mathcal{S}_d)^{-1}(-\mathcal{S}_d + \mathcal{S}_d\mathcal{C}\mathcal{S}_d) \tag{2.7}$$

$$= I - \mathcal{C}\mathcal{S}_d + (\mathcal{C} + \mathcal{S}_d)^{-1}(\mathcal{C} + \mathcal{S}_d)\mathcal{C}\mathcal{S}_d = I. \tag{2.8}$$

The last two equalities show that indeed $\mathcal{C}_d^{-1} = -\mathcal{C} + \tilde{\mathcal{S}}_d$ with $\tilde{\mathcal{S}}_d$ given by (2.6). □

Lemma 2.5. *Let $\mathfrak{b} \in C^{p,\alpha}_{2\pi}$ with $p \geq 1$ an integer and $\alpha \in (0,1)$. Let $\mathcal{Q}_d$ denote the mapping*

$$\mathfrak{b} \to \mathcal{Q}_d(\mathfrak{b}) = \mathfrak{b}\mathcal{C}_d(\mathfrak{b}') - \mathcal{C}_d(\mathfrak{b}\mathfrak{b}'),$$

We then have that $\mathcal{Q}_d(\mathfrak{b}) \in C^{p,\delta}_{2\pi}$ for any $\delta \in (0,\alpha)$.

Lemma 2.6. *Let $p \geq 1$ be an integer, $\alpha \in (0,1)$ and $d > 0$. If $f \in C^{p,\alpha}_{2\pi}$ and $g \in C^{p-1,\alpha}_{2\pi}$ then*

$$f\mathcal{C}_d(g) - \mathcal{C}_d(fg) \in C^{p,\delta}_{2\pi} \text{ for all } \delta \in (0,\alpha).$$

Proof. The proof follows the line of the proofs of Lemma 3.2 and of Lemma B1 from the paper [12]. □

The reformulation of the original free boundary value problem.

Definition 2.7. *We say that a solution (Ω, ψ) of the water wave equation* (1.13) *is of class $C^{2,\alpha}$ if the free surface satisfies* (1.1), (1.2) *and* (1.3) *with $a, b \in C^{2,\alpha}$ and $\psi \in C^\infty(\Omega) \cap C^{2,\alpha}(\overline{\Omega})$.*

The following definition concerns the so-called *conformal mean depth* which recalls the notion of the classical conformal modulus for doubly connected domains, see [31].

Definition 2.8.
- *We say that $\Omega \subset \mathbb{R}^2$ is an L-periodic strip like domain if it is contained in the upper half (X,Z)-plane and if its boundary consists of the real axis $\mathcal{B}$ and a parametric curve $\mathcal{S}$ defined by* (1.1) *and which satisfies* (1.2) *and* (1.3).
- *For any such domain, the conformal mean depth is defined to be unique positive number h such that there exists an onto conformal mapping $\tilde{A} + i\tilde{B} : \mathcal{R}_h \to \Omega$ which admits an extension between the closures of these domains, with onto mappings*

$$\{(x,0) : x \in \mathbb{R}\} \to \mathcal{S},$$

and

$$\{(x,-h) : x \in \mathbb{R}\} \to \mathcal{B},$$

and such that

$$\begin{aligned} \tilde{A}(x+L,z) &= \tilde{A}(x,z) + L, \\ \tilde{B}(x+L,z) &= \tilde{B}(x,z), \end{aligned} \quad (x,z) \in \mathcal{R}_h \tag{2.9}$$

For a proof of the existence and uniqueness of such an h we refer the reader to Appendix A of the paper [12].

Theorem 2.9. *Let ψ be a solution of* (1.13) *within Ω and let $k = \frac{2\pi}{L}$. Then there are an $h > 0$, a positive function $b \in C^{2,\alpha}_{2\pi}$ with $[b] = h$ such that*

$$\left\{\frac{m}{kh} + \gamma\left(\mathcal{G}_{kh}(\frac{b^2}{2}) - b\mathcal{G}_{kh}(b)\right)\right\}^2 = \left(Q + 2\sigma\frac{\mathcal{G}_{kh}(b)b'' - \mathcal{G}_{kh}(b')b'}{(b'^2 + \mathcal{G}_{kh}(b)^2)^{3/2}} - 2(g - 2\omega c)b\right)\left(b'^2 + \mathcal{G}_{kh}(b)^2\right) \tag{2.10}$$

satisfying

$$b'(x)^2 + \mathcal{G}_{kh}(b)(x)^2 \neq 0 \text{ for all } x \in \mathbb{R},$$

together with the injectivity on $\mathbb{R}$ of the map

$$x \to \left(\frac{x}{k} + \mathcal{C}_{kh}(b - h)(x), b(x)\right).$$

Moreover, the free surface $\mathcal{S}$ can be represented as

$$\mathcal{S} = \left\{\left(q + \frac{x}{k} + \mathcal{C}_{kh}(b - h)(x), b(x)\right) : x \in \mathbb{R}\right\}, \tag{2.11}$$

where $q \in \mathbb{R}$ denotes a constant. Conversely, assuming that $\mathcal{S}$ is defined by (2.11) *with an arbitrary $q \in \mathbb{R}$, Ω be the domain whose boundary consists of $\mathcal{S}$ and of the real axis $\mathcal{B}$, $h > 0$ and the positive function $v \in C^{2,\alpha}_{2\pi}$ are such that* (2.10) *holds. Then there exists a function ψ in Ω such that (Ω, ψ) is a solution of* (1.13) *of class $C^{2,\alpha}$.*

Proof. We start with the necessity. Let (Ω, ψ) be a solution of class $C^{2,\alpha}$ of (1.13). Let h be the conformal mean depth of Ω and let $\tilde{A} + i\tilde{B}$ the conformal mapping associated to Ω. If we consider the mapping $A + iB : \mathcal{R}_{kh} \to \Omega$ given by

$$\begin{aligned} A(x,z) &= \tilde{A}(\tfrac{x}{k}, \tfrac{z}{k}), \\ B(x,z) &= \tilde{B}(\tfrac{x}{k}, \tfrac{z}{k}), \end{aligned} \quad (x,z) \in \mathcal{R}_{kh}, \tag{2.12}$$

where $k = \frac{2\pi}{L}$ then following the proof of Theorem 2.2 in [12] we see that $A, B \in C^{2,\alpha}(\overline{\mathcal{R}_h})$ and $A + iB$ is a conformal mapping from $\mathcal{R}_{kh}$ onto Ω which extends homeomorphically to the closures of these domains, with onto mappings

$$\{(x, 0) : x \in \mathbb{R}\} \to \mathcal{S},$$

and

$$\{(x, -kh) : x \in \mathbb{R}\} \to \mathcal{B}.$$

Moreover,

$$A_x^2(x,0) + B_x^2(x,0) \neq 0 \text{ for all } x \in \mathbb{R}. \tag{2.13}$$

Setting

$$b(x) = B(x,0) \text{ for all } x \in \mathbb{R}, \quad a(x) = A(x,0) \text{ for all } x \in \mathbb{R}, \tag{2.14}$$

we have immediately that

$$a = \mathcal{C}_{kh}(b),$$

and from (2.3) it follows that

$$a' = \mathcal{G}_{kh}(b) \text{ and } a'' = \mathcal{G}_{kh}(b') \tag{2.15}$$

□

It is a consequence of [12] that $v \in C^{2,\alpha}_{2\pi}$ and

$$[b] = h \tag{2.16}$$

$$b(x) > 0 \text{ for all } x \in \mathbb{R}, \tag{2.17}$$

$$\text{the mapping } x \to \left(\frac{x}{k} + \mathcal{C}_{kh}(b-h)(x), b(x)\right) \text{ is injective on } \mathbb{R}, \tag{2.18}$$

$$\mathcal{S} = \left\{\left(q + \frac{x}{k} + \mathcal{C}_{kh}(b-h)(x), b(x)\right) : x \in \mathbb{R}\right\}, \tag{2.19}$$

for some $q \in \mathbb{R}$, whose presence in the formula (2.19) is due to the invariance of problem (1.13) to horizontal translations. From (2.13) and the Cauchy-Riemann equations it follows that

$$b'(x)^2 + \mathcal{G}_{kh}(b)(x)^2 \neq 0 \text{ for all } x \in \mathbb{R}. \tag{2.20}$$

