EXISTENCE OF CAPILLARY-GRAVITY WATER WAVES WITH PIECEWISE CONSTANT VORTICITY

CALIN IULIAN MARTIN AND BOGDAN–VASILE MATIOC

ABSTRACT. In this paper we construct small-amplitude periodic capillary-gravity water waves with a piecewise constant vorticity distribution. They describe water waves traveling on superposed linearly sheared currents that have different vorticities. This is achieved by associating to the height function formulation of the water wave problem a diffraction problem where we impose suitable transmission conditions on each line where the vorticity function has a jump. The solutions of the diffraction problem, found by using local bifurcation theory, are the desired solutions of the hydrodynamical problem.

1. INTRODUCTION

We are concerned in this paper with the existence of steady periodic rotational waves interacting with currents that possess a discontinuous vorticity distribution, a situation accounting for sudden changes in a current and whose numerical simulations have only recently been undertaken, see [27, 28]. More precisely, we establish the existence of capillary-gravity waves propagating at constant speed over a flat bed and interacting with several vertically superposed and linearly sheared currents of different (constant) vorticities. On physical grounds we can justify this situation by the fact that rotational waves generated by wind possess a thin layer of high vorticity that is adjacent to the wave surface [41, 43], while in the near bed region there may exist currents resulting from sediment transport along the ocean bed [42].

A rotational fluid is not only interesting as an intricate mathematical problem but also serves a very concrete physical situation since it models wave-current interactions among other phenomena [5, 25, 45]. The waves we consider here are two-dimensional, have an apriori unknown free surface, and the vorticity function is piecewise constant. Though the vorticity distribution considered in the context of pure gravity waves in [9] corresponds to a merely bounded vorticity function, being more general than ours, we have in addition to gravity also the surface tension as a restoring force. This has the effect of adding a second order term in the top boundary condition of the height function formulation of the problem, situation that makes the analysis more intricate. We enhance that surface tension appears in the dynamics of water waves in many physical situations one of which is that of wind blowing over a still fluid surface and giving rise to two-dimensional small amplitude wave trains driven by capillarity [26] which grow larger and turn into capillary-gravity waves.

In the irrotational regime, the local bifurcation picture was described in [22] for waves traveling over a fluid layer of finite depth, respectively in [23, 24, 44] for waves of infinite depth. A particular feature in the irrotational case and for waves with constant vorticity [34] is that sometimes a mode interacts with another one of half its size giving rise to waves

with two crests within a period, so-called Wilton ripples. Capillary-gravity water waves with a constant vorticity and stagnation points have been shown to exist in [33] by using techniques related to the ones employed in [10] for pure gravity waves. Allowing for a general Hölder continuous vorticity distribution, the existence theory is further developed in [47] for flows without stagnation points. The methods presented herein were recently further developed to establish the existence of capillary-gravity flows with arbitrary bounded [40] and even unbounded vorticity [35]. It is worth to mention that the many properties of capillary-gravity water waves, such as the regularity of the streamlines and of the wave profile [3, 17, 18, 37, 40], or the description of particle trajectories within the fluid [16], were only recently considered (see [4, 7, 11, 36] for the case when surface tension is neglected).

In this paper we consider a different context than in [47], namely that of waves with a step function like vorticity. In order to prove our result we use the height function formulation of the water wave problem which is obtained via the Dubreil-Jacotin transformation (see [9] for details) and which has the advantage that the original free boundary problem is rendered into a quasilinear elliptic problem in a fixed domain. While in [9] the authors worked with a weak formulation of this problem, we are not able to do so here. This is due to the fact that the top boundary condition is nonlinear and contains second order derivatives of the unknown. We overcome this difficulty by associating to the height function formulation a diffraction (or transmission) problem where we impose suitable transmission conditions on each horizontal line where the vorticity has a jump. Then, using existing results for diffraction problems together with a Fourier multiplier argument, we are able to recast the mathematical problem as an abstract bifurcation problem in a functional analytic context which enables us to use local bifurcation theory. One of the difficulties in doing this is due to the lack or rigorous results concerning the $C^{2+\alpha}$-regularity of solutions to diffraction problems close to the interface where transmission conditions are imposed. We emphasize that diffraction problems are not seldom, they appear when multiphase flows are considered, as is the case of the Muskat problem [13]. The solutions that we find solve the boundary conditions of the problem in classical sense and the quasilinear equation in the weak sense defined in [9] and almost everywhere in the transformed fluid domain.

In addition to proving existence of waves with the already mentioned properties we derive, in the case of two underlying currents, the dispersion relation which is an implicit equation relating: the mean depth, the average thickness of the currents, the wavelength, the constant vorticities of the currents, and the relative speed at the surface of the laminar flow.

The outline of the paper is as follows: in Section 2 we present the mathematical model and the main result. Section 3 is devoted to recasting the equation as an abstract bifurcation problem and to the study of the Fredholm property of an operator associated to a diffraction problem. In Section 4 we find necessary conditions for local bifurcation, and in Section 5 we prove the main result Theorem 2.1 and derive the dispersion relation.

2. The mathematical model and the main result

The mathematical model. We consider herein two-dimensional periodic waves over a rotational, inviscid, and incompressible fluid, that are driven by the interplay of gravity and surface tension forces. Moreover, the waves are assumed to move at constant wave speed $c > 0$. In a reference frame which moves in the same direction as the wave and with speed c, the free surface of the wave is considered to be the graph $y = \eta(x)$.

Assuming that the fluid is homogeneous (with unit density), in the fluid domain

$$\Omega_\eta := \{(x,y) \,:\, x \in \mathbb{S} \text{ and } -d < y < \eta(x)\},$$

the equations of motion are the steady-state Euler equations

$$\begin{cases} (\mathbf{u} - c)\mathbf{u}_x + \mathbf{v}\mathbf{u}_y &= -\mathbf{P}_x, \\ (\mathbf{u} - c)\mathbf{v}_x + \mathbf{v}\mathbf{v}_y &= -\mathbf{P}_y - g, \\ \mathbf{u}_x + \mathbf{v}_y &= 0. \end{cases} \tag{2.1a}$$

We use $\mathbf{P}$ to denote the dynamic pressure, $(\mathbf{u}, \mathbf{v})$ is the velocity field, g stands for the gravity constant, and d is the average depth of the water. Furthermore, $\mathbb{S}$ is the unit circle, meaning that $\eta, (\mathbf{u}, \mathbf{v}), \mathbf{P}$ are 2π-periodic in x. The equations (2.1a) are supplemented by the following boundary conditions

$$\begin{cases} \mathbf{P} &= \mathbf{P}_0 - \sigma\eta''/(1 + \eta'^2)^{3/2} & \text{on } y = \eta(x), \\ \mathbf{v} &= (\mathbf{u} - c)\eta' & \text{on } y = \eta(x), \\ \mathbf{v} &= 0 & \text{on } y = -d, \end{cases} \tag{2.1b}$$

with $\mathbf{P}_0$ denoting the constant atmospheric pressure and $\sigma > 0$ being the surface tension coefficient. Since the flow is rotational, the vorticity of the flow is the scalar function

$$\omega := \mathbf{u}_y - \mathbf{v}_x \qquad \text{in } \overline{\Omega}_\eta. \tag{2.1c}$$

The problem (2.1) can also be reformulated as a free boundary value problem, by introducing the stream function $\psi : \overline{\Omega}_\eta \to \mathbb{R}$ by the relation

$$\psi(x,y) := -p_0 + \int_{-d}^{y} (\mathbf{u}(x,s) - c)\,ds \qquad \text{for } (x,y) \in \overline{\Omega}_\eta.$$

We have that $\nabla\psi = (-\mathbf{v}, \mathbf{u} - c)$ and, as shown in [5, 8], the problem (2.1) is equivalent to the following system

$$\begin{cases} \Delta\psi &= \gamma(-\psi) & \text{in } \Omega_\eta, \\ |\nabla\psi|^2 + 2g(y + d) - 2\sigma\dfrac{\eta''}{(1 + \eta'^2)^{3/2}} &= Q & \text{on } y = \eta(x), \\ \psi &= 0 & \text{on } y = \eta(x), \\ \psi &= -p_0 & \text{on } y = -d, \end{cases} \tag{2.2}$$

the constant $p_0 < 0$ representing the relative mass flux, and $Q \in \mathbb{R}$ being related to the so-called total head. The function γ, called the vorticity function, is obtained by making the additional assumption that the horizontal velocity of each fluid particle is less than the wave speed

$$\mathbf{u} - c < 0 \qquad \text{in } \overline{\Omega}_\eta, \tag{2.3}$$

condition valid for waves that are not near breaking [46]. This assumption is needed in order to show, cf. [8, 38], that

$$\omega(x,y) = \gamma(-\psi(x,y)) \qquad \text{in } \overline{\Omega}_\eta.$$

The assumption (2.3) is also crucial when obtaining the third equivalent formulation of the problem (2.1), the height function formulation (2.4). Namely, because of (2.3), the function $\Phi : \overline{\Omega}_\eta \to \overline{\Omega}$ defined by

$$\Phi(x,y) := (q,p)(x,y) := (x, -\psi(x,y)), \qquad (x,y) \in \overline{\Omega}_\eta,$$

where $\Omega := \mathbb{S} \times (p_0, 0)$, is a diffeomorphism. Then, defining the height function $h : \overline{\Omega} \to \mathbb{R}$ by $h(q, p) := y + d$ for $(q, p) \in \overline{\Omega}$, the problem (2.2)-(2.3) is equivalent to the nonlinear boundary value problem

$$
\begin{cases}
(1 + h_q^2)h_{pp} - 2h_p h_q h_{pq} + h_p^2 h_{qq} - \gamma(p)h_p^3 = 0 & \text{in } \Omega, \\
1 + h_q^2 + (2gh - Q)h_p^2 - 2\sigma \dfrac{h_p^2 h_{qq}}{(1 + h_q^2)^{3/2}} = 0 & \text{on } p = 0, \\
h = 0 & \text{on } p = p_0,
\end{cases} \tag{2.4}
$$

subjected to the condition

$$
\min_{\overline{\Omega}} h_p > 0. \tag{2.5}
$$

The main result. We construct in this paper solutions of the problem (2.4)-(2.5) in the case when the vorticity function is a step function. More precisely, we assume that there exists an integer $N \geq 2$, real constants $\gamma_1, \ldots \gamma_N$ with $\gamma_{i-1} \neq \gamma_i$ for $1 \leq i \leq N$, and real numbers $p_0 < p_1 < \ldots < p_N = 0$ such that

$$
\gamma(p) := \gamma_i \qquad \text{for } 1 \leq i \leq N \text{ and } p \in (p_{i-1}, p_i). \tag{2.6}
$$

At each $p = p_i$, $0 \leq i \leq N$, the vorticity function may have (has if $1 \leq i \leq N - 1$) a discontinuity of the first kind. Our solutions satisfy the boundary conditions of (2.4) in classical sense and the first equation almost everywhere in Ω and in the following weak sense

$$
\int_{\Omega} \left(\frac{h_q}{h_p} \phi_q - \left(\Gamma + \frac{1 + h_q^2}{h_p^2} \right) \phi_p \right) d(q, p) = 0 \qquad \text{for all } \phi \in C_0^1(\Omega). \tag{2.7}
$$

Hereby, the function Γ is the anti-derivative of the vorticity function

$$
\Gamma(p) := \int_0^p \gamma(s) \, ds \qquad \text{for } p_0 \leq p \leq 0. \tag{2.8}
$$

Our main result is the following theorem.

