ON CRAPPER'S WAVE

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Abstract. We show that the parametric representation of the flow beneath the Crapper wave is dynamically possible, in the sense that it is a global diffeomorphism from the parameter domain $\{(\alpha, \beta) : \alpha \in \mathbb{R}, \beta > 0\}$ to the infinite depth fluid domain below the surface wave $y = \eta(x, t)$.

1. Introduction

Despite the everyday and overall occurence of water waves, rigorous mathematical foundations of hydrodynamics were set up only in the 18^{th} century through the works of Bernoulli, Euler, Lagrange and d'Alembert. Although their seminal works followed by the major achievements of Navier and Stokes in the 19^{th} century have resulted in a tremendous development of mathematical sciences there are still many unturned stones in the field of fluid dynamics even in the idealised situation treating a perfect fluid. One of the shortcomings of the water wave theory is the existence of very few explicit solutions for the governing equations of a perfect fluid with a non-flat free surface. The first such explicit solution in the case of gravity waves was described by Gerstner [12] back in 1802. However, the first rigorous analysis of Gerstner's wave was performed by Constantin [2] who showed that the evolution of the fluid domain under the passage of Gerstner's wave is consistent with the governing equations. Adapting the methods from [2], the same author gave in [1] an explicit solution of the governing equations for the propagation of edge-waves along sloping beaches. In spite of the recent advance concerning the determination of particle trajectories [3, 5, 16], existence theorems for waves with continuous or discontinuous vorticity [7, 8], regularity of the free surface [6], symmetry [4], the treatment of water waves allowing stagnation points [9, 14], explicit solutions have not been found until the middle of the 20^{th} century in the context of pure capillary waves, i.e., when the only restorative force acting on the fluid is surface tension. We are refering here to the works of Crapper [10] for infinite depth and of Kinnersley [15] in the case of finite depth, see also [11]. After briefly summarizing in Section 2 the equations of motions we prove in Section 3 that Crapper's solutions provide a global diffeomorphism from $\{(\alpha, \beta) : \alpha \in \mathbb{R}, \beta > 0\}$ to the infinite depth fluid domain below the surface wave $y = \eta(x, t)$.

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2. Equations of motion

We consider two-dimensional irrotational travelling waves on the surface of an ideal fluid of infinite depth. We neglect gravity and assume that the only restorative force applied to the fluid is surface tension. We assume the waves to be two dimensional, which means that the motion is identical in any direction parallel to the crest line. We therefore need to consider a cross section of the flow in a direction perpendicular to the crest line. Following

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Crapper [10] we choose the Cartesian axes so that x is measured horizontally to the left and y points vertically downwards. The fluid is steady and moving in the positive x direction with the wave speed c. We summarize now the water wave equations.
The irrotationality of the flow implies the existence of the velocity potential of the flow ϕ which satisfies the equation of motion

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{2.1}$$

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Moreover, we also work with the stream function ψ satisfying also equation (2.1). At the free surface the flow satisfies Bernoulli's condition

$$\frac{P}{\rho} + \frac{1}{2}|\nabla \phi|^2 = \frac{1}{2}c^2, \tag{2.2}$$

Bernoulli

where P represents the difference of pressure from its hydrostatic value, while ρ denotes the constant density. At infinite depth there are further conditions, namely

$$\left\{ \begin{array}{lcl} \phi_x & = & \psi_y \to c \\ \phi_y & = & -\psi_x \to 0, \end{array} \right. \quad \text{as } y \to \infty. \tag{2.3}$$

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Since the surface tension creates a difference of pressure along the free surface we have that

$$P - P_0 = \frac{\sigma}{R},$$

whereby P is the pressure in the fluid at the surface, P_0 is the atmospheric pressure, σ denotes the coefficient of surface tension, and R is the radius of curvature of the surface, counted positive when the center of curvature lies inside the fluid. For positive R we have

$$\frac{1}{R} = \frac{y''(x)}{(1 + y'(x)^2)^{\frac{3}{2}}}.$$

Hence, the Bernoulli's condition becomes

$$\frac{\sigma}{\rho} \frac{y''(x)}{(1 + y'(x)^2)^{\frac{3}{2}}} + \frac{1}{2}|\nabla \phi|^2 = \frac{1}{2}c^2, \tag{2.4}$$