Now let $\xi : \mathcal{R}_{kh} \to \mathbb{R}$ be defined by

$$\xi(x,z) = \psi(A(x,z), B(x,z)),\ (x,z) \in \mathcal{R}_{kh}. \tag{2.21}$$

Note now that the first equation in (1.13) implies that the function $(x,z) \to \psi(x,z) + \frac{\gamma}{2}z^2$ is harmonic in Ω. Due to the invariance of harmonic functions under conformal mappings we have that

$$\xi + \frac{\gamma}{2}B^2 \text{ is harmonic in } \mathcal{R}_{kh}. \tag{2.22}$$

The chain rule and the Cauchy-Riemann equations imply that

$$\xi_x^2 + \xi_z^2 = (\psi_x^2(A,B) + \psi_z^2(A,B))(B_x^2 + B_z^2) \text{ in } \overline{\mathcal{R}_{kh}}.$$

From the last equation in (1.13) and (2.15) it follows that

$$\xi_x^2 + \xi_z^2 = \left(Q + 2\sigma\frac{\mathcal{G}_{kh}(b)b'' - \mathcal{G}_{kh}(b')b'}{(b'^2 + \mathcal{G}_{kh}(b)^2)^{3/2}} - 2(g - 2\omega c)b\right)\left(b'^2 + \mathcal{G}_{kh}(b)^2\right). \tag{2.23}$$

Define $\zeta : \mathcal{R}_{kh} \to \mathbb{R}$ through

$$\zeta = \xi + m + \frac{\gamma}{2}B^2. \tag{2.24}$$

Using the boundary conditions from (1.13) we obtain the following system

$$\begin{cases} \Delta\zeta &= 0 \text{ in } \mathcal{R}_{kh}, \\ \zeta(x,-kh) &= 0 \text{ for all } x \in \mathbb{R}, \\ \zeta(x,0) &= m + \frac{\gamma}{2}b^2(x) \text{ for all } x \in \mathbb{R}, \\ (\zeta_z - \gamma BB_z)^2 &= \left(Q + 2\sigma\frac{\mathcal{G}_{kh}(b)b'' - \mathcal{G}_{kh}(b')b'}{(b'^2+\mathcal{G}_{kh}(b)^2)^{3/2}} - 2(g-2\omega c)b\right)\left(b'^2 + \mathcal{G}_{kh}(b)^2\right) \text{ at } (x,0) \\ & \quad\text{for all } x \in \mathbb{R}, \end{cases} \tag{2.25}$$

which by means of (2.2) and using the Dirichlet-Neumann operator can be reformulated as

$$\left\{\frac{m}{kh} + \gamma\left(\mathcal{G}_{kh}(\frac{b^2}{2}) - b\mathcal{G}_{kh}(b)\right)\right\}^2 = \left(Q + 2\sigma\frac{\mathcal{G}_{kh}(b)b'' - \mathcal{G}_{kh}(b')b'}{(b'^2 + \mathcal{G}_{kh}(b)^2)^{3/2}} - 2(g - 2\omega c)b\right)\left(b'^2 + \mathcal{G}_{kh}(b)^2\right). \tag{2.26}$$

For the sufficiency suppose that the positive number h and the function $b \in C^{2,\alpha}_{2\pi}$ satisfy (2.10). Let B be the harmonic function on $\mathcal{R}_{kh}$ which satisfies

$$B(x, -kh) = 0$$

and

$$B(x, 0) = b(x) \text{ for all } x \in \mathbb{R},$$

and let $A : \mathcal{R}_{kh} \to \mathcal{R}$ be such that $A + iB$ is holomorphic. An application of Lemma 2.1 from [12] yields that $A + iB \in C^{2,\alpha}(\overline{\mathcal{R}_{kh}})$. From $[b] = h$ we obtain

$$\begin{cases} A(x + 2\pi, z) = A(x, z) + \frac{2\pi}{k}, \\ B(x + 2\pi, z) = B(x, z). \end{cases} \quad (x, z) \in \mathcal{R}_{kh}. \tag{2.27}$$

Moreover, the curve (2.19) is non-self-intersecting by the injectivity of the mapping $x \to \big(\frac{x}{k} + \mathcal{C}_{kh}(b - h)(x), b(x)\big)$ and lies in the upper half-plane since $b(x) > 0$. If Ω denotes the domain whose boundary consists of $\mathcal{S}$ and $\mathcal{B}$, it follows from Theorem 3.4 in [32] that $A + iB$ is a conformal mapping from $\mathcal{R}_{kh}$ onto Ω, which extends to a homeomorphism between the closures of these domains, with onto mappings

$$\{(x, 0) : x \in \mathbb{R}\} \to \mathcal{S},$$

and

$$\{(x, -kh) : x \in \mathbb{R}\} \to \mathcal{B}.$$

The latter together with (2.27) imply that Ω is a L-periodic strip-like domain, with $L = 2\pi/k$. The properties of the mapping $\tilde{A} + i\tilde{B} : \mathcal{R}_h \to \Omega$, with $\tilde{A}, \tilde{B}$ given by (2.12) ensure that the conformal mean depth of Ω is h. Let now ζ be defined as the unique solution of the system

$$\begin{cases} \Delta\zeta & = & 0 \text{ in } \mathcal{R}_{kh}, \\ \zeta(x, -kh) & = & 0 \text{ for all } x \in \mathbb{R}, \\ \zeta(x, 0) & = & m + \frac{\gamma}{2} b^2(x) \text{ for all } x \in \mathbb{R}. \end{cases} \tag{2.28}$$

We then have that $\zeta \in C^{2,\alpha}(\overline{\mathcal{R}_{kh}}) \cap C^\infty(\mathcal{R}_{kh})$. Defining ξ by (2.24) and ψ by (2.21) we obtain that ψ satisfies the first three equations in (1.13). Finally, from the first equation in (2.10) we see that the last equation from (1.13) also holds.

3. The bifurcation equation

This section uses bifurcation theory in the spirit of Crandall-Rabinowitz to prove the existence of a local bifurcation curve of solutions of (2.10). We will invoke the following theorem due to Crandall and Rabinowitz.

Theorem 3.1 (Crandall-Rabinowitz). *Let $\mathbb{X}, \mathbb{Y}$ be real Banach spaces, $I \subset \mathbb{R}$ an open interval, and let $F : I \times \mathbb{X} \to \mathbb{Y}$ be a real analytic map satisfying:*

(a) *$F(\lambda, 0) = 0$ for all $\lambda \in I$;*

(b) *There exists $\lambda_* \in I$ such that Fréchet derivative $F_{\mathfrak{u}}(\lambda_*, 0)$ is a Fredholm operator of index zero with a one-dimensional kernel and*

$$\operatorname{Ker} F_{\mathfrak{u}}(\lambda_*, 0) = \{s\mathfrak{u}_0 : s \in \mathbb{R}, 0 \neq \mathfrak{u}_0 \in \mathbb{X}\};$$

(c) *The transversality condition holds*

$$F_{\lambda\mathfrak{u}}(\lambda_*, 0)\mathfrak{u}_0 \notin \operatorname{Im}(F_{\mathfrak{u}}(\lambda_*, 0)).$$

Then, $(\lambda_*, 0)$ is a bifurcation point in the sense that there exists $\epsilon > 0$ and a real-analytic curve $(\lambda, \mathfrak{u}) : (-\varepsilon, \varepsilon) \to I \times \mathbb{X}$ consisting only of solutions of the equation $F(\lambda, \mathfrak{u}) = 0$. Moreover, as $s \to 0$, we have that

$$\lambda(s) = \lambda_* + O(s) \qquad \textit{and} \qquad \mathfrak{u}(s) = s\chi(s) + O(s^2).$$

Furthermore there exists an open set $U \subset I \times \mathbb{X}$ with $(\lambda_*, 0) \in U$ and

$$\{(\lambda, \mathfrak{u}) \in U : F(\lambda, \mathfrak{u}) = 0, \mathfrak{u} \neq 0\} = \{(\lambda(s), \mathfrak{u}(s)) : 0 < |s| < \epsilon\}.$$