Theorem 2.1. *Let $N \in \mathbb{N}, N \geq 2$, $p_0 < p_1 < \ldots < p_N = 0$, $(\gamma_1, \ldots, \gamma_N) \in \mathbb{R}^N$ be given such that $\gamma_{i-1} \neq \gamma_i$ for $1 \leq i \leq N$, and let the vorticity function γ be defined by (2.6).*

Then, given $\alpha \in (0, 1)$, there exists a positive integer $n_{\min}$ and for each $n \in \mathbb{N}$ with $n \geq n_{\min}$ a real-analytic curve $\mathcal{C}_n$ in the space $C^{1+\alpha}(\overline{\Omega})$, consisting only of solutions h of problem (2.4)-(2.5) with the additional regularity $h \in C^{2-}(\overline{\Omega})$, $h(\cdot, 0) \in C^{\infty}(\mathbb{S})$, and

$$
h \in C^{2+\alpha}(\mathbb{S} \times [p_{i-1}, p_i]) \cap C^{\infty}(\mathbb{S} \times (p_{i-1}, p_i)) \qquad \text{for all } 1 \leq i \leq N. \tag{2.9}
$$

Each curve $\mathcal{C}_n$ contains a laminar flow solution (all the streamlines being parallel to the flat bed) and all the other points on the curve correspond to solutions that have minimal period $2\pi/n$, only one crest and trough per period, and are symmetric with respect to the crest line.

These solutions solve the last two equations of (2.4) in classical sense and the first equation almost everywhere in Ω (more precisely in $\Omega \setminus (\{\mathbb{S} \times \{p_i\} : 1 \leq i \leq N - 1\})$) and in the weak sense defined by (2.7).

Remark 2.2. The integer $n_{\min}$ in Theorem 2.1 may be chosen to be $n_{\min} = 1$ provided that the condition (4.27) is satisfied.

Though our result is true for any arbitrary finite number $N \geq 2$, any finite sequence $p_0 < p_1 < \ldots < p_N = 0$, and any tuple $(\gamma_1, \ldots, \gamma_N) \in \mathbb{R}^n$, we will prove first the result when $N = 2$ and $(\gamma_1, \gamma_2) \in \mathbb{R}^2$ satisfies $\gamma_1 \neq \gamma_2$. The corresponding result for $\gamma_1 = \gamma_2$, has been already established in [47] in the context of waves which satisfy also condition

(2.5), respectively in [33] for waves that possess stagnation points. For a characterization of the global continua corresponding to the local branches obtained in these references we refer to [39]. The proof of Theorem 2.1, with $N \geq 3$, will be discussed after proving the corresponding statement for $N = 2$.

Before of that, let us reconsider the problems (2.1) and (2.2) and interpret our solutions in the light of these formulations. It is well-known that the streamlines of the flow coincide with the level curves of the stream function ψ and that they are parametrized by the mappings $x \mapsto h(x, p) - d$, $p_0 \leq p \leq 0$ (the free wave surface corresponding to $p = 0$), cf. [8]. Using this property, we have the following result.

Corollary 2.3. *Given a solution h of the problem* (2.4)-(2.5) *as found in Theorem 2.1, we define $\eta_i(x) := h(x, p_i) - d$, $0 \leq i \leq N$, and*

$$\Omega_\eta^i := \{(x, y) \,:\, \eta_{i-1}(x) < y < \eta_i(x)\} \qquad \text{for } 1 \leq i \leq N.$$

Then, we have $\eta_i \in C^{2+\alpha}(\mathbb{S})$ for $1 \leq i \leq N-1$. The other streamlines are parametrized by smooth functions $h(\cdot, p) - d$, with $p \in [p_0, 0] \setminus \{p_i \,:\, 1 \leq i \leq N-1\}$, the wave surface being the graph $y = \eta(x) := \eta_N(x)$. Furthermore, the stream function ψ satisfies

$$\psi \in C^{2-}(\overline{\Omega}_\eta) \cap C^{2+\alpha}(\overline{\Omega_\eta^i}) \cap C^\infty(\Omega_\eta^i) \qquad \text{for all } 1 \leq i \leq N, \tag{2.10}$$

and we have

$$\mathbf{u}, \mathbf{v}, \mathbf{P} \in C^{1-}(\overline{\Omega}_\eta) \cap C^{1+\alpha}(\overline{\Omega_\eta^i}) \cap C^\infty(\Omega_\eta^i) \qquad \text{for all } 1 \leq i \leq N. \tag{2.11}$$

Proof. The stream function ψ and the height function h are coupled by the following relation: given $x \in \mathbb{S}$, the function $\psi(x, \cdot)$ is the solution of the initial value problem

$$\begin{cases} \psi_y(x, y) &=& -\dfrac{1}{h_p(x, -\psi(x, y))} \qquad \text{for } y \leq \eta(x), \\ \psi(x, \eta(x)) &=& 0, \end{cases}$$

cf. [5]. This proves (2.10). The property (2.11) follows from (2.10) and from Bernoulli's law

$$\frac{(c - \mathbf{u})^2 + \mathbf{v}^2}{2} + gy + \mathbf{P} + \Gamma(-\psi) = const. \qquad \text{in } \overline{\Omega}_\eta.$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3. The associated diffraction problem

In the following we choose $N = 2$, $p_0 < p_1 < 0$, $(\gamma_1, \gamma_2) \in \mathbb{R}^2$ with $\gamma_1 \neq \gamma_2$, and define the vorticity function γ by (2.6). In order to prove our main result Theorem 2.1, we associate to (2.4) the following diffraction (or transmission) problem

$$\begin{cases} (1 + u_q^2)u_{pp} - 2u_p u_q u_{pq} + u_p^2 u_{qq} - \gamma_1 u_p^3 &=& 0 & \text{in } \Omega_1, \\ (1 + U_q^2)U_{pp} - 2U_p U_q U_{pq} + U_p^2 U_{qq} - \gamma_2 U_p^3 &=& 0 & \text{in } \Omega_2, \\ 1 + U_q^2 + (2gU - Q)U_p^2 - 2\sigma \dfrac{U_p^2 U_{qq}}{(1 + U_q^2)^{3/2}} &=& 0 & \text{on } p = 0, \\ u &=& U & \text{on } p = p_1, \\ u_p &=& U_p & \text{on } p = p_1, \\ u &=& 0 & \text{on } p = p_0, \end{cases} \tag{3.1}$$

where we used the notation $\Omega_1 := \mathbb{S} \times (p_0, p_1)$ and $\Omega_2 := \mathbb{S} \times (p_1, 0)$. The reason for defining (3.1) is twofold. First of all, the nonlinear boundary condition on $p = 0$ contains second

order derivatives of the unknown, so that we cannot consider a weak formulation of (2.4) in a similar manner as in [9]. Secondly, using (2.4) we can still work in a classical Hölder setting and incorporate arbitrarily many jumps of the vorticity function into the problem. This is due to the fact that each solution of (3.1) defines a solution of (2.4).

Lemma 3.1. *Assume that* $(u, U) \in C^{2+\alpha}(\overline{\Omega}_1) \times C^{2+\alpha}(\overline{\Omega}_2)$ *is a solution of* (3.1). *Then, the function* $h : \overline{\Omega} \to \mathbb{R}$ *defined by*

$$h := \begin{cases} u & in\ \overline{\Omega}_1, \\ U & in\ \overline{\Omega}_2 \end{cases} \tag{3.2}$$

belongs to $C^{2-}(\overline{\Omega})$ *and solves the last two equations of* (2.4) *pointwise and the first equation of* (2.4) *in* $\Omega_1 \cup \Omega_2$. *Moreover, if* h *satisfies* (2.5), *then we additionally have* $h \in C^{\infty}(\Omega_i)$, $i = 1, 2$, *and* $h(\cdot, 0) \in C^{\infty}(\mathbb{S})$.

Proof. Because of the transmission conditions (equations four and five) of (3.1), we easily see that the first order derivatives of h are Lipschitz continuous, that is $h \in C^{2-}(\overline{\Omega}_1)$. Clearly, h solves also the last two equations of (2.4) and the first one in $\Omega_1 \cup \Omega_2$. To finish the proof, we note that condition (2.5) ensures that the first two equations of (3.1) are uniformly elliptic. Then, by elliptic regularity, cf. [15], we obtain that $h \in C^{\infty}(\Omega_i)$, $i = 1, 2$. The property that the wave surface is smooth, or equivalently that $h(\cdot, 0) \in C^{\infty}(\mathbb{S})$, follows from Theorem 7.3 in [1], by using the uniform ellipticity of the first equation of (2.4) together with the fact that the linearization of the boundary condition of (2.4) on $p = 0$ satisfies the complementing condition, see e.g. [20].   □

In the remainder of this paper we shall seek solutions of (3.1) that satisfy (2.5). According to Lemma 3.1, they are the solutions described in Theorem 2.1. To this end, we recast the problem (3.1) as an abstract bifurcation problem, and use then bifurcation tools to prove the existence of local bifurcation curves consisting of solutions of (3.1) and (2.5).

Laminar flow solutions. We introduce now an additional parameter λ into the problem (3.1) which is used to parametrize the trivial solutions of (3.1). These solutions describe water waves with a flat surface and parallel streamlines, and we call them laminar flow solutions of (3.1). The head Q corresponding to a laminar flow solution also depends on λ and therefore we are left with only one free parameter in (3.1).

We shall denote the laminar flow solutions by $(\overline{u}, \overline{U})$. Assuming that $(\overline{u}, \overline{U})$ depends only upon the variable p, we see that $(\overline{u}, \overline{U})$ solves (3.1) and (2.5) if and only if it is a solution of the system

$$\begin{cases} \overline{u}'' &=& \gamma_1 \overline{u}'^3 & \text{in } p_0 < p < p_1, \\ \overline{U}'' &=& \gamma_2 \overline{U}'^3 & \text{in } p_1 < p < 0, \\ 1 + (2g\overline{U}(0) - Q)\overline{U}'^2(0) &=& 0, \\ \overline{u}(p_1) &=& \overline{U}(p_1), \\ \overline{u}'(p_1) &=& \overline{U}'(p_1), \\ \overline{u}(p_0) &=& 0. \end{cases} \tag{3.3}$$

Whence, there exists $\lambda > 2\max_{[p_0,0]}\Gamma \geq 0$ such that we have

$$\overline{u}(p) := \overline{u}(p;\lambda) := \int_{p_0}^{p} \frac{1}{\sqrt{\lambda - 2\Gamma(s)}}\, ds, \qquad p \in [p_0, p_1],$$

$$\overline{U}(p) := \overline{U}(p;\lambda) := \int_{p_0}^{p} \frac{1}{\sqrt{\lambda - 2\Gamma(s)}}\, ds, \qquad p \in [p_1, 0]. \tag{3.4}$$

We observe that $(\overline{u}, \overline{U}) \in C^\infty([p_0, p_1]) \times C^\infty([p_1, 0])$, and that $(\overline{u}, \overline{U})$ verify the system (3.3) exactly when

$$Q = Q(\lambda) := \lambda + 2g \int_{p_0}^{0} \frac{1}{\sqrt{\lambda - 2\Gamma(p)}}\, dp. \tag{3.5}$$