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at the free surface. The problem simplifies if we set (ϕ, ψ) as new independent variables and (τ, θ) new dependent variables, where $\tau = \log q$, $q = |\nabla \phi|$ and $(q \cos \theta, \sin \theta)$ are the velocity components. In the new notation the equation (2.4) can be written as

$$\frac{\sigma}{\rho} q \frac{\partial \theta}{\partial \phi} + \frac{1}{2}q^2 = \frac{1}{2}c^2, \text{ on } \psi = 0.$$

Taking $\sigma/\rho c^2$ as unit of length and c as unit of velocity we obtain the surface boundary condition in the variables (ϕ, ψ) as

$$\frac{\partial \theta}{\partial \phi} = -\sinh \tau, \text{ on } \psi = 0. \tag{2.5}$$

From the holomorphy property of τ and θ the water wave problem is finally recasted as the problem

$$\left\{ \begin{array}{c} \frac{\partial^2 \tau}{\partial \phi^2} + \frac{\partial^2 \tau}{\partial \psi^2} = 0, \\ \frac{\partial \tau}{\partial \psi} = -\sinh(\tau), \text{ on } \psi = 0, \\ \tau \to 0, \ \theta \to 0 \text{ as } \psi \to \infty. \end{array} \right. \tag{2.6}$$

rec

Denoting $z = x + iy$ and $w = \phi + i\psi$ and returning to the original length and velocity units it is proved in [10] that the solutions to the capillary wave problem are given by

$$z = \frac{w}{c} - \frac{4i}{k}\frac{1}{1 + Ae^{\frac{ikw}{c}}} + \frac{4i}{k}, \tag{2.7}$$

zw

where A and $k \geq 0$ are constants such that $A^2 = \frac{k - \frac{\rho c^2}{\sigma}}{k + \frac{\rho c^2}{\sigma}}$, the latter relation accounting for Bernoulli's condition. From (2.7) we see that an increse of w by $\frac{2\pi c}{k}$ results in an increase of z by $\frac{2\pi}{k}$, therefore for the wavelength λ we have

$$\lambda = \frac{2\pi}{k} = 2\pi \left(\frac{1 - A^2}{1 + A^2}\right)\frac{\sigma}{\rho c^2}. \tag{2.8}$$

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The solutions in (2.7) can equivalently be written as

$$\begin{aligned} \frac{x}{\lambda} &= \alpha - \frac{2}{\pi}\frac{Ae^{-2\pi\beta}\sin(2\pi\alpha)}{1+2Ae^{-2\pi\beta}\cos(2\pi\alpha)+A^2e^{-4\pi\beta}} \\ \frac{y}{\lambda} &= \beta - \frac{2}{\pi}\frac{1+Ae^{-2\pi\beta}\cos(2\pi\alpha)}{1+2Ae^{-2\pi\beta}\cos(2\pi\alpha)+A^2e^{-4\pi\beta}} + \frac{2}{\pi} \end{aligned} \tag{2.9}$$

Crappermap

where $\alpha = \frac{\phi}{c\lambda}$, $\beta = \frac{\psi}{c\lambda}$. It was shown in [17] that under certain positivity assumptions they are the only solutions to the capillary water wave problem.

Remark 2.1. If a denotes the amplitude of the wave, defined as the vertical height between trough and crest, it is a consequence of the formula (2.9) that

$$\frac{a}{\lambda} = \frac{4|A|}{\pi(1 - A^2)}. \tag{2.10}$$

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3. Crapper's wave as a global diffeomorphism

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Theorem 3.1. *The map (2.9) defines for all $A \in [-3 + 2\sqrt{2}, 3 - 2\sqrt{2}]$ a diffeomorphism from the domain $\{(\alpha, \beta) : \alpha \in \mathbb{R}, \beta > 0\}$ in the (x, y) plane to the domain below the surface $y = \eta(x)$.*