Using properties of the Dirichlet-Neumann map $\mathcal{G}_{kh}$ and of the Hilbert transform $\mathcal{C}_{kh}$ we will rewrite the equation (2.10) in a more suitable form. We first set $b = \mathfrak{b} + h$ where $[\mathfrak{b}] = 0$. From (2.3) we obtain that

$$\begin{aligned} \mathcal{G}_{kh}(\mathfrak{b} + h) &= \mathcal{G}_{kh}(\mathfrak{b}) + \mathcal{G}_{kh}(h) = \tfrac{[\mathfrak{b}]}{kh} + \mathcal{C}_{kh}(\mathfrak{b}') + \tfrac{h}{kh} = \tfrac{1}{k} + \mathcal{C}_{kh}(\mathfrak{b}') \\ \mathcal{G}_{kh}(b') &= \mathcal{G}_{kh}(\mathfrak{b}') = \tfrac{[\mathfrak{b}']}{kh} + \mathcal{C}_{kh}(\mathfrak{b}'') = \mathcal{C}_{kh}(\mathfrak{b}''), \end{aligned} \tag{3.1}$$

where in the latter relation we used that $\mathfrak{b}$ is periodic. By the above relations we can write the equation (2.10) as

$$\left\{\frac{m}{kh} + \gamma\left(\frac{[\mathfrak{b}^2]}{2kh} - \frac{\mathfrak{b}}{k} - \frac{h}{2k} + \mathcal{C}_{kh}(\mathfrak{b}\mathfrak{b}') - \mathfrak{b}\mathcal{C}_{kh}(\mathfrak{b}')\right)\right\}^2 \tag{3.2}$$

$$= \left\{Q + 2\sigma \frac{\frac{\mathfrak{b}''}{k} + \mathfrak{b}''\mathcal{C}_{kh}(\mathfrak{b}') - \mathfrak{b}'\mathcal{C}_{kh}(\mathfrak{b}'')}{\left(\mathfrak{b}'^2 + \left(\frac{1}{k} + \mathcal{C}_{kh}(\mathfrak{b}')\right)^2\right)^{3/2}} - 2(g - 2\omega c)(h + \mathfrak{b})\right\}\left\{\mathfrak{b}'^2 + \left(\frac{1}{k} + \mathcal{C}_{kh}(\mathfrak{b}')\right)^2\right\} \tag{3.3}$$

$$\tag{3.4}$$

with the properties

$$\begin{cases} [\mathfrak{b}] = 0 \\ \mathfrak{b}(x) > -h \text{ for all } x \in \mathbb{R}, \\ \text{the mapping } x \to \left(\frac{x}{k} + \mathcal{C}_{kh}(\mathfrak{b})(x), \mathfrak{b}(x) + h\right) \text{ is injective on } \mathbb{R}, \\ \mathfrak{b}'(x)^2 + \left(\frac{1}{k} + \mathcal{C}_{kh}(\mathfrak{b}')(x)\right)^2 \neq 0 \text{ for all } x \in \mathbb{R}. \end{cases} \tag{3.5}$$

Note that a necessary and sufficient condition for $\mathfrak{b} = 0 \in C^{2,\alpha}_{2\pi,o}$ to be a solution of (3.2) is that

$$Q = 2(g - 2\omega c)h + \left(\frac{m}{h} - \frac{\gamma h}{2}\right)^2.$$

The above relation leads us to set

$$\begin{aligned} \lambda &= \tfrac{m}{h} - \tfrac{\gamma h}{2}, \\ \mu &= Q - 2(g - 2\omega c)h - \left(\tfrac{m}{h} - \tfrac{\gamma h}{2}\right)^2, \end{aligned} \tag{3.6}$$

which together with the remark that the map $(m, Q) \to (\lambda, \mu)$ is a bijection from $\mathbb{R}^2$ onto itself transforms the equation (3.2) in

$$\begin{aligned} &\left\{\tfrac{\lambda}{k} + \gamma\left(\tfrac{[\mathfrak{b}^2]}{2kh} - \tfrac{\mathfrak{b}}{k} + \mathcal{C}_{kh}(\mathfrak{b}\mathfrak{b}') - \mathfrak{b}\mathcal{C}_{kh}(\mathfrak{b}')\right)\right\}^2 \\ &= \left\{\lambda^2 + \mu + 2\sigma \tfrac{\frac{\mathfrak{b}''}{k} + \mathfrak{b}''\mathcal{C}_{kh}(\mathfrak{b}') - \mathfrak{b}'\mathcal{C}_{kh}(\mathfrak{b}'')}{\left(\mathfrak{b}'^2 + \left(\frac{1}{k} + \mathcal{C}_{kh}(\mathfrak{b}')\right)^2\right)^{3/2}} - 2(g - 2\omega c)\mathfrak{b}\right\}\left\{\mathfrak{b}'^2 + \left(\tfrac{1}{k} + \mathcal{C}_{kh}(\mathfrak{b}')\right)^2\right\}, \end{aligned} \tag{3.7}$$

for $\mathfrak{b} \in C^{2,\alpha}_{2\pi,o}$, $\mu \in \mathbb{R}, \lambda \in \mathbb{R}$. We will further apply the Crandall-Rabinowitz theorem on bifurcation from simple eigenvalues, [13]. To bring equation (3.7) in a form suitable for the application of the Crandall-Rabinowitz theorem we write it as $F(\lambda,(\mu,\mathfrak{b})) = 0$ with $F : \mathbb{R} \times \mathbb{X} \to \mathbb{Y}$ given by

$$\begin{aligned} F(\lambda,(\mu,\mathfrak{b})) &= \gamma^2\left(\mathcal{C}_{kh}(\mathfrak{b}\mathfrak{b}') - \mathfrak{b}\mathcal{C}_{kh}(\mathfrak{b}') - \frac{\mathfrak{b}}{k} + \frac{[\mathfrak{b}^2]}{2kh}\right)^2 \\ &\quad + \frac{2\lambda\gamma}{k}\left(\mathcal{C}_{kh}(\mathfrak{b}\mathfrak{b}') - \mathfrak{b}\mathcal{C}_{kh}(\mathfrak{b}') - \frac{\mathfrak{b}}{k} + \frac{[\mathfrak{b}^2]}{2kh}\right) \\ &\quad - \left(\mu + 2\sigma\frac{\frac{\mathfrak{b}''}{k} + \mathfrak{b}''\mathcal{C}_{kh}(\mathfrak{b}') - \mathfrak{b}'\mathcal{C}_{kh}(\mathfrak{b}'')}{\left(\mathfrak{b}'^2 + \left(\frac{1}{k} + \mathcal{C}_{kh}(\mathfrak{b}')\right)^2\right)^{3/2}} - 2(g-2\omega c)\mathfrak{b}\right)\left(\mathfrak{b}'^2 + \left(\frac{1}{k} + \mathcal{C}_{kh}(\mathfrak{b}')\right)^2\right) \\ &\quad - \lambda^2\left(\mathfrak{b}'^2 + \frac{2}{k}\mathcal{C}_{kh}(\mathfrak{b}') + \left(\mathcal{C}_{kh}(\mathfrak{b}')\right)^2\right), \end{aligned} \tag{3.8}$$

whereby

$$\mathbb{X} = \mathbb{R} \times C^{p+1,\alpha}_{2\pi,o,e}, \ \mathbb{Y} = C^{p,\alpha}_{2\pi,e},$$

the meaning of "e" being the even functions from $C^{p+1,\alpha}_{2\pi,o}$ and $C^{p,\alpha}_{2\pi}$, respectively. Clearly, $F(\lambda,(0,0)) = 0$ for all $\lambda \in \mathbb{R}$, condition (a) from Theorem 3.1 being thus checked. We now proceed with checking condition (b).