Let us also observe that the constant λ is related to the speed at the flat free surface $y = 0$ of the laminar flow. Namely, we have that

$$\sqrt{\lambda} = \frac{1}{\overline{U}_p(0)} = (c - \mathbf{u})\big|_{y=0}.$$

The functional analytic setting. We define now an abstract functional analytic setting which allows us to recast the problem (3.1) as a bifurcation problem. Therefore, we choose $\alpha \in (0, 1)$ and define the Banach spaces:

$$\mathbb{X} := \left\{ (v, V) \in C^{2+\alpha}(\overline{\Omega}_1) \times C^{2+\alpha}(\overline{\Omega}_2) : v = V, \ v_p = V_p \text{ on } p = p_1, \text{ and } v\big|_{p=p_0} = 0 \right\},$$

$$\mathbb{Y}_1 := C^\alpha(\overline{\Omega}_1) \times C^\alpha(\overline{\Omega}_2), \qquad \mathbb{Y}_2 := C^\alpha(\mathbb{S}),$$

whereby we have identified, when defining $\mathbb{Y}_2$, the unit circle $\mathbb{S}$ with $p = 0$. Moreover, we introduce the operator $(\mathcal{F}, \mathcal{G}) : (2\max_{[p_0,0]}\Gamma, \infty) \times \mathbb{X} \to \mathbb{Y} := \mathbb{Y}_1 \times \mathbb{Y}_2$, with $\mathcal{F} := (\mathcal{F}_1, \mathcal{F}_2)$, by the following expressions:

$$\mathcal{F}_1(\lambda, v) := (1 + v_q^2)(v_{pp} + \overline{u}'') - 2(v_p + \overline{u}')v_q v_{pq} + (v_p + \overline{u}')^2 v_{qq} - \gamma_1(v_p + \overline{u}')^3,$$

$$\mathcal{F}_2(\lambda, V) := (1 + V_q^2)(V_{pp} + \overline{U}'') - 2(V_p + \overline{U}')V_q V_{pq} + (V_p + \overline{U}')^2 V_{qq} - \gamma_2(V_p + \overline{U}')^3,$$

$$\mathcal{G}(\lambda, V) := 1 + V_q^2 + (2g(V + \overline{U}) - Q(\lambda))(V_p + \overline{U}')^2 - 2\sigma \frac{(V_p + \overline{U}')^2 V_{qq}}{(1 + V_q^2)^{3/2}},$$

with $Q(\lambda)$ given by (3.5). Let us observe that the function $(\mathcal{F}, \mathcal{G})$ is well-defined and that it depends real-analytically on its arguments, that is

$$(\mathcal{F}, \mathcal{G}) \in C^\omega((2\max_{[p_0,0]}\Gamma, \infty) \times \mathbb{X}, \mathbb{Y}). \tag{3.6}$$

With this notation, the problem (3.1) is equivalent to the following abstract operator equation

$$(\mathcal{F}, \mathcal{G})(\lambda, (v, V)) = 0 \qquad \text{in } \mathbb{Y}. \tag{3.7}$$

Since when $(v, V) = 0$ the problem (3.7) is equivalent to the system (3.3), we have that

$$(\mathcal{F}, \mathcal{G})(\lambda, (0, 0)) = 0 \qquad \text{for all } \lambda \in (2\max_{[p_0,0]}\Gamma, \infty). \tag{3.8}$$

Moreover, if $(\lambda, (v, V))$ is a solution of (3.7), then the function $(u, U) := (\overline{u} + v, \overline{U} + V)$ solves the diffraction problem (3.1) when $Q = Q(\lambda)$ and, according to Lemma 3.1, it defines a solution h of the water wave problem (2.4). We remark that if (v, V) are sufficiently small, the associated solution h of (2.4) satisfies also the condition (2.5).

The Fredholm property for the linearized operator. Our main tool in determining non-laminar solutions of problem (3.7) is the theorem on bifurcation from simple eigenvalues due to Crandall and Rabinowitz, cf. [12].

Theorem 3.2. *Let $\mathbb{X}, \mathbb{Y}$ be real Banach spaces, $I \subset \mathbb{R}$ an open interval, and let the mapping $H \in C^\omega(I \times \mathbb{X}, \mathbb{Y})$ satisfy:*

(a) *$H(\lambda, 0) = 0$ for all $\lambda \in I$;*

(b) *There exists $\lambda_* \in I$ such that Fréchet derivative $\partial_x H(\lambda_*, 0)$ is a Fredholm operator of index zero with a one-dimensional kernel and*

$$\operatorname{Ker} \partial_x H(\lambda_*, 0) = \{sx_0 : s \in \mathbb{R}, 0 \neq x_0 \in \mathbb{X}\};$$

(c) *The tranversality condition holds*

$$\partial_{\lambda x} H(\lambda_*, 0)x_0 \notin \operatorname{Im} \partial_x H(\lambda_*, 0).$$

Then, $(\lambda_*, 0)$ is a bifurcation point in the sense that there exists $\varepsilon > 0$ and a real-analytic curve $(\lambda, x) : (-\varepsilon, \varepsilon) \to I \times \mathbb{X}$ consisting only of solutions of the equation $H(\lambda, x) = 0$. Moreover, as $s \to 0$, we have that

$$\lambda(s) = \lambda_* + O(s) \qquad and \qquad x(s) = sx_0 + O(s^2).$$

Furthermore there exists an open set $U \subset I \times \mathbb{X}$ with $(\lambda_*, 0) \in U$ and

$$\{(\lambda, x) \in U : H(\lambda, x) = 0, x \neq 0\} = \{(\lambda(s), x(s)) : 0 < |s| < \epsilon\}.$$

In order to use Theorem 3.2 in the context of the bifurcation problem (3.6)-(3.8), we choose $H := (\mathcal{F}, \mathcal{G})$. Therefore, we need to determine particular values of λ for which $\partial_{(v,V)}(\mathcal{F}, \mathcal{G})(\lambda, 0) \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$ is a Fredholm operator of index zero with a one-dimensional kernel. We prove first that $\partial_{(v,V)}(\mathcal{F}, \mathcal{G})(\lambda, 0)$ is a Fredholm operator of index zero for every value of $\lambda \in (2 \max_{[p_0,0]} \Gamma, \infty)$. The other hypotheses of Theorem 3.2 are achieved later on by choosing appropriate values for λ and by restricting the operator $(\mathcal{F}, \mathcal{G})$ to certain subspaces of $\mathbb{X}$ and $\mathbb{Y}$.

It is not difficult to see that the Fréchet derivative $\partial_{(v,V)}(\mathcal{F}, \mathcal{G})(\lambda, 0)$ is the linear operator $(L, T) \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$, with $L := (L_1, L_2)$, given by

$$
\begin{aligned}
L_1 v :&= v_{pp} + \overline{u}'^2 v_{qq} - 3\gamma_1 \overline{u}'^2 v_p, \\
L_2 V :&= V_{pp} + \overline{U}' V_{qq} - 3\gamma_2 \overline{U}'^2 V_p, \qquad\qquad\qquad \text{for } (v, V) \in \mathbb{X}. \qquad (3.9) \\
TV :&= 2\left[(2g\overline{U} - Q(\lambda))\overline{U}' V_p + g\overline{U}'^2 V - \sigma \overline{U}'^2 V_{qq} \right]\Big|_{p=0},
\end{aligned}
$$

Showing that (L, T) is a Fredholm operator does not follow from the existing theory on diffraction problems, cf. [29]. This is due to the fact that when solving the linear diffraction problem $(L, T)(v, V) = ((f, F), \varphi) \in \mathbb{Y}$, there are, as far as we know, no results that guarantee that (v, V) belong to $\mathbb{X}$. The problem is generated by the transmission conditions on $p = p_1$ which can be used to only show that $v \in C^{2+\alpha}(\mathbb{S} \times [p_0, p_1]) \cap C^{1+\alpha}(\overline{\Omega}_1)$ and $V \in C^{2+\alpha}(\mathbb{S} \times (p_1, p_0]) \cap C^{1+\alpha}(\overline{\Omega}_2)$. We will overcome this difficulty by using an approach based on the elliptic theory for linear boundary value problems with Venttsel boundary condition [32] together with a Fourier multiplier theorem for operators on Hölder spaces of periodic functions [14, 30].

Before proceeding, we observe that T can be re-expressed by the formula

$$TV = \frac{2}{\lambda}\left[gV - \lambda^{3/2} V_p - \sigma V_{qq} \right]\Big|_{p=0}. \qquad (3.10)$$

Theorem 3.3. *Assume that $(\gamma_1, \gamma_2) \in \mathbb{R}^2$ and that $\gamma_1 \neq \gamma_2$. Then, for every constant $\lambda \in (2 \max_{[p_0,0]} \Gamma, \infty)$, the Fréchet derivative $\partial_{(v,V)}(\mathcal{F}, \mathcal{G})(\lambda, 0) \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$ is a Fredholm operator of index zero.*

Proof. Let us presuppose that $(L, T_0) \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$, where

$$T_0 V := [V - V_{qq}]\big|_{p=0}$$

is an isomorphism. Then, it is obvious that we also have $(L, 2\sigma\lambda^{-1}T_0) \in \mathrm{Isom}(\mathbb{X}, \mathbb{Y})$. Observing that

$$(L, T)(v, V) = (L, 2\sigma\lambda^{-1}T_0)(v, V) + \left(0, \frac{2}{\lambda}\left[(g - \sigma)V - \lambda^{3/2}V_p\right]\big|_{p=0}\right)$$

for all $(v, V) \in \mathbb{X}$, and since the operator

$$\left[V \mapsto \frac{2}{\lambda}\left[(g - \sigma)V - \lambda^{3/2}V_p\right]\big|_{p=0}\right] \in \mathcal{L}(\mathbb{X}, \mathbb{Y}_2)$$

is compact, we deduce that (L, T) is a Fredholm operator of index zero.

Whence, we are left to prove that $(L, T_0) \in \mathrm{Isom}(\mathbb{X}, \mathbb{Y})$. First, we observe that the kernel of the operator (L, T_0) consists only of the zero vector. Indeed, if $(L, T_0)(v, V) = 0$ in $\mathbb{Y}$ and V has a positive maximum at $(\bar{q}, \bar{p})$, then $\bar{p} \in \{p_1, 0\}$, and because $T_0 V = 0$ we must have $\bar{p} = p_1$. But, then also v has a positive maximum at $(\bar{q}, \bar{p})$. Applying Hopf's lemma in each domain Ω_1 and Ω_2, we find that $V_p(\bar{q}, \bar{p}) < 0$ and $v_p(\bar{q}, \bar{p}) > 0$. This contradicts the transmission condition $v_p = V_p$ on $p = p_1$.