Proof. We begin by showing that (2.9) provides a local diffeormorphism from the domain $\{(\alpha, \beta) : \alpha \in \mathbb{R}, \beta > 0\}$ onto its image. The differential of (2.9) at a fixed point (α, β) with $\beta > 0$ equals

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{4Ae^{-2\pi\beta}}{b^2}\begin{pmatrix} \cos(2\pi\alpha)(1 + A^2e^{-4\pi\beta}) + 2Ae^{-2\pi\beta} & \sin(2\pi\alpha)(A^2e^{-4\pi\beta} - 1) \\ -\sin(2\pi\alpha)(A^2e^{-4\pi\beta} - 1) & \cos(2\pi\alpha)(1 + A^2e^{-4\pi\beta}) + 2Ae^{-2\pi\beta} \end{pmatrix}, \tag{3.1}$$

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with $b = 1 + 2Ae^{-2\pi\beta}\cos(2\pi\alpha) + A^2e^{-4\pi\beta}$.

b

Remark 3.2. It is not difficult to see that $b > 0$. Indeed, denoting $x = Ae^{-2\pi\beta}$ it follows that $b = x^2 + 2x\cos(2\pi\alpha) + 1$. The discriminant of the latter second order expression equals $4\cos^2(2\pi\alpha) - 4 < 0$ for $\alpha \in (0, 1) \setminus \{\frac{1}{2}\}$. If $\alpha = 0$ or $\alpha = \frac{1}{2}$ we see that $b = 0$ is equivalent to $x = 1$ or $x = -1$. However, since $\beta \geq 0$, we have that $|x| = |A|e^{-2\pi\beta} \leq |A| \leq 1$. From the formula for A we see that $|A| = 1$ if and only if $A = \pm i$. But then $x = \pm 1 \Leftrightarrow \pm ie^{-2\pi\beta} = \pm 1$, which is impossible. This shows that $b > 0$ for all $\alpha \in [0, 1)$ and for all $\beta > 0$.

Hence the determinant of the matrix in (3.1) is

$$d = \left(1 - \frac{4Ae^{-2\pi\beta}}{b^2}\left(\cos(2\pi\alpha)(1+A^2e^{-4\pi\beta}) + 2Ae^{-2\pi\beta}\right)\right)^2 + \left(\frac{4Ae^{-2\pi\beta}}{b^2}\right)^2\left(\sin(2\pi\alpha)(A^2e^{-4\pi\beta}-1)\right)^2 \tag{3.2}$$

By the same argument as in Remark 3.2 we have that $A^2e^{-4\pi\beta} - 1 \neq 0$. Therefore we see at once that $d > 0$ for all $A \neq 0$, $\alpha \in (0,1) \setminus \frac{1}{2}$ and all $\beta \geq 0$. If $\alpha = \frac{1}{2}$ we have with $x = Ae^{-2\pi\beta}$ that

$$\begin{aligned} d = \frac{1}{b^4}\left((1-x)^4 - 4x(-1-x^2+2x)\right)^2 &= \frac{1}{b^4}\left((1-x)^4 + 4x(1-x)^2\right)^2 \\ &= \frac{1}{b^4}\left((1-x)^2(1+x)^2\right)^2 > 0, \end{aligned} \tag{3.3}$$

since $|x| \leq |A| \leq 3 - 2\sqrt{2} < 1$ for all $\beta \geq 0$. The case $\alpha = 0$ can be treated similarly. Due to periodicity we then see that $d > 0$ for all $\alpha \in \mathbb{R}$, $\beta \geq 0$ and all $A \in (-3+2\sqrt{2}, 3-2\sqrt{2})$. From the inverse function theorem we infer that (2.9) is a local diffeomorphism from $\{(\alpha,\beta) : \alpha \in \mathbb{R}, \beta > 0\}$ onto its image. We now want to prove that (2.9) is a global diffeomorphism. To prove the injectivity we set $\xi = (\alpha, \beta)$ and we write the right-hand side of (2.9) as $F(\xi) = \xi + f(\xi)$, where where

$$f(\alpha,\beta) = \begin{pmatrix} f_1(\alpha,\beta) \\ f_2(\alpha,\beta) \end{pmatrix} = \begin{pmatrix} -\frac{2}{\pi}\frac{Ae^{-2\pi\beta}\sin(2\pi\alpha)}{1+2Ae^{-2\pi\beta}\cos(2\pi\alpha)+A^2e^{-4\pi\beta}} \\ -\frac{2}{\pi}\frac{1+Ae^{-2\pi\beta}\cos(2\pi\alpha)}{1+2Ae^{-2\pi\beta}\cos(2\pi\alpha)+A^2e^{-4\pi\beta}} + \frac{2}{\pi} \end{pmatrix}.$$