The kernel of the linearization. We compute

$$\begin{aligned} F_{(\mu,\mathfrak{b})}(\lambda,(0,0))(\nu,\mathfrak{f}) &= -\frac{2\lambda\gamma}{k^2}\mathfrak{f} + \frac{2(g-2\omega c)\mathfrak{f}}{k^2} - \frac{\nu}{k^2} - 2\sigma\mathfrak{f}'' - 2\frac{\lambda^2}{k}\mathcal{C}_{kh}(\mathfrak{f}') \\ &= -\frac{2}{k^2}\left(\lambda\gamma\mathfrak{f} - (g-2\omega c)\mathfrak{f} + \lambda^2 k\mathcal{C}_{kh}(\mathfrak{f}') + \sigma k^2\mathfrak{f}''\right) - \frac{\nu}{k^2} \end{aligned} \tag{3.9}$$

From representation (2.5) it follows further that

$$F_{(\mu,\mathfrak{b})}(\lambda,(0,0))(\nu,\mathfrak{f}) = -\frac{2}{k^2}\sum_{n=1}^{\infty}(kn\coth(nkh)\lambda^2 + \gamma\lambda - (g-2\omega c) - \sigma k^2n^2)a_n\cos(nx) - \frac{\nu}{k^2}. \tag{3.10}$$

if

$$\mathfrak{f} = \sum_{n=1}^{\infty} a_n\cos(nx).$$

By means of Lemma 2.4 it follows that the bounded linear operator $F_{(\mu,\mathfrak{b})}(\lambda,(0,0)) : \mathbb{X} \to \mathbb{Y}$ is invertible if and only if

$$kn\coth(nkh)\lambda^2 + \gamma\lambda - (g-2\omega c) - \sigma k^2n^2 \neq 0 \text{ for any integer } n \geq 1. \tag{3.11}$$

Thus, in order to find the bifurcation values λ for (3.7) we have to look at the solutions of the equation

$$kn\coth(nkh)\lambda^2 + \gamma\lambda - (g-2\omega c) - \sigma k^2n^2 = 0, \tag{3.12}$$

for some integer $n \geq 1$. We denote by

$$\lambda_{\pm}^{*n} = -\frac{\gamma \tanh(nkh)}{2kn} \pm \sqrt{\frac{\gamma^2 \tanh^2(nkh)}{4k^2n^2} + (nk\sigma + \frac{g-2\omega c}{kn}) \tanh(nkh)},$$

denote the two solutions of equation (3.12).

Remark 3.2. Since

$$\lambda_{\pm}^{*1}(nk) = \lambda_{\pm}^{*n}(k), \tag{3.13}$$

for all integers $n \geq 1$ and all $k > 0$, it make sense to look at solutions of (3.7) of minimal period 2π, task that we perform in the next lemmas.

Lemma 3.3. *Let* $\lambda(k) = \lambda_{+}^{*1}(k)$.

(i) *If* $\frac{\sigma}{(g-2\omega c)h^2} > \frac{\gamma^2 h}{6(g-2\omega c)} + \frac{1}{3} - \frac{\gamma}{6(g-2\omega c)}\sqrt{\gamma^2h^2 + 4(g-2\omega c)h}$ *then the function* λ *is strictly increasing.*

(ii) *If* $\frac{\sigma}{(g-2\omega c)h^2} < \frac{\gamma^2 h}{6(g-2\omega c)} + \frac{1}{3} - \frac{\gamma}{6(g-2\omega c)}\sqrt{\gamma^2h^2 + 4(g-2\omega c)h}$ *then the function* λ *has a maximum at* $k = 0$ *and a unique local extrema, namely a local minimum at* $k = k_0 > 0$. *Moreover, there is a strictly decreasing sequence* $(k_n)_{n\geq 2}$ *such that*

$$\lambda(k) = \lambda(nk)$$

for $k > 0$ *if and only if* $k = k_n$.

Proof. Let us denote $H(x) = \frac{\gamma^2h^2}{4}\frac{\tanh^2(hk)}{(hk)^2}$ and $G(k) = \frac{\sigma}{h}hk\tanh(hk) + (g-2\omega c)h\frac{\tanh(hk)}{hk}$. We can then write

$$\lambda(k) = -\frac{\gamma h}{2}\frac{\tanh(hk)}{hk} + \sqrt{H(k) + G(k)}.$$

We have that $\frac{d}{dk}\frac{\tanh(hk)}{hk}|_{k=0} = 0$ and $H'(0) = G'(0) = 0$. This implies $\lambda'(0) = 0$ and

$$\begin{aligned} \lambda''(0) &= -\frac{\gamma h}{2}\frac{d^2}{dk^2}\frac{\tanh(hk)}{hk}|_{\{k=0\}} + \frac{2(H+G)(H''+G'') - (H'+G')^2}{4(H+G)\sqrt{H+G}}|_{\{k=0\}} \\ &= \frac{\gamma}{3}h^3 + \frac{H''+G''}{2\sqrt{H+G}}|_{\{k=0\}} \\ &= \frac{\gamma}{3}h^3 + \frac{-\frac{\gamma^2}{3}h^4 + 2h\sigma - \frac{2}{3}(g-2\omega c)h^3}{\sqrt{\gamma^2h^2 + 4(g-2\omega c)h}} \end{aligned} \tag{3.14}$$

Set now $f(k) = k\coth(hk)$. From the formula for $\lambda_{+}^{*1}(k)$ we obtain that

$$\lambda = -\frac{\gamma}{2f} + \sqrt{\frac{\gamma^2}{4f^2} + \frac{k^2\sigma + (g-2\omega c)}{f}}$$

which by squaring leads further to

$$\lambda^2 f = k^2\sigma + (g-2\omega c) - \lambda\gamma. \tag{3.15}$$

Implicit differentiation of (3.15) gives

$$\lambda'(\gamma + 2\lambda f) = 2k\sigma - \lambda^2 f' \tag{3.16}$$

Again by implicit differentiation of (3.16) we get

$$\lambda''(\gamma + 2\lambda f) = 2\sigma - 2(\lambda')^2 f - 4\lambda\lambda' f' - \lambda^2 f''. \tag{3.17}$$

Let $k=k_0$ be a potential critical point of λ. It follows then from (3.16) that $\lambda^2(k_0)=\frac{2\sigma k_0}{f'(k_0)}$ and therefore

$$\lambda''(k_0)(\gamma+2\lambda f(k_0))=2\sigma-\lambda^2 f''(k_0)=2\sigma\frac{f'(k_0)-k_0 f''(k_0)}{f'(k_0)} \tag{3.18}$$

where $f'(k)-kf''(k)=\frac{\sinh^2(hk)\cosh(hk)+hk\sinh(hk)-2(hk)^2\cosh(hk)}{\sinh^3(hk)}$.
Denoting $q(x)=\sinh^2(x)\cosh(x)+x\sinh(x)-2x^2\cosh(x)$ a calculation shows that

$$\begin{aligned} q^{(2)}(x)&=9\sinh^2(x)\cosh(x)-7x\sinh(x)-2x^2\cosh(x)\\ &=2(\sinh^2(x)-x^2)\cosh(x)+\frac{7}{2}\sinh(x)(\sinh(2x)-2x) \end{aligned}$$

Using that $\sinh(y)>y$ for all $y>0$ we obtain from the latter relation that $q^{(2)}(x)>0$ for all $x>0$ and since $q(0)=q'(0)=0$ we have that $q(x)>0$ for all $x>0$. The latter together with $\gamma+2\lambda f(k_0)>0$ and $f'(k_0)>0$ imply via (3.18) that $\lambda''(k_0)>0$. Hence λ has at most one critical point for $k>0$, which is then a minimum point. Note now that the requirement in (i) is equivalent to asking $\lambda''(0)>0$ which in turn says that λ has a local minimum at $k=0$. The latter together with $\lim_{k\to\infty}\lambda(k)=\infty$ and the fact λ can not have a local maximum at $k>0$ implies the conclusion. The condition in (ii) is equivalent to asking $\lambda''(0)<0$ which in turn says that λ has a local maximum at $k=0$. Since $\lim_{k\to\infty}\lambda(k)=\infty$ and because λ can not have a local maximum at $k>0$ we have that λ has a unique local minimum at some $k_0>0$.
It remains to prove the existence of the sequence k_n with the asserted property. For $k\in(0,k_0)$ let $\tilde{k}>k_0$ be such that $\lambda(k)=\lambda(\tilde{k})$. Define now $\varphi:(0,k_0)\to(1,\infty)$, $\varphi(k)=\frac{\tilde{k}}{k}$. It is easy to see that φ is strictly decreasing with $\varphi(k)\to\infty$ as $k\to 0$ and $\varphi(k)\to 1$ as $k\to k_0$. Set $k_n=\varphi^{-1}(n)$. Since φ^{-1} is decreasing it follows that k_n is well-defined and decreases to 0. Moreover, the only point k with $\lambda(k)=\lambda(nk)$ is $k=k_n$. ☐