It remains to show that (L, T_0) is onto. To this end, let $((f, F), \varphi) \in \mathbb{Y}$ be given. The results established in [32] for second order elliptic equations with Venttsel boundary conditions imply that there exists a unique solution $W \in C^{2+\alpha}(\overline{\Omega}_2)$ of the problem

$$\begin{cases} L_2 W & = & F & \text{in } \Omega_2, \\ T_0 W & = & \varphi & \text{on } p = 0, \\ W & = & 0 & \text{on } p = p_1. \end{cases} \tag{3.11}$$

This property can be obtained also by referring to [15]. Indeed, it is not difficult to see that the problem $T_0 \widetilde{W} = \widetilde{W} - \widetilde{W}'' = \varphi$ possesses a unique solution $\widetilde{W} \in C^{2+\alpha}(\mathbb{S})$. Therefore, the function W solving (3.11) is the solution of the Dirichlet problem

$$L_2 W = F \quad \text{in } \Omega_2 \qquad W = \widetilde{\varphi} \quad \text{on } \partial\Omega_2,$$

whereby $\widetilde{\varphi} \in C^{2+\alpha}(\overline{\Omega}_2)$ is defined by $\widetilde{\varphi}(q, p) = (1 - p/p_1)\widetilde{W}(q)$. Moreover, we introduce the function $w \in C^{2+\alpha}(\overline{\Omega}_1)$ as being the unique solution of the Dirichlet problem

$$\begin{cases} L_1 w & = & f & \text{in } \Omega_1, \\ w & = & 0 & \text{on } p = p_1, \\ w & = & 0 & \text{on } p = p_0. \end{cases} \tag{3.12}$$

With this notation, it suffices to show that for every $\xi \in C^{1+\alpha}(\mathbb{S})$, the diffraction problem

$$\begin{cases} L_1 z & = & 0 & \text{in } \Omega_1, \\ L_2 Z & = & 0 & \text{in } \Omega_2, \\ T_0 Z & = & 0 & \text{on } p = 0, \\ z & = & Z & \text{on } p = p_1, \\ z_p - Z_p & = & \xi & \text{on } p = p_1, \\ z & = & 0 & \text{on } p = p_0, \end{cases} \tag{3.13}$$

possesses a (unique) solution $(z, Z) \in C^{2+\alpha}(\overline{\Omega}_1) \times C^{2+\alpha}(\overline{\Omega}_2)$. Indeed, if this is true, let (z, Z) denote the solution corresponding to $\xi = (W_p - w_p)\big|_{p=p_1}$. Then, $(v, V) := (w + z, W + Z)$ belongs to $\mathbb{X}$ and it solves the equation $(L, T_0)(v, V) = ((f, F), \varphi)$.

Hence, we are left to study the solvability of the problem (3.13). Using elliptic maximum principles as we did before, it is easy to see that (3.13) has for each $\xi \in C^{1+\alpha}(\mathbb{S})$ at most a classical solution. From $T_0 Z = 0$ on $p = 0$ we conclude that in fact $Z = 0$ on $p = 0$. These facts and the Theorems 16.1 and 16.2 of [29] ensure the existence of a unique solution (z, Z) of (3.13) with $z \in C^{1+\alpha}(\overline{\Omega}_1) \cap C^{2+\alpha}(\mathbb{S} \times [p_0, p_1))$ and $Z \in C^{1+\alpha}(\overline{\Omega}_2) \cap C^{2+\alpha}(\mathbb{S} \times (p_1, 0])$. Let us remark that we only need to show that the restriction $z|_{p_1} \in C^{2+\alpha}(\mathbb{S})$. This property together with the first and last equation of (3.13) show that $z \in C^{2+\alpha}(\overline{\Omega}_1)$. Since $z = Z$ on $p = p_1$, we also have $Z \in C^{2+\alpha}(\overline{\Omega}_2)$. To finish the proof, we represent the mapping

$$C^{1+\alpha}(\mathbb{S}) \ni \xi \mapsto z|_{p=p_1} \in C^{1+\alpha}(\mathbb{S}) \tag{3.14}$$

as a Fourier multiplier. To this end, we introduce the functions $a := 1/\overline{u}' \in C^\infty([p_0, p_1])$ and $A := 1/\overline{U}' \in C^\infty([p_1, 0])$, where $(\overline{u}, \overline{U})$ denote the solutions of (3.3). Similarly as in [8], the first two equations of (3.13) can be written in the more concise form

$$a^3 L_1 z = (a^3 z_p)_p + (a z_q)_q = 0 \quad \text{in } \Omega_1, \qquad A^3 L_2 Z = (A^3 Z_p)_p + (A Z_q)_q = 0 \quad \text{in } \Omega_2. \tag{3.15}$$

Considering now the Fourier expansions of $\xi, z,$ and Z:

$$\xi(q) = \sum_{k \in \mathbb{Z}} a_k e^{ikq}, \quad z(q, p) = \sum_{k \in \mathbb{Z}} z_k(p) e^{ikq}, \quad Z(q, p) = \sum_{k \in \mathbb{Z}} Z_k(p) e^{ikq},$$

we find that the functions (z_k, Z_k), $k \in \mathbb{Z}$, solve the following problem:

$$\begin{cases} (a^3 z_k')' - k^2 a z_k &= 0 & p_0 < p < p_1, \\ (A^3 Z_k')' - k^2 A Z_k &= 0 & p_1 < p < 0, \\ Z_k(0) &= 0, \\ z_k(p_1) &= Z_k(p_1), \\ z_k'(p_1) - Z_k'(p_1) &= a_k, \\ z_k(p_0) &= 0. \end{cases} \tag{3.16}$$

We already know from the solvability of (3.13) that the problem (3.16) possesses for each $k \in \mathbb{Z}$ a unique solution (z_k, Z_k) of regularity $z_k \in C^{2+\alpha}((p_0, p_1)) \cap C^{1+\alpha}([p_0, p_1])$ and $Z_k \in C^{2+\alpha}((p_1, 0]) \cap C^{1+\alpha}([p_1, 0])$. Moreover, these functions can be computed explicitly. Indeed, when $\gamma_1 \gamma_2 \neq 0$, using a substitution similar to that used in [8, Section 8], we find from the first two equations of (3.16) that

$$z_k = \frac{2\gamma_1}{a} \left(\beta e^{-|k|a/\gamma_1} + \delta e^{|k|a/\gamma_1} \right) \qquad \text{and} \qquad Z_k = \frac{2\gamma_2}{A} \left(\theta e^{-|k|A/\gamma_2} + \vartheta e^{|k|A/\gamma_2} \right), \tag{3.17}$$

with real constants $\beta, \delta, \theta, \vartheta$ that can be determine by solving the last four equations of (3.16). After some tedious, though elementary, computations we obtain that $z_k(p_1) = \lambda_k a_k$, whereby

$$\lambda_k := \frac{a^2(p_1)}{\gamma_1 - \gamma_2 + a(p_1) \left[\coth\left(\Theta_1 |k|\right) + \coth\left(\Theta_2 |k|\right) \right] |k|}, \qquad k \in \mathbb{Z}, \tag{3.18}$$

and Θ_1 and Θ_2 are the positive expressions

$$\Theta_1 := \frac{a(p_0) - a(p_1)}{\gamma_1} \qquad \text{and} \qquad \Theta_2 := \frac{A(p_1) - A(0)}{\gamma_2}. \tag{3.19}$$

When $k = 0$, the value λ_0 should be understood as the limit $\lim_{k \to 0} \lambda_k$. The formula (3.18) is still true when $\gamma_1 = 0$ or $\gamma_2 = 0$ with the mention that if $\gamma_1 = 0$, then we have to replace Θ_1 by its limit $\lim_{\gamma_1 \to 0} \Theta_1$, which is again a positive number (similarly when $\gamma_2 = 0$). We note that the solvability of (3.16) ensures that the denominator of the right-hand side of (3.18) has to be different from zero. With this observation, it is not difficult to see that

$$\sup_{k \in \mathbb{Z}} |k||\lambda_k| < \infty \qquad \text{and} \qquad \sup_{k \in \mathbb{Z}} |k|^2 |\lambda_{k+1} - \lambda_k| < \infty. \tag{3.20}$$

Since the mapping (3.14) can be identified with the Fourier multiplier

$$\sum_{k \in \mathbb{Z}} a_k e^{ikq} \mapsto \sum_{k \in \mathbb{Z}} \lambda_k a_k e^{ikq}$$

we infer from (3.20) and [30, Theorem 2.1] that it belongs to $\mathcal{L}(C^{1+\alpha}(\mathbb{S}), C^{2+\alpha}(\mathbb{S}))$. Consequently, the trace $z\big|_{p=p_1} \in C^{2+\alpha}(\mathbb{S})$, and this completes our argument. $\qquad \square$

4. The kernel of $\partial_{(v,V)}(\mathcal{F}, \mathcal{G})(\lambda, 0)$

In this section we merely assume that $\gamma_1 \in C^\alpha([p_0, p_1])$ and that $\gamma_2 \in C^\alpha([p_0, p_1])$. In order to identify the kernel of $\partial_{(v,V)}(\mathcal{F}, \mathcal{G})(\lambda, 0)$, we introduce a family of Sturm-Liouville operators and determine in a first step the kernel of these operators, cf. Lemmas 4.1 and 4.2. This will reduce our task to finding the zeros of the Wronskian determinant (4.6), which will be accomplished in Lemmas 4.3-4.7. In the case of two layers of constant vorticity there is a shorter argument relying on the dispersion relation (5.11). Nevertheless, this argument cannot be generalized to the case of three or more layers of constant vorticity. Compared to [47], where for the study of the kernel of the Sturm-Liouville operators, abstract functional analytic results for eigenvalue problems in the context of Pontryagin spaces are used, the approach presented herein is self-contained. It is based on direct estimates for the solution of Sturm-Liouville problems with non-constant coefficients, one of the coefficients growing linearly in dependence of a parameter. We emphasize that our approach can be generalized to arbitrary bounded [40] and even unbounded vorticities [35]. From the analysis in Section 3 it follows that, if $(v, V) = (v_k(p) \cos(kq), V_k(p) \cos(kq)) \in \mathbb{X}$ belongs to the kernel of $\partial_{(v,V)}(\mathcal{F}, \mathcal{G})(\lambda, 0)$, then the map

$$\mathfrak{v}(p) := \begin{cases} v_k(p), & p \in [p_0, p_1], \\ V_k(p), & p \in [p_1, 0], \end{cases} \tag{4.1}$$

belongs to the real Hilbert space $H := \{\mathfrak{v} \in H^2((p_0, 0)) : \mathfrak{v}(p_0) = 0\}$, and it is also in the kernel of the Sturm-Liouville operator $R_{\lambda,\mu} : H \to L_2 \times \mathbb{R}$, where $L_2 := L_2((p_0, 0))$ and

$$R_{\lambda,\mu}\mathfrak{v} := \begin{pmatrix} (\mathfrak{a}^3 \mathfrak{v}')' - \mu \mathfrak{a} \mathfrak{v} \\ (g + \sigma\mu)\mathfrak{v}(0) - \lambda^{3/2}\mathfrak{v}'(0) \end{pmatrix},$$

provided that $\mu = k^2$. Hereby, the function

$$\mathfrak{a}(p) := \mathfrak{a}(p; \lambda) := \sqrt{\lambda - 2\Gamma(p)}, \qquad p \in [p_0, 0], \tag{4.2}$$

belongs to $C^\infty([p_0, p_1]) \cap C^\infty([p_1, 0])$ for all $\lambda \in (2\max_{[p_0,0]} \Gamma, \infty)$. We note that the first derivative of $\mathfrak{a}$ has a jump at p_1. Vice versa, if $\mathfrak{v}$ belongs to the kernel of $R_{\lambda,\mu}$ and $\mu = k^2$, then letting (v_k, V_k) be given by (4.1), the vector $(v, V) = (v_k(p) \cos(kq), V_k(p) \cos(kq)) \in \mathbb{X}$ belongs to the kernel of $\partial_{(v,V)}(\mathcal{F}, \mathcal{G})(\lambda, 0)$. This correspondence motivates us to study the kernel of $R_{\lambda,\mu}$.