Borrowing a technique from [2] we have that

$$\begin{aligned} |F(\xi_2) - F(\xi_1)| &\geq |\xi_2 - \xi_1| - |f(\xi_2) - f(\xi_1)| \\ &\geq |\xi_2 - \xi_1| - \left(\max_{s\in[0,1]} ||Df_{s\xi_1+(1-s)\xi_2}||\right)|\xi_2 - \xi_1| \end{aligned} \tag{3.4}$$

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where $||Df_{(\alpha,\beta)}|| = \max_{\theta\in[0,2\pi]} |Df_{(\alpha,\beta)}(\cos\theta, \sin\theta)|$ and

$$Df_{(\alpha,\beta)} = -\frac{4Ae^{-2\pi\beta}}{b^2}\begin{pmatrix} \cos(2\pi\alpha)(1+A^2e^{-4\pi\beta}) + 2Ae^{-2\pi\beta} & \sin(2\pi\alpha)(A^2e^{-4\pi\beta}-1) \\ -\sin(2\pi\alpha)(A^2e^{-4\pi\beta}-1) & \cos(2\pi\alpha)(1+A^2e^{-4\pi\beta}) + 2Ae^{-2\pi\beta} \end{pmatrix}. \tag{3.5}$$

We then have

$$|Df_{(\alpha,\beta)}(\cos\theta, \sin\theta)| = \frac{4|A|e^{-2\pi\beta}}{b^2}|a_{\alpha\beta}(\theta), b_{\alpha\beta}(\theta)|,$$

with

$$a_{\alpha\beta}(\theta) = \cos(2\pi\alpha+\theta) + A^2e^{-4\pi\beta}\cos(2\pi\alpha-\theta) + 2Ae^{-2\pi\beta}\cos\theta,$$

$$b_{\alpha\beta}(\theta) = \sin(2\pi\alpha+\theta) + A^2e^{-4\pi\beta}\sin(\theta-2\pi\alpha) + 2Ae^{-2\pi\beta}\sin\theta.$$

Hence

$$\begin{aligned} a_{\alpha\beta}^2(\theta) + b_{\alpha\beta}^2(\theta) &= 1 + (A^2 e^{-4\pi\beta})^2 + 4A^2 e^{-4\pi\beta} \\ &+ 2A^2 e^{-4\pi\beta}(\cos(\theta+2\pi\alpha)\cos(\theta-2\pi\alpha) + \sin(\theta+2\pi\alpha)\sin(\theta-2\pi\alpha)) \\ &+ 4A^3 e^{-6\pi\beta}(\cos(\theta-2\pi\alpha)\cos\theta + \sin(\theta-2\pi\alpha)\sin\theta) \\ &+ 4Ae^{-2\pi\beta}(\cos(\theta+2\pi\alpha)\cos\theta + \sin(\theta+2\pi\alpha)\sin\theta) \\ &= 1 + (A^2 e^{-4\pi\beta})^2 + 4A^2 e^{-4\pi\beta} + 2A^2 e^{-4\pi\beta}(2\cos^2(2\pi\alpha) - 1) \\ &+ 4A^3 e^{-6\pi\beta}\cos(2\pi\alpha) + 4Ae^{-2\pi\beta}\cos(2\pi\alpha) \\ &= 1 + (A^2 e^{-4\pi\beta})^2 + 4A^2 e^{-4\pi\beta}\cos^2(2\pi\alpha) + 2A^2 e^{-4\pi\beta} \\ &+ 4A^3 e^{-6\pi\beta}\cos(2\pi\alpha) + 4Ae^{-2\pi\beta}\cos(2\pi\alpha) \\ &= b^2. \end{aligned} \tag{3.6}$$