Lemma 3.4. *Let $\tilde{\lambda}(k)=\lambda_-^{*1}(k)$.*

(i) *If $\frac{\sigma}{(g-2\omega c)h^2}\geq\frac{\gamma^2 h}{6(g-2\omega c)}+\frac{1}{3}+\frac{\gamma}{6(g-2\omega c)}\sqrt{\gamma^2h^2+4(g-2\omega c)h}$ then the function $\tilde{\lambda}$ is strictly decreasing.*

(ii) *If $\frac{\sigma}{(g-2\omega c)h^2}<\frac{\gamma^2 h}{6(g-2\omega c)}+\frac{1}{3}+\frac{\gamma}{6(g-2\omega c)}\sqrt{\gamma^2h^2+4(g-2\omega c)h}$ then the function $\tilde{\lambda}$ has a minimum at $k=0$ and a unique local extremum, namely a local maximum at $k=\tilde{k}_0>0$. Moreover, there is a strictly decreasing sequence $(\tilde{k}_n)_{n\geq 2}$ such that*

$$\tilde{\lambda}(k)=\tilde{\lambda}(nk)$$

for $k>0$ if and only if $k=\tilde{k}_n$.

Proof. We only treat the equality case in (i). The rest is similar to Lemma 3.3. Therefore, we assume that $\frac{\sigma}{(g-2\omega c)h^2}=\frac{\gamma^2 h}{6(g-2\omega c)}+\frac{1}{3}+\frac{\gamma}{6(g-2\omega c)}\sqrt{\gamma^2h^2+4(g-2\omega c)h}$. With the notation from Lemma 3.3 we have that

$$\begin{aligned} \tilde{\lambda}''(0)&=\frac{\gamma}{3}h^3-\frac{H''+G''}{2\sqrt{H+G}}|_{\{k=0\}}\\ &=\frac{\gamma}{3}h^3-\frac{-\frac{\gamma^2}{3}h^4+2h\sigma-\frac{2}{3}(g-2\omega c)h^3}{\sqrt{\gamma^2h^2+4(g-2\omega c)h}} \end{aligned} \tag{3.19}$$

Notice that $\frac{\sigma}{(g-2\omega c)h^2} = \frac{\gamma^2 h}{6(g-2\omega c)} + \frac{1}{3} + \frac{\gamma}{6(g-2\omega c)}\sqrt{\gamma^2 h^2 + 4(g-2\omega c)h}$ is equivalent to $\tilde{\lambda}''(0) = 0$. We claim that

$$\tilde{\lambda}^{(3)}(0) = 0 \text{ and } \tilde{\lambda}^{(4)}(0) < 0. \tag{3.20}$$

This implies that $\tilde{\lambda}''$ has a local maximum at $k=0$ and since we are in the case $\tilde{\lambda}''(0) = 0$ it follows further that $\tilde{\lambda}''(k) < 0$ for $k > 0$ and close to 0. Therefore $\tilde{\lambda}'$ is strictly decreasing for $k > 0$. Since $\tilde{\lambda}'(0) = 0$ we have that $\tilde{\lambda}'(k) < 0$ for $k > 0$ close to 0. Hence $\tilde{\lambda}$ is decreasing for $k > 0$ close to 0 and since $\tilde{\lambda}$ does not have local minima and $\lim_{k\to\infty} \tilde{\lambda}(k) = -\infty$ we infer that $\tilde{\lambda}$ is strictly decreasing. We now proceed with the claim (3.20).
We have first that

$$\begin{aligned}\tilde{\lambda}^{(3)} &= -\frac{\gamma h}{2}\frac{d^3}{dk^3}\frac{\tanh(hk)}{hk} \\ &- \frac{(H+G)^2(H^{(3)}+G^{(3)})\sqrt{H+G} - [(H+G)(H''+G'') - (H'+G')^2]E}{2(H+G)^3}\end{aligned} \tag{3.21}$$

where $E = \frac{3}{2}(H'+G')\sqrt{H+G}$. Using that $H'(0) = G'(0) = H^{(3)}(0) = G^{(3)}(0) = 0$ we see that $\tilde{\lambda}^{(3)}(0) = 0$.
From (3.21) and using that

$$0 = \tilde{\lambda}''(0) = \frac{\gamma}{3}h^3 - \frac{H''+G''}{2\sqrt{H+G}}|_{\{k=0\}}$$

we have

$$\begin{aligned}\tilde{\lambda}^{(4)}(0) &= -\frac{\gamma h}{2}\frac{d^4}{dk^4}\frac{\tanh(hk)}{hk}|_{\{k=0\}} \\ &- \frac{2(H+G)^2(H^{(4)}+G^{(4)})\sqrt{H+G} - 2(H+G)(H''+G'')\frac{3}{2}\sqrt{H+G}(H''+G'')}{4(H+G)^3}|_{\{k=0\}} \\ &= -\frac{24}{15}\gamma h^5 - \frac{2(H+G)^2(H^{(4)}+G^{(4)})\sqrt{H+G} - 3(H+G)\sqrt{H+G}\frac{4}{9}(H+G)\gamma^2h^6}{4(H+G)^3}|_{\{k=0\}} \\ &= -\frac{24}{15}\gamma h^5 - \frac{(H+G)^2\sqrt{H+G}[2(H^{(4)}+G^{(4)}) - \frac{4}{3}\gamma^2h^6]}{4(H+G)^3}|_{\{k=0\}} \\ &= -\frac{24}{15}\gamma h^5 - \frac{1}{4\sqrt{H+G}}[2(H^{(4)}+G^{(4)}) - \frac{4}{3}\gamma^2h^6]|_{\{k=0\}} \\ &= \frac{-96\gamma h^5\sqrt{H+G} - 15[2(H^{(4)}+G^{(4)}) - \frac{4}{3}\gamma^2h^6]}{60\sqrt{H+G}}|_{\{k=0\}}\end{aligned} \tag{3.22}$$

A calculation shows that the numerator equals

$$\begin{aligned}&-96\gamma h^5\frac{3}{2}\frac{H''(0)+G''(0)}{\gamma h^3} - 15\left[2\left(\frac{34}{15}\gamma^2h^6 + \frac{48}{15}(g-2\omega c)h^5 - 8\sigma h^3\right) - \frac{4}{3}\gamma^2h^6\right] \\ &= -48\cdot 3h^2\left(2h\sigma - \frac{2}{3}(g-2\omega c)h^3 - \frac{1}{3}\gamma^2h^4\right) - 68\gamma^2h^6 - 96(g-2\omega c)h^5 + 240\sigma h^3 + 20\gamma^2h^6 \\ &= -48\sigma h^3 < 0\end{aligned} \tag{3.23}$$

□

Applying now Lemmas 3.3 and 3.4 and Remark 3.2 we obtain the following necessary and sufficient conditions for the one-dimensionality of the kernel $\mathcal{N}(F_{(\mu,\mathfrak{b})}(\lambda^*,0))$

Lemma 3.5. *Let λ^{*1} be a solution of (3.12) with $n=1$, i.e.,*

$$\lambda_{\pm}^{*1} = -\frac{\gamma\tanh(kh)}{2k} \pm \sqrt{\frac{\gamma^2\tanh^2(kh)}{4k^2} + (k\sigma + \frac{g-2\omega c}{k})\tanh(kh)}.$$

Then the kernel $\mathcal{N}(F_{(\mu,\mathfrak{b})}(\lambda^*,0)))$ is one-dimensional if and only if

$$\frac{\sigma}{(g-2\omega c)h^2} \geq \frac{\gamma^2 h}{6(g-2\omega c)} + \frac{1}{3} + \frac{\gamma}{6(g-2\omega c)}\sqrt{\gamma^2h^2 + 4(g-2\omega c)h} \tag{3.24}$$

or

$$\frac{\sigma}{(g-2\omega c)h^2} < \frac{\gamma^2 h}{6(g-2\omega c)} + \frac{1}{3} + \frac{\gamma}{6(g-2\omega c)}\sqrt{\gamma^2h^2 + 4(g-2\omega c)h} \tag{3.25}$$

and $k \neq k_n$ and $k \neq \tilde{k}_n$ for all $n \geq 2$

Moreover, in these situations, $\mathcal{N}(F_{(\mu,\mathfrak{b})}(\lambda^*,0))$ is generated by $(0,\mathfrak{b}^*) \in \mathbb{X}$, where $\mathfrak{b}^*(x) = \cos(x)$ for all $x \in \mathbb{R}$.