Lemma 4.1. *For every $(\lambda, \mu) \in (2 \max_{[p_0,0]} \Gamma, \infty) \times [0, \infty)$, the operator $R_{\lambda,\mu}$ is a Fredholm operator of index zero and its kernel is at most one-dimensional.*

Proof. Similarly as in the proof of Theorem 3.3, we define the compact perturbation $\mathcal{R}$ of $R_{\lambda,\mu}$ by the relation

$$\mathcal{R}\mathfrak{v} := R_{\lambda,\mu}\mathfrak{v} - \begin{pmatrix} 0 \\ (g + \sigma\mu)\mathfrak{v}(0) \end{pmatrix}, \qquad \mathfrak{v} \in H.$$

The first part of our claim follows from the fact that the operator $\mathcal{R}$ is a an isomorphism. Indeed, given $(f, z) \in L_2 \times \mathbb{C}$, if the vector $\mathfrak{v} \in H$ solves the equation $\mathcal{R}\mathfrak{v} = (f, z)$, then for all $\varphi \in H_0^1 := \{\varphi \in H^1((p_0, 0)) : \varphi(p_0) = 0\}$ we have

$$\int_{p_0}^0 \left(\mathfrak{a}^3 \mathfrak{v}' \varphi' + \mu \mathfrak{a} \mathfrak{v} \varphi\right) dp = -z\varphi(0) - \int_{p_0}^0 f\varphi \, dp. \tag{4.3}$$

The right-hand side of (4.3) defines an element of $\mathcal{L}(H_0^1, \mathbb{R})$, while the left-hand side of (4.3) defines, in view of Poincaré's inequality, a bounded coercive bilinear functional on $H_0^1 \times H_0^1$. Using the Lax-Milgram theorem, cf. [15, Theorem 5.8], we obtain a unique vector $\mathfrak{v} \in H_0^1$ which solves the variational formulation (4.3). It is immediate to see that in fact $\mathfrak{v} \in H$. This proves that indeed $\mathcal{R} \in \mathrm{Isom}(H, L_2 \times \mathbb{R})$.

To finish the proof we see that if $\mathfrak{v}_1, \mathfrak{v}_2 \in H$ are two vectors in the kernel of $R_{\lambda,\mu}$, then

$$0 = ((\mathfrak{a}^3 \mathfrak{v}_1')' - \mu\mathfrak{a}\mathfrak{v}_1)\mathfrak{v}_2 - ((\mathfrak{a}^3 \mathfrak{v}_2')' - \mu\mathfrak{a}\mathfrak{v}_2)\mathfrak{v}_1 = ((\mathfrak{a}^3(\mathfrak{v}_2\mathfrak{v}_1' - \mathfrak{v}_1\mathfrak{v}_2'))' \qquad \text{in } (p_0, 0), \tag{4.4}$$

which implies that $\mathfrak{a}^3(\mathfrak{v}_2\mathfrak{v}_1' - \mathfrak{v}_1\mathfrak{v}_2')$ is constant. Since $\mathfrak{v}_1(p_0) = \mathfrak{v}_2(p_0) = 0$ and $\mathfrak{a} > 0$, we obtain that $\mathfrak{v}_1$ and $\mathfrak{v}_2$ are linearly dependent. $\qquad\square$

In order to determine when the operator $R_{\lambda,\mu}$ has a nontrivial kernel, we define for each pair $(\lambda, \mu) \in (2 \max_{[p_0,0]} \Gamma, \infty) \times [0, \infty)$ the functions $\mathfrak{u}, \mathfrak{v} \in H^2((p_0, 0))$ with $\mathfrak{u} := \mathfrak{u}(\cdot; \lambda, \mu)$ and $\mathfrak{v} := \mathfrak{v}(\cdot; \lambda, \mu)$ as solutions of the initial value problems

$$\begin{cases} (\mathfrak{a}^3\mathfrak{u}')' - \mu\mathfrak{a}\mathfrak{u} = 0 & \text{in } (p_0, 0), \\ \mathfrak{u}(p_0) = 0, \quad \mathfrak{u}'(p_0) = 1, \end{cases} \qquad \begin{cases} (\mathfrak{a}^3\mathfrak{v}')' - \mu\mathfrak{a}\mathfrak{v} = 0 & \text{in } (p_0, 0), \\ \mathfrak{v}(0) = \lambda^{3/2}, \quad \mathfrak{v}'(0) = g + \sigma\mu. \end{cases} \tag{4.5}$$

These problems can be seen as system of first order linear ordinary differential equations, and therefore the existence and uniqueness of $\mathfrak{u}, \mathfrak{v}$ follows from the classical theory, cf. [2].

Lemma 4.2. *Given $(\lambda, \mu) \in (2 \max_{[p_0,0]} \Gamma, \infty) \times [0, \infty)$, the operator $R_{\lambda,\mu}$ has a nontrivial kernel exactly when the functions $\mathfrak{u}$ and $\mathfrak{v}$, given by (4.5), are linearly dependent.*

Proof. It is easy to see that if $\mathfrak{u}$ and $\mathfrak{v}$ are linearly dependent, then they both belong to the kernel of $R_{\lambda,\mu}$. On the other hand, if $R_{\lambda,\mu}\mathfrak{z} = 0$, then it follows from relation (4.4) that $\{\mathfrak{u}, \mathfrak{z}\}$ and $\{\mathfrak{v}, \mathfrak{z}\}$ are linearly dependent systems. $\qquad\square$

Summarizing, the previous lemmas state that $\mathrm{Ker}\, R_{\lambda,\mu}$ is non-trivial (and has dimension one) exactly when (λ, μ) is a zero of the Wronskian $\Xi : (2 \max_{[p_0,0]} \Gamma, \infty) \times [0, \infty) \to \mathbb{R}$ given by

$$\Xi(\lambda, \mu) := \lambda^{3/2}\mathfrak{u}'(0; \lambda, \mu) - (g + \sigma\mu)\mathfrak{u}(0; \lambda, \mu). \tag{4.6}$$

Invoking (4.2) and (4.5), the classical theory of ordinary differential equations ensures that the function Ξ is real-analytic. Particularly, its zeros are isolated. Of course, we are

interested only in the zeros for which $\mu = k^2$ for some $k \in \mathbb{N}$. If $\mu = 0$, then $\mathfrak{u}$ can be computed explicitly

$$\mathfrak{u}(p; \lambda, 0) := \int_{p_0}^{p} \frac{\mathfrak{a}^3(p_0)}{\mathfrak{a}^3(s)} \, ds, \qquad p \in [p_0, 0].$$

In this case, $\Xi(\lambda, 0) = 0$ if and only if λ solves the equation

$$\frac{1}{g} = \int_{p_0}^{0} \frac{1}{\mathfrak{a}^3(p)} \, dp. \tag{4.7}$$

The right-hand side of (4.7) is a strictly decreasing function of λ,

$$\int_{p_0}^{0} \frac{1}{\mathfrak{a}^3(p)} \, dp \xrightarrow[\lambda \to \infty]{} 0 \qquad \text{and} \qquad \int_{p_0}^{0} \frac{1}{\mathfrak{a}^3(p)} \, dp \xrightarrow[\lambda \to 2 \max_{[p_0,0]} \Gamma]{} \infty.$$

Consequently, there exists a unique $\lambda_0 \in (2 \max_{[p_0,0]} \Gamma, \infty)$ which satisfies (4.7). Since for $\mu > 0$ we cannot determine in general an explicit expression for $\mathfrak{u}$, determining the zeros of $\Xi(\cdot, \mu)$ is more difficult. Nevertheless, we have the following result.

Lemma 4.3. *Let λ_0 be the unique solution of $\Xi(\lambda, 0) = 0$. Then, we have*

$$(i) \quad \Xi(\lambda, 0) > 0 \qquad \text{for all } \lambda > \lambda_0; \tag{4.8}$$

$$(ii) \quad \lim_{\mu \to \infty} \Xi(\lambda, \mu) = -\infty \qquad \text{for all } \lambda \geq \lambda_0. \tag{4.9}$$

Proof. It follows readily from (4.5) that $\mathfrak{u}$ satisfies the following integral relation

$$\mathfrak{u}(p) = \int_{p_0}^{p} \frac{\mathfrak{a}^3(p_0)}{\mathfrak{a}^3(s)} \, ds + \mu \int_{p_0}^{p} \frac{1}{\mathfrak{a}^3(s)} \int_{p_0}^{s} (\mathfrak{a}\mathfrak{u})(r) \, dr \, ds, \qquad p \in [p_0, 0]. \tag{4.10}$$

Therefore, we have

$$\Xi(\lambda, 0) = g\mathfrak{a}^3(p_0) \left(\frac{1}{g} - \int_{p_0}^{0} \frac{1}{\mathfrak{a}^3(p)} \, dp \right) > 0$$

for all $\lambda > \lambda_0$, which proves (4.8).

In order to show (4.9), we fix $\lambda \geq \lambda_0$ and use (4.10) to decompose $\Xi(\lambda, \mu) = T_1 + \mu T_2$, whereby

$$T_1 := \mathfrak{a}^3(p_0) \left(1 - (g + \sigma\mu) \int_{p_0}^{0} \frac{1}{\mathfrak{a}^3(p)} \, dp \right),$$

$$T_2 := \int_{p_0}^{0} (\mathfrak{a}\mathfrak{u})(p) \, dp - (g + \sigma\mu) \int_{p_0}^{0} \frac{1}{\mathfrak{a}^3(s)} \int_{p_0}^{s} (\mathfrak{a}\mathfrak{u})(r) \, dr \, ds,$$

depend only on μ. Since $\mathfrak{a}$ does not depend on μ, we obtain that $T_1 \to -\infty$ as $\mu \to \infty$. Before studying the behavior of T_2, let us infer from (4.5) and (4.10) that $\mathfrak{u}$ and $\mathfrak{u}'$ are both positive on $(p_0, 0]$. In fact, an argument similar to that used to deduce the relations (4.13) and (4.16) below shows that for each $p \in (p_0, 0]$, $\min_{[p,0]} \mathfrak{u}$ and $\min_{[p,0]} \mathfrak{u}'$ grow at an exponential rate as $\mu \to \infty$. Thus, proving that $T_2 \to -\infty$ when $\mu \to \infty$, is not obvious. However, because $\mathfrak{a}$ does not depend on μ, integration by parts shows that $T_2 \to -\infty$ when $\mu \to \infty$ if we have

$$\lim_{\mu \to \infty} \left(\int_{p_0}^{0} \mathfrak{u}(p) \, dp - \mu^{6/7} \int_{p_0}^{0} (-p)\mathfrak{u}(p) \, dp \right) = -\infty.$$

Noticing that

$$\left(\int_{p_0}^{-\mu^{-2/3}} \mathfrak{u}(p) \, dp - \mu^{6/7} \int_{p_0}^{-\mu^{-2/3}} (-p)\mathfrak{u}(p) \, dp \right)$$

$$\leq (\mu^{2/3} - \mu^{6/7}) \int_{p_0}^{-\mu^{-2/3}} (-p)\mathfrak{u}(p) \, dp \underset{\mu\to\infty}{\to} -\infty,$$

it suffices to prove that

$$\lim_{\mu\to\infty} \left(\int_{-\mu^{-2/3}}^{0} \mathfrak{u}(p) \, dp - \mu^{6/7} \int_{-\mu^{-2/3}}^{0} (-p)\mathfrak{u}(p) \, dp \right) = -\infty. \qquad (4.11)$$

Because the interval $[-\mu^{-2/3}, 0]$ is very small when μ is large, we can approximate $\mathfrak{u}$ on this interval by solutions of some linear initial value problems with constant coefficients. To be more precise, we let $\mathfrak{w}$ denote the solution of the linear initial value problem

$$\begin{cases} \mathfrak{w}'' + C\mathfrak{w}' - \mu D\mathfrak{w} = 0 & \text{in } (-\mu^{-2/3}, 0), \\ \mathfrak{w}(-\mu^{-2/3}) = A, \quad \mathfrak{w}'(-\mu^{-2/3}) = B, \end{cases} \qquad (4.12)$$

where $C \in \mathbb{R}$ and $D > 0$ are constants, $A := \mathfrak{u}(-\mu^{-2/3})$, and $B := \mathfrak{u}'(-\mu^{-2/3})$. The solution $\mathfrak{w}$ of (4.12) is given by the following formula

$$\mathfrak{w}(p) = \frac{A}{r_1 - r_2} \left(r_1 e^{r_2(p+\mu^{-2/3})} - r_2 e^{r_1(p+\mu^{-2/3})} \right) + \frac{B}{r_1 - r_2} \left(e^{r_1(p+\mu^{-2/3})} - e^{r_2(p+\mu^{-2/3})} \right)$$
$$(4.13)$$

for $p \in [-\mu^{-2/3}, 0]$, whereby

$$r_1 := \frac{-C + \sqrt{C^2 + 4D\mu}}{2} \qquad \text{and} \qquad r_2 := \frac{-C - \sqrt{C^2 + 4D\mu}}{2}. \qquad (4.14)$$