Henceforth,

$$|Df_{(\alpha,\beta)}(\cos\theta, \sin\theta)| = \frac{4|A|e^{-2\pi\beta}}{b^2} \cdot b = \frac{4|A|e^{-2\pi\beta}}{b},$$

for all $\alpha \in [0,1), \beta \geq 0$ and for all $\theta \in [0, 2\pi]$. Denoting $\xi_1 = (\alpha_1, \beta_1)$ and $\xi_2 = (\alpha_2, \beta_2)$ we get that

$$\max_{s\in[0,1]} ||Df_{s\xi_1+(1-s)\xi_2}|| = \frac{4|A|e^{-2\pi\beta_0}}{b(\alpha_0, \beta_0)}, \tag{3.7}$$ maxnorm

where $\alpha_0 \in [\min(\alpha_1, \alpha_2), \max(\alpha_1, \alpha_2)]$, $\beta_0 \in [\min(\beta_1, \beta_2), \max(\beta_1, \beta_2)]$ and $b(\alpha_0, \beta_0)$ is obtained by replacing α with α_0 and β with β_0 in the formula for b. Looking at (3.4) and (3.7) we see that

$$|F(\xi_2) - F(\xi_1)| \geq \left(1 - \frac{4|A|e^{-2\pi\beta_0}}{b(\alpha_0, \beta_0)}\right)|\xi_2 - \xi_1|, \tag{3.8}$$

for all $\xi_1 = (\alpha_1, \beta_1)$ and $\xi_2 = (\alpha_2, \beta_2)$ with $\alpha_1, \alpha_2 \in [0,1)$ and $\beta_1, \beta_2 \geq 0$. Therefore, in order to prove injectivity, we only need to show that $b > 4|A|e^{-2\pi\beta}$ for all $\alpha \in [0,1)$ and all $\beta > 0$. We distinguish two cases.

Case $A \in [0, 3 - 2\sqrt{2}]$:

We note that $b > 4|A|e^{-2\pi\beta}$ is equivalent to

$$A^2 e^{-4\pi\beta} + (2\cos(2\pi\alpha) - 4)Ae^{-2\pi\beta} + 1 > 0, \tag{3.9}$$ secordineq

which, after setting $x = Ae^{-2\pi\beta}$, is equivalent to showing

$$x^2 + (2\cos(2\pi\alpha) - 4)x + 1 > 0, \tag{3.10}$$

for all $\alpha \in [0,1)$ and all $\beta > 0$. Note that

$$x_1(\alpha) := 2 - \cos(2\pi\alpha) - \sqrt{(1 - \cos(2\pi\alpha))(3 - \cos(2\pi\alpha))}$$

is the smallest of the two positive roots of the equation $x^2 + (2\cos(2\pi\alpha) - 4)x + 1 = 0$. Since the function

$$[-1,1] \to \mathbb{R},\ t \to 2 - t - \sqrt{(1-t)(3-t)}$$

is strictly increasing, we have that

$$\min_{\alpha\in[0,1)} x_1(\alpha) = 3 - 2\sqrt{2} \in (0,1).$$

Because $A \in [0, 3-2\sqrt{2}]$ we see, since $\beta > 0$, that $x = Ae^{-2\pi\beta} \in [0, 3-2\sqrt{2})$ and therefore (3.9) is verified for all $\alpha \in [0,1)$ and all $\beta > 0$.
Case $A \in [-3+2\sqrt{2}, 0)$:
The inequality $b > 4|A|e^{-2\pi\beta}$ is equivalent to

$$A^2 e^{-4\pi\beta} + (2\cos(2\pi\alpha) + 4)Ae^{-2\pi\beta} + 1 > 0. \tag{3.11}$$

secsecordineq

As before, with the notation $y = Ae^{-2\pi\beta}$ we see that (3.11) reduces to showing that

$$y^2 + (2\cos(2\pi\alpha) + 4)y + 1 > 0$$

for all $\alpha \in [0,1)$ and all $\beta > 0$. Note that

$$y_2(\alpha) := -2 - \cos(2\pi\alpha) + \sqrt{(1+\cos(2\pi\alpha))(3+\cos(2\pi\alpha))}$$