Remark 3.6. Note that setting $\gamma = 0$ and $\omega = 0$ we recover the necessary and sufficient condition for the one-dimensionality of the kernel $\mathcal{N}(F_{(\mu,\mathfrak{b})}(\lambda^*,0))$ found in [23] in the context of irrotational waves.

We give in the next Lemma a sufficient condition for the one dimensionality of the kernel $\mathcal{N}(F_{(\mu,\mathfrak{b})}(\lambda^*,0))$. This sufficient condition has the advantage of being easy to verify and as we shall see will prove to be useful later on when we establish the existence of stagnation points.

Lemma 3.7. *Let λ^{*1} be a solution of (3.12) with $n=1$, i.e.,*

$$\lambda_{\pm}^{*1} = -\frac{\gamma\tanh(kh)}{2k} \pm \sqrt{\frac{\gamma^2\tanh^2(kh)}{4k^2} + (k\sigma + \frac{(g-2\omega c)}{k})\tanh(kh)}.$$

Assume that $k^3 \geq 2\frac{\gamma^2}{\sigma}$, $k^2 \geq 4\frac{(g-2\omega c)}{\sigma}$ and $kh \geq \frac{1}{2}$. Then it follows from (3.10) and Lemma 2.4 that the kernel $\mathcal{N}(F_{(\mu,\mathfrak{b})}(\lambda^*,0))$ is one-dimensional being generated by $(0,\mathfrak{b}^*) \in \mathbb{X}$, where $\mathfrak{b}^*(x) = \cos(x)$ for all $x \in \mathbb{R}$.

Proof. The proof is very similar to the proof of Lemma 6 in [24]. Therefore, we omit here. $\square$

The transversality condition. It is routine to check that

$$\mathcal{R}(F_{(\mu,\mathfrak{b})}(\lambda^*,0)) = \left\{ f \in \mathbb{Y} : \int_{-\pi}^{\pi} f(x)\cos(x)dx = 0 \right\},$$

and the latter is a closed subspace of $\mathbb{Y}$. Consequently, $\mathbb{Y}/\mathcal{R}(F_{(\mu,\mathfrak{b})}(\lambda^*,0))$ is the one dimensional subspace of $\mathbb{Y}$ generated by $\mathfrak{b}^*(x) = \cos(x)$. For $\mathfrak{b}^* = \cos(x)$ we use that $\mathcal{C}_{kh}(\mathfrak{b}^*) = \coth(kh)\sin(x)$ and $\mathcal{C}_{kh}(\mathfrak{b}^{*\prime}) = (\mathcal{C}_{kh}(\mathfrak{b}^*))' = \coth(kh)\cos(x) = \coth(kh)\mathfrak{b}^*$, to compute that

$$
\begin{aligned}
F_{\lambda,(\mu,\mathfrak{b})}(\lambda^*,(0,0))(1,(0,\mathfrak{b}^*)) &= -\frac{2}{k^2}\gamma\mathfrak{b}^* - \frac{4}{k^2}\lambda^* k\mathcal{C}_{kh}(\mathfrak{b}^{*\prime}) \\
&= \frac{2}{k^2}(-\gamma - 2\lambda^* k\coth(kh))\mathfrak{b}^* \\
&\notin \mathcal{R}(F_{(\mu,\mathfrak{b})}(\lambda^*,(0,0))),
\end{aligned} \tag{3.26}
$$

since the equation giving λ^* gives

$$
-\gamma - 2\lambda^* k\coth(kh) = -\lambda^*\left(k\coth(kh) + \frac{\sigma k^2 + g - 2\omega c}{(\lambda^*)^2}\right) \neq 0.
$$

We have thus shown that the transversality condition from Theorem 3.1 takes place in our case. We are now ready to prove the existence of a curve of solutions to the system (1.13) bifurcating from the trivial solutions with a flat free surface.

Theorem 3.8. *For every $h > 0, \sigma > 0, \gamma \in \mathbb{R}, c < 0$ satisfying* (3.24) *or* (3.25) *and $m, k \in \mathbb{R}$ there exists laminar flows with a flat free surface in water of depth h, of constant vorticity γ. The laminar flows of mass flux*

$$
m_{\pm} = \frac{\gamma h^2}{2} - \frac{\gamma h\tanh(kh)}{2k} \pm h\sqrt{\frac{\gamma^2\tanh^2(kh)}{4k^2} + \frac{k^2\sigma + g - 2\omega c}{k}\tanh(kh)} \tag{3.27}
$$

are exactly those whose horizontal speeds at the flat free surface equal to

$$
\lambda_{\pm} = -\frac{\gamma\tanh(kh)}{2k} \pm \sqrt{\frac{\gamma^2\tanh^2(kh)}{4k^2} + \frac{k^2\sigma + g - 2\omega c}{k}\tanh(kh)}. \tag{3.28}
$$

The values of $m_{\pm}$ of the flux given by (3.27) *give rise to equatorial geophysical periodic steady capillary-gravity waves of small amplitude, with period $\frac{2\pi}{k}$ and conformal mean depth h, which have a smooth profile with one crest and one trough per period, monotone between consecutive crests and troughs and symmetric about any crest line.*

Proof. Arguing as in [25] we see that $\mathfrak{b} = 0$ gives rise to laminar flows in the fluid domain bounded above by the flat surface $z = h$ and below by the bed $\mathcal{B}$. In terms of the stream function the laminar flow solutions are given by

$$
\psi(x,z) = -\frac{\gamma}{2}z^2 + \left(\frac{m}{h} + \frac{\gamma h}{2}\right)z - m,\ x \in \mathbb{R},\ 0 \le z \le h,
$$

with the velocity field

$$
(\psi_z, -\psi_x) = \left(-\gamma z + \frac{m}{h} + \frac{\gamma h}{2}, 0\right),\ z \in \mathbb{R},\ 0 \le z \le h. \tag{3.29}
$$

Remembering that $\lambda = \frac{m}{h} - \frac{\gamma h}{2}$ we see that the velocity field can be rewritten as

$$
(\psi_z, -\psi_x) = (\lambda_{\pm} + \gamma(h - z), 0),\ x \in \mathbb{R},\ 0 \le z \le h, \tag{3.30}
$$

relation which shows that for laminar flows the horizontal fluid velocity at the free surface coincides with $\lambda_{\pm}$. Since $\lambda_+ > 0$ and $\lambda_- < 0$ we see that the laminar flows do not possess stagnation points (i.e. points (x,z) where $\psi_z(x,z) = u - c = 0$) at the free surface. We can now establish the existence of waves of small amplitude with the properties mentioned in the statement of the theorem. By means of the Crandall-Rabinowitz Theorem there exists a local bifurcation curve

$$\{(\lambda(s), (0 + o(s), s\cos(x) + o(s))) : |s| < \varepsilon\} \subset \mathbb{R} \times X$$

consisting of solutions of (3.7) with $\lambda_\pm$ given by (3.28).
Choosing ε sufficiently small and using Lemma 2.4 we can ensure that

$$\mathfrak{b}(x) > -h \text{ for all } x \in \mathbb{R},$$

and

$$\frac{1}{k} + \mathcal{C}_{kh}(\mathfrak{b}')(x) > 0 \text{ for all } x \in \mathbb{R}. \tag{3.31}$$