The idea of considering the problem (4.12) is the following: defining the μ-dependent functions

$$\underline{C} := \max_{[-\mu^{-2/3}, 0]} \frac{3\mathfrak{a}'}{\mathfrak{a}}, \quad \overline{C} := \min_{[-\mu^{-2/3}, 0]} \frac{3\mathfrak{a}'}{\mathfrak{a}}, \quad \underline{D} := \min_{[-\mu^{-2/3}, 0]} \frac{1}{\mathfrak{a}^2}, \quad \overline{D} := \max_{[-\mu^{-2/3}, 0]} \frac{1}{\mathfrak{a}^2}, \quad (4.15)$$

it is not difficult to see by subtracting the equations of (4.12) from those satisfied by $\mathfrak{u}$ that we have

$$\underline{\mathfrak{w}} \leq \mathfrak{u} \leq \overline{\mathfrak{w}} \qquad \text{on } [-\mu^{-2/3}, 0]. \qquad (4.16)$$

Hereby, $\underline{\mathfrak{w}}$ and $\overline{\mathfrak{w}}$ are the solutions of (4.12) corresponding to $(\underline{C}, \underline{D})$, and $(\overline{C}, \overline{D})$, respectively. Therefore, the relation (4.11) is fulfilled if we show that

$$\lim_{\mu\to\infty} \left(\int_{-\mu^{-2/3}}^{0} \overline{\mathfrak{w}}(p) \, dp - \mu^{6/7} \int_{-\mu^{-2/3}}^{0} (-p)\underline{\mathfrak{w}}(p) \, dp \right) = -\infty. \qquad (4.17)$$

An elementary computation now gives

$$\int_{-\mu^{-2/3}}^{0} \overline{\mathfrak{w}}(p) \, dp - \mu^{6/7} \int_{-\mu^{-2/3}}^{0} (-p)\underline{\mathfrak{w}}(p) \, dp = AT_A + BT_B,$$

whereby

$$T_A := \frac{1}{\overline{r}_1 - \overline{r}_2}\left(\frac{\overline{r}_1}{\overline{r}_2}(e^{\overline{r}_2/\mu^{2/3}} - 1) - \frac{\overline{r}_2}{\overline{r}_1}(e^{\overline{r}_1/\mu^{2/3}} - 1)\right) - \frac{\mu^{6/7-2/3}}{\underline{r}_1 - \underline{r}_2}\left(\frac{\underline{r}_2}{\underline{r}_1} - \frac{\underline{r}_1}{\underline{r}_2}\right)$$
$$- \frac{\mu^{6/7}}{\underline{r}_1 - \underline{r}_2}\left(\frac{\underline{r}_1}{\underline{r}_2^2}(e^{\underline{r}_2/\mu^{2/3}} - 1) - \frac{\underline{r}_2}{\underline{r}_1^2}(e^{\underline{r}_1/\mu^{2/3}} - 1)\right),$$

$$T_B := \frac{1}{\overline{r}_1 - \overline{r}_2}\left(\frac{1}{\overline{r}_1}(e^{\overline{r}_1/\mu^{2/3}} - 1) - \frac{1}{\overline{r}_2}(e^{\overline{r}_2/\mu^{2/3}} - 1)\right) - \frac{\mu^{6/7-2/3}}{\underline{r}_1 - \underline{r}_2}\left(\frac{1}{\underline{r}_2} - \frac{1}{\underline{r}_1}\right)$$
$$- \frac{\mu^{6/7}}{\underline{r}_1 - \underline{r}_2}\left(\frac{1}{\underline{r}_1^2}(e^{\underline{r}_1/\mu^{2/3}} - 1) - \frac{1}{\underline{r}_2^2}(e^{\underline{r}_2/\mu^{2/3}} - 1)\right),$$

and with $(\underline{r}_1, \underline{r}_2)$ and $(\overline{r}_1, \overline{r}_2)$ being defined by (4.14) with (C, D) being replaced by $(\underline{C}, \underline{D})$, and $(\overline{C}, \overline{D})$, respectively. The claim (4.17) follows from the following properties

$$T_A \to_{\mu\to\infty} -\infty \qquad \text{and} \qquad \mu^{2/3}T_B \to_{\mu\to\infty} -\infty. \tag{4.18}$$

We establish first the claim for T_A. Clearly, it suffices to show that

$$\frac{\underline{r}_1 - \underline{r}_2}{\overline{r}_1 - \overline{r}_2}\left(\frac{\overline{r}_1}{\overline{r}_2}(e^{\overline{r}_2/\mu^{2/3}} - 1) - \frac{\overline{r}_2}{\overline{r}_1}(e^{\overline{r}_1/\mu^{2/3}} - 1)\right) - \mu^{6/7-2/3}\left(\frac{\underline{r}_2}{\underline{r}_1} - \frac{\underline{r}_1}{\underline{r}_2}\right)$$
$$- \mu^{6/7}\left(\frac{\underline{r}_1}{\underline{r}_2^2}(e^{\underline{r}_2/\mu^{2/3}} - 1) - \frac{\underline{r}_2}{\underline{r}_1^2}(e^{\underline{r}_1/\mu^{2/3}} - 1)\right) =: E_1 - E_2 - E_3 \to_{\mu\to\infty} -\infty.$$

Let us now observe that $(\underline{C}, \underline{D})$ and $(\overline{C}, \overline{D})$ converge, when $\mu \to \infty$, towards the constant pair $(C, D) := (3\mathfrak{a}'(0)/\mathfrak{a}(0), 1/\mathfrak{a}^2(0))$. Because $\overline{r}_i/\mu^{2/3} \to 0$ as $\mu \to \infty$, we easily see that $E_1 \to 0$. Moreover, there exists a constant K, independent of μ, such that

$$|E_2| = \mu^{6/7-2/3}\left|\frac{\underline{r}_2}{\underline{r}_1} - \frac{\underline{r}_1}{\underline{r}_2}\right| = \mu^{6/7-2/3}\frac{|\underline{r}_2^2 - \underline{r}_1^2|}{\underline{r}_1\underline{r}_2} \leq K\mu^{6/7-2/3-1/2}$$

and we are left to consider E_3. To this end, we write $E_3 = E_{3a} + E_{3b}$ where we set

$$E_{3a} := \mu^{6/7}\left(\frac{\sqrt{D\mu}}{D\mu}(e^{-\sqrt{D}\mu^{-1/6}} - 1) + \frac{\sqrt{D\mu}}{D\mu}(e^{\sqrt{D}\mu^{-1/6}} - 1)\right).$$

We note that E_{3a} is obtained by replacing in the definition of E_3 the constants $(\underline{C}, \underline{D})$ and $(\overline{C}, \overline{D})$ by their limit (C, D) and retaining only the highest order terms in μ. Since by the mean value theorem we have

$$\max\{|\overline{D} - D|, |\underline{D} - D|\} \leq K\mu^{-2/3},$$

where K is again independent of μ, one can show that $E_{3b} \to_{\mu\to\infty} 0$. Furthermore, using l'Hopitals rule we get

$$\lim_{\mu\to\infty} E_{3a} = \frac{7}{15}\lim_{\mu\to\infty}\frac{e^{\sqrt{D}\mu^{-1/6}} - e^{-\sqrt{D}\mu^{-1/6}}}{\mu^{-1/6}}\mu^{1/42} = \infty.$$

Thus, $E_{3a} \to \infty$, and we conclude that $T_A \to_{\mu\to\infty} -\infty$. The second claim of (4.18) follows similarly. This finishes the proof. $\qquad\square$

Invoking Lemma 4.3, we find for every $\lambda \geq \lambda_0$ a unique constant $\mu(\lambda) \in [0, \infty)$ such that

$$
\begin{aligned}
&\Xi(\lambda, \mu(\lambda)) = 0; \\
&\Xi(\lambda, \mu) < 0 \text{ for all } \mu > \mu(\lambda).
\end{aligned}
\tag{4.19}
$$

Since $\Xi(\lambda, 0) > 0$ when $\lambda > \lambda_0$, any zero of $\Xi(\lambda, \cdot)$ is positive when $\lambda > \lambda_0$. We next prove that $\lambda \mapsto \mu(\lambda)$ is a real-analytic and strictly increasing curve.

Lemma 4.4. *Assume that $(\overline{\lambda}, \overline{\mu}) \in [\lambda_0, \infty) \times (0, \infty)$ satisfies $\Xi(\overline{\lambda}, \overline{\mu}) = 0$. Then, we have*

$$
\Xi_\lambda(\overline{\lambda}, \overline{\mu}) > 0 \qquad and \qquad \Xi_\mu(\overline{\lambda}, \overline{\mu}) < 0.
\tag{4.20}
$$

Before proving the lemma, let us observe that since $\Xi(\lambda, 0) > 0$ for all $\lambda > \lambda_0$, the second relation of (4.20) ensures additionally to (4.19) that

$$
\Xi(\lambda, \mu) > 0 \text{ for all } \mu \in [0, \mu(\lambda)).
\tag{4.21}
$$

Particularly, if $\lambda > \lambda_0$, then $\Xi(\lambda, \mu) = 0$ if and only if $\mu = \mu(\lambda)$.