is the greatest of the two negative roots of the equation $y^2 + (2\cos(2\pi\alpha) + 4)y + 1 = 0$. It turns out that

$$\max_{\alpha\in[0,1)} y_2(\alpha) = -3 + 2\sqrt{2} \in (-1, 0).$$

Since $A \in [-3+2\sqrt{2}, 0)$ we have, since $\beta > 0$, that $y = Ae^{-2\pi\beta} \in (-3+2\sqrt{2}, 0]$ and (3.11) is verified for all $\alpha \in [0,1)$ and all $\beta > 0$.
In order to prove surjectivity we appeal like in [13] to the "Invariance of the Domain" theorem. We only state this theorem here and refer the interested reader to [19]. Let Ω be an open set (not necessarily bounded) of $\mathbb{R}^n$. If $F : \Omega \to \mathbb{R}^n$ is one to one and continuous, then $F(\Omega)$ is open and $F(\partial\Omega) = \partial F(\Omega)$.
To apply the above mentioned Invariance of the Domain Theorem we set $\Omega := \{(\alpha,\beta) : \beta > 0\}$ and let F be the map (2.9). We then infer that $F(\Omega)$ is open and that the boundary of $F(\Omega)$ is $F(\partial\Omega)$ which is the curve

$$\left\{\left(\lambda\left(\alpha - \frac{2}{\pi}\frac{A\sin(2\pi\alpha)}{1+A^2+2A\cos(2\pi\alpha)}\right), \lambda\left(\frac{2}{\pi} - \frac{2}{\pi}\frac{1+A\cos(2\pi\alpha)}{1+A^2+2A\cos(2\pi\alpha)}\right)\right) : \alpha \in \mathbb{R}\right\}, \tag{3.12}$$

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obtained by setting $\beta = 0$ in (2.9), i.e., the free surface, which does not have self intersections as we can infer from the injectivity up to $\beta = 0$ of the map (2.9). Since the boundary of $F(\Omega)$ is the free surface it follows that $F(\Omega)$ is one of the domains separated by it. Putting $\beta = \infty$ it follows that $y = \infty$. Therefore $F(\Omega)$ is the domain which contains $y = \infty$, namely the water domain. This shows the surjectivity of the map (2.9).
Adding the previous considerations we have altogether that (2.9) is a global diffeomorphism. □

A few remarks are in order.

Remark 3.3. A glance at the formula (2.10) shows that for the wave amplitude a of the flows that we found to be dynamically possible the following inequality is true

$$0 \le a \le \frac{\sqrt{2}}{2\pi}\lambda.$$

Remark 3.4. It is a fact ([18], page 46) that Crapper's wave can be represented as $y = \eta(x)$ with a single valued function η if and only if $|A| < \sqrt{2} - 1$. It would be then interesting to know whether we can extend the result of Theorem 3.1 beyond the value $3 - 2\sqrt{2}$.

Remark 3.5. We want now to determine the value of A for which the flows given by the map (2.9) have the flat free surface. Setting $\beta = 0$ in formula (2.9) we have for the vertical coordinate y of a point on the free surface the formula

$$\frac{y}{\lambda} = \frac{2}{\pi} - \frac{2}{\pi}\frac{1 + A\cos(2\pi\alpha)}{1 + A^2 + 2A\cos(2\pi\alpha)} \tag{3.13}$$

We then see that

$$\frac{2}{\pi} - \frac{2}{\pi}\frac{1 + A\cos(2\pi\alpha)}{1 + A^2 + 2A\cos(2\pi\alpha)}$$

is constant if and only if

$$\frac{1 + A\cos(2\pi\alpha)}{1 + A^2 + 2A\cos(2\pi\alpha)} = m = \text{const},$$

which is true if and only if $(2m - 1)A = 0$. The case $m = \frac{1}{2}$ leads to $A^2 = 1$ which, due to (2.8), leads further to $\lambda = 0$, which is impossible. It remains the case $A = 0$ which gives the flow with the flat free surface $x = \lambda\alpha\,(\alpha \in \mathbb{R}), y = 0$.

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