The inequality (3.31) implies that the corresponding non-flat free surface $\mathcal{S}$ given by (2.11) with $b = \mathfrak{b} + h$ is the graph of a smooth function, symmetric with respect to the points obtained for the values $x = n\pi, n \in \mathbb{Z}$. From

$$\mathfrak{b}(x; s) = s\cos(x) + o(s) \text{ in } C_{2\pi}^{p+1,\alpha},$$

we have that

$$s\mathfrak{b}'(x; s) < 0 \text{ for all } x \in (0, \pi),\ 0 < |s| < \varepsilon,$$

for $\varepsilon > 0$ sufficiently small and $p \geq 1$. Using the evenness of $x \to \mathfrak{b}(x; s)$ we conclude the proof of the assertion about the free surface $\mathcal{S}$, i.e., $\mathcal{S}$ has one crest and one trough per minimal period and is monotone between consecutive crests and troughs. □

Remark 3.9. Note that setting $\sigma = 0$ in formula (3.28) for λ_- we recover the dispersion relation for gravity equatorial wind waves obtained in [5]. Setting $\omega = 0$ in (3.28) we recover the formula for λ_- for the context of capillary-gravity waves from [33]

Theorem 3.10. *For any $h > 0$, $k > 0$, $\gamma \in \mathbb{R}$ and $m \in \mathbb{R}$ satisfying $k^3 \geq 2\frac{\gamma^2}{\sigma}$, $k^2 \geq 4\frac{g-2\omega c}{\sigma}$ and $kh \geq \frac{1}{2}$ there exists laminar flows with a flat free surface in water of depth h, of constant vorticity γ and relative mass flux m. The laminar flows of flux $m_\pm$ are exactly those with horizontal speeds at the flat free surface equal to $\lambda_\pm$ given by* (3.28)*. The values of $m_\pm$ of the flux given by* (3.27) *trigger the appearance of equatorial geophysical steady periodic capillary-gravity waves of small amplitude, with period $\frac{2\pi}{k}$ and conformal mean depth h, which have a smooth profile with one crest and one trough per period, monotone between consecutive crests and troughs and symmetric about any crest line.*

Proof. The conditions on k, h and γ ensure via Lemma 3.7 that the kernel of $F_{(\mu,\mathfrak{b})}(\lambda^*, 0)$ is one dimensional and is generated by the element $(0, \cos(x))$. The rest of the proof follows the one in Theorem 3.8. □

We end this section with a result concerning the physical relevance of the bifurcation equation (3.12) satisfied by the bifurcation parameter $\lambda_+^{*1} =: \lambda_0$.

Theorem 3.11. *If u_0 is the strength of the current at the depth $z = 0$, then the wave speed c of the bifurcating laminar flow equals*

$$c = u_0 - \gamma h - \frac{2\omega - \gamma}{k}\tanh(kh) - \sqrt{\frac{(2\omega-\gamma)^2}{4k^2}\tanh^2(kh) + \frac{\tanh(kh)}{k}\left[k^2\sigma + g - 2\omega(u_0 - \gamma h)\right]} \tag{3.32}$$

Proof. As noted in the proof of Theorem 3.8 the solution λ_0 of the equation (3.12) with $n = 1$ gives rise to the bifurcating laminar flow whose horizontal velocity u satisfies

$$u(z) - c = \psi_z = \lambda_0 + \gamma(h - z),$$

cf. (3.30). Denoting $u(0) = u_0$ we obtain that $c = u_0 - \lambda_0 - \gamma h$. Inserting the latter in the formula (3.28) for λ_0 we obtain the polynomial equation of degree two in the unknown λ_0

$$\lambda_0^2 + \frac{\gamma - 2\omega}{k} \tanh(kh)\lambda_0 - \frac{\tanh(kh)}{k}\left[k^2\sigma + g - 2\omega(u_0 - \gamma h)\right] = 0$$

whose only positive root is

$$\begin{aligned}\lambda_0 = {} & \frac{2\omega - \gamma}{k} \tanh(kh) \\ & + \sqrt{\frac{(2\omega-\gamma)^2}{4k^2}\tanh^2(kh) + \frac{\tanh(kh)}{k}\left[k^2\sigma + g - 2\omega(u_0 - \gamma h)\right]}.\end{aligned}$$

The formula (3.32) for c is thus proved. □

Remark 3.12. Note that setting $\sigma = 0$ we recover formula (3.9) for c obtained in [5] in the context of equatorial gravity waves.

4. Existence of stagnation points

We first prove the existence of stagnation points in the laminar flows. We see from (3.30) that stagnation points exists in laminar flows if and only if the equation

$$\lambda_\pm + \gamma(h - z) = 0 \tag{4.1}$$

has at least a solution $z \in [0, h]$. The latter condition is equivalent to

$$\lambda_\pm(\lambda_\pm + \gamma h) \leq 0. \tag{4.2}$$

It is immediate from (4.2) that for $\gamma > 0$ the flow corresponding to λ_+ does not contain stagnation points, while the flow corresponding to λ_- possesses stagnation points if and only if $\lambda_- + \gamma h \geq 0$, condition which is equivalent to

$$\tanh(kh) \leq \frac{\gamma^2 h^2 k}{\gamma^2 h + k^2\sigma + g - 2\omega c}. \tag{4.3}$$

The case $\gamma < 0$ mirrors the previous one. Namely, $\lambda_-(\lambda_- + \gamma h) > 0$ for $\gamma < 0$ and therefore the flow corresponding to λ_- does not contain stagnation points. The flow corresponding to λ_+ contains stagnation points if and only if $\lambda_+ + \gamma h \leq 0$, inequality is also equivalent to (4.3).

Remark 4.1. Note that (4.3) is false for $k \to \infty$. This just shows that a flow with the wavelength $L \to 0$ does not have stagnation points.

Lemma 4.2. *If the vorticity γ is such that*

$$\gamma^2 \geq \frac{4\sigma h + \sqrt{16\sigma^2 h^2 + 16 h^4 \sigma (g - 2\omega c)}}{2h^4} \tag{4.4}$$

there are values $k_1 \leq k_2$ with the property that (4.3) *holds true whenever $k \in [k_1, k_2]$. Moreover, in this situation, the stagnation points are neither on the free surface nor on the flat bed.*

Proof. Note that

$$\gamma_{\pm} = \pm\sqrt{\frac{4\sigma h + \sqrt{16\sigma^2 h^2 + 16h^4\sigma(g-2\omega c)}}{2h^4}}$$

are the only real solutions of the equation

$$h^4\gamma^4 - 4\sigma h\gamma^2 - 4\sigma(g-2\omega c) = 0,$$

which besides $\gamma_{\pm}$ has another two complex conjugate solutions. It follows that

$$h^4\gamma^4 - 4\sigma h\gamma^2 - 4\sigma(g-2\omega c) \geq 0, \tag{4.5}$$

for all $\gamma \in (-\infty, \gamma_-] \cup [\gamma_+, \infty)$. But (4.5) ensures that the equation

$$\sigma k^2 - \gamma^2 h^2 k + \gamma^2 h + (g-2\omega c) = 0$$

has two (not necessarily distinct) solutions

$$k_1 := \frac{\gamma^2 h^2 - \sqrt{\gamma^4 h^4 - 4\sigma\gamma^2 h - 4\sigma(g-2\omega c)}}{2\sigma}$$

$$k_2 := \frac{\gamma^2 h^2 + \sqrt{\gamma^4 h^4 - 4\sigma\gamma^2 h - 4\sigma(g-2\omega c)}}{2\sigma}$$

We then have that

$$\sigma k^2 - \gamma^2 h^2 k + \gamma^2 h + (g-2\omega c) \leq 0,$$

for all $k \in [k_1, k_2]$ which implies that

$$\frac{\gamma^2 h^2 k}{\gamma^2 h + k^2\sigma + (g-2\omega c)} \geq 1. \tag{4.6}$$

Therefore (4.3) holds true for all $k \in [k_1, k_2]$. Concerning the position of the stagnation points we already proved that they can not appear on the free surface. To show that they can not be on the flat bed note that (4.6) implies that

$$\tanh(kh) < \frac{\gamma^2 h^2 k}{\gamma^2 h + k^2\sigma + g - 2\omega c} \quad \text{for all} \quad k \in [k_1, k_2]$$

which is equivalent to $\lambda_- + \gamma h > 0$(for $\gamma > 0$) and $\lambda_+ + \gamma h < 0$(for $\gamma < 0$). The last two inequalities show that $\psi_z|_{z=0} \neq 0$ which proves the claim. $\square$

Lemma 4.3. *Let γ and h be such that*

$$\gamma^2 \geq \frac{4\sigma h + \sqrt{16\sigma^2 h^2 + 64h^4\sigma(g-2\omega c)}}{2h^4}. \tag{4.7}$$

Let k_1 and k_2 be the values from Lemma 4.2. If $k \in \left[\frac{k_1+k_2}{2}, k_2\right]$ then the sufficient conditions for the existence of laminar flows from Theorem 3.10 are satisfied. In addition, these flows posses stagnation points as the proof of Lemma 4.2 shows.