Proof of Lemma 4.4. Because $\Xi(\overline{\lambda}, \overline{\mu}) = 0$, it follows from the Lemmas 4.1-4.2, and the discussion following them, that $\operatorname{Ker} R_{\overline{\lambda}, \overline{\mu}}$ is spanned by the function $\mathfrak{u}$ defined by (4.5) when $(\lambda, \mu) = (\overline{\lambda}, \overline{\mu})$. Thus, $\mathfrak{u}$ solves the following system of equations

$$
\begin{cases}
(\mathfrak{a}^3 \mathfrak{u}')' - \overline{\mu} \mathfrak{a} \mathfrak{u} = 0 & \text{in } (p_0, 0), \\
\mathfrak{u}(p_0) = 0, \quad \mathfrak{u}'(p_0) = 1, \\
(g + \sigma \overline{\mu}) \mathfrak{u}(0) - \overline{\lambda}^{3/2} \mathfrak{u}'(0) = 0.
\end{cases}
\tag{4.22}
$$

Differentiating the equations of (4.5) with respect to μ shows that the Fréchet derivative $\mathfrak{u}_\mu := \mathfrak{u}_\mu(\cdot, \overline{\lambda}, \overline{\mu})$ is the solution of the problem

$$
\begin{cases}
(\mathfrak{a}^3 \mathfrak{u}_\mu')' - \overline{\mu} \mathfrak{a} \mathfrak{u}_\mu = \mathfrak{a} \mathfrak{u} & \text{in } (p_0, 0), \\
\mathfrak{u}_\mu(p_0) = 0, \quad \mathfrak{u}_\mu'(p_0) = 0.
\end{cases}
\tag{4.23}
$$

First, we establish that

$$
\Xi_\mu(\overline{\lambda}, \overline{\mu}) = \overline{\lambda}^{3/2} \mathfrak{u}_\mu'(0) - \sigma \mathfrak{u}(0) - (g + \sigma \overline{\mu}) \mathfrak{u}_\mu(0) < 0.
\tag{4.24}
$$

Multiplying the first equation of (4.22) by $\mathfrak{u}_\mu$ and the first equation of (4.23) by $\mathfrak{u}$, we obtain after integrating by parts that

$$
\mathfrak{u}(0) \left(\overline{\lambda}^{3/2} \mathfrak{u}_\mu'(0) - (g + \sigma \overline{\mu}) \mathfrak{u}_\mu(0) \right) = \int_{p_0}^0 \mathfrak{a} \mathfrak{u}^2 \, dp.
\tag{4.25}
$$

Moreover, if we multiply the first equation of (4.22) by $\mathfrak{u}$ and integrate it by parts we find that

$$
\int_{p_0}^0 \mathfrak{a}^3 \mathfrak{u}'^2 \, dp - g \mathfrak{u}^2(0) = \overline{\mu} \left(\sigma \mathfrak{u}^2(0) - \int_{p_0}^0 \mathfrak{a} \mathfrak{u}^2 \, dp \right).
$$

But, since $\lambda \geq \lambda_0$ and $\mu > 0$,

$$
g \mathfrak{u}^2(0) = g \left(\int_{p_0}^0 (\mathfrak{a}^{3/2} \mathfrak{u}') \frac{1}{\mathfrak{a}^{3/2}} \, dp \right)^2 < g \int_{p_0}^0 \mathfrak{a}^3 \mathfrak{u}'^2 \, dp \int_{p_0}^0 \frac{1}{\mathfrak{a}^3} \, dp \leq \int_{p_0}^0 \mathfrak{a}^3 \mathfrak{u}'^2 \, dp,
$$

cf. (4.7), which implies that the right-hand side of (4.25) is bounded from above by $\sigma \mathfrak{u}^2(0)$. This proves (4.24).

For the first claim of (4.24) we note that

$$\Xi_\lambda(\overline{\lambda}, \overline{\mu}) = \overline{\lambda}^{3/2}\mathfrak{u}'_\lambda(0) + \frac{3}{2}\overline{\lambda}^{1/2}\mathfrak{u}'(0) - (g + \sigma\overline{\mu})\mathfrak{u}_\lambda(0),$$

whereby $\mathfrak{u}_\lambda$ is the solution of

$$\begin{cases} (\mathfrak{a}^3\mathfrak{u}'_\lambda)' - \overline{\mu}\mathfrak{a}\mathfrak{u}_\lambda = -(3\mathfrak{a}^2\mathfrak{a}_\lambda\mathfrak{u}')' + \overline{\mu}\mathfrak{a}_\lambda\mathfrak{u} & \text{in } (p_0, 0), \\ \mathfrak{u}_\lambda(p_0) = 0, \quad \mathfrak{u}'_\lambda(p_0) = 0, \end{cases} \tag{4.26}$$

and $\mathfrak{a}_\lambda = 1/(2\mathfrak{a})$. Similarly as before, we multiply the first equation of (4.22) by $\mathfrak{u}_\lambda$ and the first equation of (4.26) by $\mathfrak{u}$ to obtain, after integrating by parts, that

$$\Xi_\lambda(\overline{\lambda}, \overline{\mu}) = \frac{1}{\mathfrak{u}(0)} \int_{p_0}^0 \left(\frac{3\mathfrak{a}}{2}\mathfrak{u}'^2 + \frac{\overline{\mu}}{2\mathfrak{a}}\mathfrak{u}^2 \right) dp > 0.$$

This completes our argument. □

Remark 4.5. The restriction $\mu > 0$ in Lemma 4.4 was needed just to prove the second claim of (4.20). But, when $\mu = 0$, we obtain easily from (4.24), (4.25), (4.7), and the explicit expression for $\mathfrak{u}(\cdot; \lambda_0, 0)$ that $\Xi_\mu(\lambda_0, 0) < 0$ if and only if

$$\int_{p_0}^0 \mathfrak{a}(p) \left(\int_{p_0}^p \frac{1}{\mathfrak{a}^3(s)} ds \right)^2 dp < \frac{\sigma}{g^2}. \tag{4.27}$$

Combining the previous lemmas, we obtain the following result.

Lemma 4.6. *The function*

$$[\lambda_0, \infty) \ni \lambda \mapsto \mu(\lambda) \in [0, \infty)$$

is continuous, real-analytic in (λ_0, ∞), and strictly increasing.

Proof. Let $\lambda_1 > \lambda_0$. Because of $\Xi(\lambda_1, \mu(\lambda_1)) = 0$ and $\Xi_\mu(\lambda_1, \mu(\lambda_1)) < 0$ there exists a real-analytic function $\overline{\mu}$ such that $\overline{\mu}(\lambda_1) = \mu(\lambda_1)$ and $\Xi(\lambda, \overline{\mu}(\lambda)) = 0$ for all λ close to λ_1. Recalling (4.20), we see that the function $\overline{\mu}$ is strictly increasing, as we have

$$\Xi_\lambda(\lambda, \overline{\mu}(\lambda)) + \Xi_\mu(\lambda, \overline{\mu}(\lambda))\overline{\mu}'(\lambda) = 0$$

The conclusion follows now from the relations (4.19)-(4.21). □

The next lemma ensures that the function $\mu : [\lambda_0, \infty) \to [\mu(\lambda_0), \infty)$ is bijective.

Lemma 4.7. *We have that*

$$\lim_{\lambda \to \infty} \frac{\mu(\lambda)}{\lambda} = \infty. \tag{4.28}$$

Proof. Assume by contradiction that there exists a sequence $\lambda_n \to \infty$ and a constant $K > 0$ such that $0 < \mu(\lambda_n)/\lambda_n \leq K$ for all $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, we denote by $\mathfrak{u}_n$ the function that spans the kernel of $R_{\lambda_n, \mu(\lambda_n)}$ and solves the first system of (4.5). Then it follows readily form (4.10) that there exists a constant $\widetilde{K}$, independent of n, such that

$$0 \leq \mathfrak{u}_n(p) \leq \widetilde{K} \left(1 + \int_{p_0}^p \mathfrak{u}_n(s) ds \right) \qquad \text{for all } p \in [p_0, 0] \text{ and } n \in \mathbb{N}.$$

Using Gronwall's inequality, we conclude that the sequence $(\mathfrak{u}_n)_n \subset C([p_0, 0])$ is bounded. Since by (4.6), (4.10), and our assumption we have

$$\Xi(\lambda_n, \mu(\lambda_n)) \geq \mathfrak{a}^3(p_0) - (g + \sigma\mu(\lambda_n))\mathfrak{u}_n(0) \to_{n\to\infty} \infty,$$

we obtain a contradiction with the properties defining the map $\mu(\cdot)$, cf. (4.19). □

5. Proof of the main result

The case $N = 2$. We now come back to the setting presented in Theorem 2.1 and assume that $N = 2$. We summarize from the Lemmas 4.3-4.7 that there exists a smallest positive integer $n_{\min}$ with the property that

$$n_{\min}^2 > \mu(\lambda_0). \tag{5.1}$$

Particularly, for each $n \geq n_{\min}$ there exists a constant $\lambda_n > \lambda_0$ such that

$$\mu(\lambda_n) := n^2. \tag{5.2}$$

The sequence $(\lambda_n)_{n \geq n_{\min}}$ is in view of Lemma 4.7 strictly increasing to infinity. Whence, we have for $n, p \geq n_{\min}$ that $\Xi(\lambda_n, p^2) = 0$ if and only if $n = p$. Moreover, it follows from the Remark 4.5 and (4.19)-(4.21) that the integer $n_{\min}$ can be chosen to be $n_{\min} = 1$ if (4.27) is satisfied. This is due to the fact that $\mu(\lambda_0) = 0$ if (4.27) holds true.

Since we want to determine nontrivial solutions of the water wave problem (2.4) we study the existence of bifurcation branches consisting of solutions of (3.7) that arise from $(\lambda_n, 0)$, with $n \geq n_{\min}$. In the following we denote by $\widehat{\mathbb{X}}$ and $\widehat{\mathbb{Y}}$ the subspaces of $\mathbb{X}$ and $\mathbb{Y}$, respectively, that consist only of $(2\pi/n)$-periodic and even functions in the variable q. Then, it follows readily from the definition of $(\mathcal{F}, \mathcal{G})$ that we have

$$(\mathcal{F}, \mathcal{G}) \in C^\omega((2 \max_{[p_0, 0]} \Gamma, \infty) \times \widehat{\mathbb{X}}, \widehat{\mathbb{Y}}). \tag{5.3}$$

Moreover, the arguments used in the proof of Theorem 3.3 show that the Fréchet derivative $\partial_{(v,V)}(\mathcal{F}, \mathcal{G})(\lambda, 0) \in \mathcal{L}(\widehat{\mathbb{X}}, \widehat{\mathbb{Y}})$ is a Fredholm operator of index zero for every value $\lambda \in (2 \max_{[p_0, 0]} \Gamma, \infty)$. Taking into account that the kernel of $\partial_{(v,V)}(\mathcal{F}, \mathcal{G})(\lambda, 0)$ is finite dimensional, by the choice of the sequence $(\lambda_n)_{n \geq n_{\min}}$, we know that $\partial_{(v,V)}(\mathcal{F}, \mathcal{G})(\lambda_n, 0)$ has a one-dimension kernel. More precisely,

$$\mathrm{Ker}\, \partial_{(v,V)}(\mathcal{F}, \mathcal{G})(\lambda_n, 0) = \mathrm{span}\{(v^0, V^0) := (v_n^0(p) \cos(nq), V_n^0(p) \cos(nq))\}$$

whereby (v_n^0, V_n^0) defines, cf. (4.1), a vector $\mathfrak{v}^0 \in H^2((p_0, 0))$ that spans the one-dimensional kernel of the operator R_{λ_n, n^2}. This vector $\mathfrak{v}^0$ is colinear to the solutions of both initial value problems (4.5).

In order to apply the bifurcation theorem of Crandall and Rabinowitz to the equation $(\mathcal{F}, \mathcal{G})(\lambda, (v, V)) = 0$ in $\widehat{\mathbb{Y}}$, which will give us, via Lemma 3.1, the desired result from Theorem 2.1, we are left check that

$$\partial_{\lambda(v,V)}(\mathcal{F}, \mathcal{G})(\lambda_n, 0)(v^0, V^0) \notin \mathrm{Im}\, \partial_{(v,V)}(\mathcal{F}, \mathcal{G})(\lambda_n, 0) \tag{5.4}$$

if $0 \neq (v^0, V^0) \in \mathrm{Ker}\, \partial_{(v,V)}(\mathcal{F}, \mathcal{G})(\lambda_n, 0)$. To this end, we need to characterize the range $\mathrm{Im}\, \partial_{(v,V)}(\mathcal{F}, \mathcal{G})(\lambda_n, 0)$.

Lemma 5.1. *Let $n_{\min}$ be given by (5.1). Given $n \geq n_{\min}$, the vector $((f, F), \varphi) \in \widehat{\mathbb{Y}}$ belongs to $\mathrm{Im}\, \partial_{(v,V)}(\mathcal{F}, \mathcal{G})(\lambda_n, 0)$ if and only if we have*

$$\int_{\Omega_1} a^3 v^0 f \, d(q, p) + \int_{\Omega_2} A^3 V^0 F \, d(q, p) + \int_{\mathbb{S} \times \{0\}} \frac{A^2 V^0 \varphi}{2} \, dq = 0 \tag{5.5}$$

for all $(v^0, V^0) \in \mathrm{Ker}\, \partial_{(v,V)}(\mathcal{F}, \mathcal{G})(\lambda_n, 0)$.