Proof. We have to show that for all such k we have that $k^3 \geq \frac{\gamma^2}{\sigma}$, $k^2 \geq 4\frac{(g-2\omega c)}{\sigma}$ and $kh \geq \frac{1}{2}$. We only need to show that the last two inequalities hold true for $k = \frac{k_1+k_2}{2} = \frac{\gamma^2 h^2}{2\sigma}$ and then the rest follows. Note first that (4.7) implies that

$$\gamma^2 \geq 4\frac{\sigma}{h^3}$$

and

$$\gamma^2 \geq \frac{\sqrt{64h^4\sigma(g-2\omega c)}}{2h^4} = 4\frac{\sqrt{\sigma(g-2\omega c)}}{h^2}.$$

If $k = \frac{k_1+k_2}{2} = \frac{\gamma^2 h^2}{2\sigma}$ then we have

$$k^3 = \left(\frac{\gamma^2 h^2}{2\sigma}\right)^3 = \frac{\gamma^6 h^6}{8\sigma^3} = \frac{\gamma^2}{\sigma}\cdot\frac{\gamma^4 h^6}{8\sigma^2} \geq \frac{\gamma^2}{\sigma}\cdot\frac{16\sigma^2}{8\sigma^2} = 2\frac{\gamma^2}{\sigma},$$

$$k^2 = \left(\frac{\gamma^2 h^2}{2\sigma}\right)^2 = \frac{\gamma^4 h^4}{4\sigma^2} \geq \frac{16\sigma(g-2\omega c)}{4\sigma^2} = 4\frac{(g-2\omega c)}{\sigma},$$

$$kh = \frac{\gamma^2 h^3}{2\sigma} \geq 2.$$

$\square$

Remark 4.4. Whenever the stagnation points are present in laminar flows form in fact horizontal lines. The stagnation line $z = z_0$ satisfies

$$h - z_0 = \frac{\tanh(kh)}{2k} + \sqrt{\frac{\tanh^2(kh)}{4k^2} + \frac{k\sigma}{\gamma^2}\tanh(kh) + \frac{g-2\omega c}{\gamma^2}\frac{\tanh(kh)}{k}}.$$

Remark 4.5. Small amplitude waves bifurcation from laminar flows that possess an inner stagnation line (i.e. a line which is neither the free surface nor the flat bed) also have a critical layer. This can be seen by an elementary analysis similar to that in [34].

5. Regularity

This section contains a regularity result concerning the free surface. The proof relies on some commutator properties for the Hilbert transform.

Theorem 5.1. *Let $h > 0$ and $b \in C^{2,\alpha}_{2\pi}$ be a solution of* (2.10)*. Then $b \in C^{\infty}_{2\pi}$.*

Proof. From (2.3) we find that

$$\mathcal{G}_{kh}(b)b'' - \mathcal{G}_{kh}(b')b' = \left(\frac{1}{k} + \mathcal{C}_{kh}(b')\right)b'' - b'\mathcal{C}_{kh}(b'') \tag{5.1}$$

From (2.2) and the second equation of (2.10) we have that

$$\mathcal{G}_{kh}(\frac{b^2}{2}) - b\mathcal{G}_{kh}(b) = \frac{[b^2]}{2kh} + \mathcal{C}_{kh}(bb') - \frac{b}{k} - b\mathcal{C}_{kh}(b') = \frac{[b^2]}{2kh} - \frac{b}{k} - \mathcal{Q}_{kh}(b), \tag{5.2}$$

where $\mathcal{Q}_{kh}(b) = b\mathcal{C}_{kh}(b') - \mathcal{C}_{kh}(bb')$. From Lemma 2.6 we have that $\mathcal{Q}_{kh}(b) \in C^{2,\alpha/3}_{2\pi}$ since $b \in C^{2,\alpha}_{2\pi}$. The latter fact together with the formulas (5.1), (5.2), (2.10) and using $b'^2 + \mathcal{G}_{kh}(b)^2 \in C^{1,\alpha/3}_{2\pi}$ yield

$$\left(\frac{1}{k} + \mathcal{C}_{kh}(b')\right)b'' - b'\mathcal{C}_{kh}(b'') \in C^{1,\alpha/3}_{2\pi} \tag{5.3}$$

Now from Lemma 2.6 with $f = -b' \in C^{1,\alpha}_{2\pi}$ and $g = \mathcal{C}_{kh}(b'') \in C^{0,\alpha}_{2\pi}$ it follows that

$$-b'\mathcal{C}_{kh}(\mathcal{C}_{kh}(b'')) - \mathcal{C}_{kh}(-b'\mathcal{C}_{kh}(b'')) \in C^{1,\alpha/3}_{2\pi}. \tag{5.4}$$

Taking into account that $\mathcal{C}_{kh} = \mathcal{C} + \mathcal{S}_{kh}$, where $\mathcal{C}$ is the familiar Hilbert transform and $\mathcal{S}_{kh}$ is a smoothing operator (see Lemma 2.4) we obtain using also Remark 2.3 that

$$\mathcal{C}_{kh}(\mathcal{C}_{kh}(b'')) = -b'' + \mathcal{R}_{kh}(b''),$$

where $\mathcal{R}_{kh} := \mathcal{C}\mathcal{S}_{kh} + \mathcal{S}_{kh}\mathcal{C} + \mathcal{S}_{kh}^2$ is also a smoothing operator. Hence, using (5.4) we have that

$$b'b'' - \mathcal{C}_{kh}(-b'\mathcal{C}_{kh}(b'')) \in C_{2\pi}^{1,\alpha/3}. \tag{5.5}$$

By applying $\mathcal{C}_{kh}$ to (5.3) and using (5.5) we get

$$\frac{1}{k}\mathcal{C}_{kh}(b'') + C_{kh}(b''C_{kh}(b')) + b'b'' \in C_{2\pi}^{1,\alpha/3} \tag{5.6}$$

Setting $f = \mathcal{C}_{kh}(b') \in C_{2\pi}^{1,\alpha}$ and $g = b'' \in C_{2\pi}^{0,\alpha}$ we get by applying Lemma 2.6

$$\mathcal{C}_{kh}(b')\mathcal{C}_{kh}(b'') - \mathcal{C}_{kh}(b''\mathcal{C}_{kh}(b')) \in C_{2\pi}^{1,\alpha/3} \tag{5.7}$$

Adding up (5.6) and (5.7) yields

$$\left(\frac{1}{k} + \mathcal{C}_{kh}(b')\right)\mathcal{C}_{kh}(b'') + b'b'' \in C_{2\pi}^{1,\alpha/3} \tag{5.8}$$

We now multiply (5.3) by $\frac{1}{k} + \mathcal{C}_{kh}(b') \in C_{2\pi}^{1,\alpha}$ and (5.8) by $b' \in C_{2\pi}^{1,\alpha}$ and by adding up the resulting expressions we obtain

$$\left(\left(\frac{1}{k} + \mathcal{C}_{kh}(b')\right)^2 + b'^2\right) b'' \in C_{2\pi}^{1,\alpha/3} \tag{5.9}$$

Since the expression in the bracket on the left-hand side of (5.9) is strictly positive and belongs to $C_{2\pi}^{1,\alpha}$ we obtain that $b'' \in C_{2\pi}^{1,\alpha/3}$. Therefore $b \in C_{2\pi}^{3,\alpha/3}$. An iteration of this method shows that $v \in C_{2\pi}^{\infty}$. □

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