Proof. We pick $0 \neq (v^0, V^0) \in \operatorname{Ker} \partial_{(v,V)}(\mathcal{F}, \mathcal{G})(\lambda_n, 0)$, and presuppose that there exists a pair $(v, V) \in \widehat{\mathbb{X}}$ with the property that $\partial_{(v,V)}(\mathcal{F}, \mathcal{G})(\lambda_n, 0)(v, V) = ((f, F), \varphi)$, that is

$$(L_1 v, L_2 V) = (f, F) \quad \text{in } \widehat{\mathbb{Y}}_1, \qquad TV = \varphi \quad \text{in } \widehat{\mathbb{Y}}_2, \tag{5.6}$$

whereby L_1, L_2, T are given by (3.9) and (3.10). Invoking (3.15), we use integration and the fact that $(v^0, V^0) \in \widehat{\mathbb{X}}$ solves (5.6) when $((f, F), \varphi) = 0$, to find

$$\int_{\Omega_1} a^3 v^0 f \, d(q, p) + \int_{\Omega_2} A^3 V^0 F \, d(q, p) + \int_{\mathbb{S} \times \{0\}} \frac{A^2 V^0 \varphi}{2} \, dq$$

$$= -\int_{\Omega_1} (a^3 v v_p^0 + a v_q v_q^0) \, d(q, p) - \int_{\Omega_2} (A^3 V V_p^0 + A V_q V_q^0) \, d(q, p)$$

$$+ \int_{\mathbb{S} \times \{0\}} (g + \sigma n^2) V^0 V \, dq = 0,$$

the last equality being obtain by using once more the fact that $(v^0, V^0) \in \widehat{\mathbb{X}}$ solves (5.6) when $((f, F), \varphi) = 0$. Taking into account that the relation (5.5) defines a closed subspace of $\widehat{\mathbb{Y}}$ which has codimension one and contains the image $\operatorname{Im} \partial_{(v,V)}(\mathcal{F}, \mathcal{G})(\lambda_n, 0)$, which has itself codimension one, we obtain the desired claim. $\qquad \square$

Lemma 5.2. *Let $n \geq n_{\min}$ be given. The transversality condition (5.4) holds true for all vectors $0 \neq (v^0, V^0) \in \operatorname{Ker} \partial_{(v,V)}(\mathcal{F}, \mathcal{G})(\lambda_n, 0)$.*

Proof. Let $0 \neq (v^0, V^0) \in \operatorname{Ker} \partial_{(v,V)}(\mathcal{F}, \mathcal{G})(\lambda_n, 0)$ be given. Then, we infer from (3.9), (3.10), and (3.15), that

$$\partial_{\lambda(v,V)} \mathcal{F}_1(\lambda_n, 0)(v^0, V^0) = -\frac{2 a_\lambda}{a^3} v_{qq}^0 + \frac{3 a a_\lambda' - 3 a_\lambda a'}{a^2} v_p^0 =: f,$$

$$\partial_{\lambda(v,V)} \mathcal{F}_2(\lambda_n, 0)(v^0, V^0) = -\frac{2 A_\lambda}{A^3} V_{qq}^0 + \frac{3 A A_\lambda' - 3 A_\lambda A'}{A^2} V_p^0 =: F,$$

$$\partial_{\lambda(v,V)} \mathcal{G}(\lambda_n, 0)(v^0, V^0) = -\left[\frac{3}{A} V_p^0 \right] \Big|_{p=0} =: \varphi.$$

Our claim is equivalent to showing that $((f, F), \varphi)$ does not satisfy (5.5). Observing that $a_\lambda = 1/(2a)$ and $A_\lambda = 1/(2A)$, we compute

$$\int_{\Omega_1} a^3 v^0 f \, d(q, p) + \int_{\Omega_2} A^3 V^0 F \, d(q, p) + \int_{\mathbb{S} \times \{0\}} \frac{A^2 V^0 \varphi}{2} \, dq$$

$$= -\int_{\Omega_1} \left(a^{-1} v_{qq}^0 v^0 + 3 a' v_p^0 v^0 \right) d(q, p) - \int_{\Omega_2} \left(A^{-1} V_{qq}^0 V^0 + 3 A' V_p^0 V^0 \right) d(q, p)$$

$$- \int_{\mathbb{S} \times \{0\}} \frac{3A}{2} V_p^0 V^0 \, dq.$$

On the other hand, using the fact that $(v^0, V^0) \in \mathrm{Ker}\, \partial_{(v,V)}(\mathcal{F}, \mathcal{G})(\lambda_k, 0)$ and integration by parts, we obtain that

$$
\int_{\Omega_1} a' v_p^0 v^0 \, d(q,p) + \int_{\Omega_2} A' V_p^0 V^0 \, d(q,p)
$$
$$
= -\int_{\mathbb{S}\times\{0\}} \frac{A}{2} V_p^0 V^0 \, dq + \int_{\Omega_1} \left(\frac{a}{2}(v_p^0)^2 + \frac{1}{2a}(v_q^0)^2 \right) d(q,p)
$$
$$
+ \int_{\Omega_2} \left(\frac{A}{2}(V_p^0)^2 + \frac{1}{2A}(V_q^0)^2 \right) d(q,p),
$$

which together with the previous relation gives

$$
\int_{\Omega_1} a^3 v^0 f \, d(q,p) + \int_{\Omega_2} A^3 V^0 F \, d(q,p) + \int_{\mathbb{S}\times\{0\}} \frac{A^2 V^0 \varphi}{2} \, dq
$$
$$
= -\int_{\Omega_1} \left(\frac{3a}{2}(v_p^0)^2 + \frac{1}{2a}(v_q^0)^2 \right) d(q,p) - \int_{\Omega_2} \left(\frac{3A}{2}(V_p^0)^2 + \frac{1}{2A}(V_q^0)^2 \right) d(q,p) < 0.
$$

This proves the lemma. □

To conclude the proof of Theorem 2.1 when $N = 2$, it only remains to show that each solution on $\mathcal{C}_n$, $n \geq n_{\min}$, has only one crest and trough per period, the symmetry property being then a result of the choice of the space $\mathbb{X}$. This follows directly from the asymptotic expressions for the bifurcation curves from Theorem 3.2, as $\mathcal{C}_n$ is parametrized in $\mathbb{X}$ by

$$
(u_n(s), U_n(s)) = (\overline{u}(\overline{\lambda}_n(s)), \overline{U}(\overline{\lambda}_n(s))) + s(v_n^0(p)\cos(nq), V_n^0(p)\cos(nq)) + O(s^2)
$$

for small $|s|$, with the parameter curve $\overline{\lambda}_n$ satisfying $\overline{\lambda}_n(s) := \lambda_n + O(s)$ for such s.
The case $N \geq 3$. When the vorticity function has several jumps, we proceed as in Section 3 and associate to problem (2.4) a diffraction problem

$$
\begin{cases}
(1 + u_{i,q}^2)u_{i,pp} - 2u_{i,p}u_{i,q}u_{i,pq} + u_{i,p}^2 u_{i,qq} - \gamma_i u_{i,p}^3 = 0 & \text{in } \Omega_i,\ i \in \{1, N\}, \\
1 + u_{N,q}^2 + (2gu_N - Q)u_{N,p}^2 - 2\sigma \dfrac{u_{N,p}^2 u_{N,qq}}{(1 + u_{N,q}^2)^{3/2}} = 0 & \text{on } p = 0, \\
\hspace{10em} u_i = u_{i+1} & \text{on } p = p_i,\ i \in \{1, N-1\}, \\
\hspace{10em} u_{i,p} = u_{i+1,p} & \text{on } p = p_i,\ i \in \{1, N-1\}, \\
\hspace{10em} u_1 = 0 & \text{on } p = p_0,
\end{cases}
$$

(5.7)

where $\Omega_i := \mathbb{S} \times (p_{i-1}, p_i)$ for all $1 \leq i \leq N$. Then, similarly to Lemma 3.1 we see that each solution of (3.1) defines a solution of problem (2.4). For $N \geq 3$, our main result is obtained by following the lines of the proof when $N = 2$ with the evident modifications. There is only one point where the analysis is different, namely when showing that the Fréchet derivative $\partial_{(v_1,\ldots,v_N)}(\mathcal{F}, \mathcal{G})(\lambda, 0) \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$ [1] is a Fredholm operator for all $\lambda \in (2\max_{[p_0,0]} \Gamma, \infty)$. As

---

[1]For $N \geq 3$ it is natural to define:

$$
\mathbb{X} := \left\{ (v_1, \ldots, v_N) \in \Pi_{i=1}^N C^{2+\alpha}(\overline{\Omega}_i) \ : \ v_i = v_{i+1},\ v_{i,p} = v_{i+1,p} \text{ on } p = p_i,\ i \in \{1, N-1\},\ 0 = v_1\big|_{p=p_0} \right\},
$$
$$
\mathbb{Y}_1 := \Pi_{i=1}^N C^{\alpha}(\overline{\Omega}_i), \quad \mathbb{Y}_2 := C^{\alpha}(\mathbb{S}).
$$

in the proof of Theorem 3.3 one can show that $\partial_{(v_1,\ldots,v_N)}(\mathcal{F},\mathcal{G})(\lambda,0)$ is a Fredholm operator of index zero provided that the unique solution $(z_1,\ldots,z_N)$ of the diffraction problem

$$
\begin{cases}
(a_i^3 z_{i,p})_p + (a_i z_{i,q})_q &=& 0 & \text{in } \Omega_i,\ i \in \{1,N\}, \\
T_0 z_N &=& 0 & \text{on } p = 0, \\
z_i &=& z_{i+1} & \text{on } p = p_i,\ i \in \{1, N-1\}, \\
z_{i,p} &=& z_{i+1,p} & \text{on } p = p_i,\ i \in \{1, N-1\} \setminus \{l\}, \\
z_{l,p} - z_{l+1,p} &=& \varphi & \text{on } p = p_l, \\
z_1 &=& 0 & \text{on } p = p_0.
\end{cases}
\tag{5.8}
$$

belongs to $\Pi_{i=1}^{N} C^{2+\alpha}(\overline{\Omega}_i)$ for all $1 \leq l \leq N-1$ and all $\varphi \in C^{1+\alpha}(\mathbb{S})$. Hereby, we set $a_i := \sqrt{\lambda - 2\Gamma} \in C^{\infty}([p_{i-1}, p_i])$ for all $1 \leq i \leq N$. Existence of a solution $(z_1,\ldots,z_N)$ of the linear diffraction problem (5.8) in the class

$$
\left(C^{2+\alpha}(\mathbb{S} \times [p_0, p_1]) \times \Pi_{i=2}^{N-1} C^{2+\alpha}(\Omega_i) \times C^{2+\alpha}(\mathbb{S} \times (p_{N-1}, 0]) \right) \cap \Pi_{i=1}^{N} C^{1+\alpha}(\overline{\Omega}_i)
$$

is obtained by using elliptic maximum principles and the results of [29]. The problem lies in ensuring $C^{2+\alpha}$ regularity at the interfaces where we have transmission conditions, an argument as in Theorem 3.3 being impossible because N is arbitrary. Nevertheless we can use the result established in Theorem 3.3. Indeed, if $1 \leq l \leq N-1$, then (z_l, z_{l+1}) solves the diffraction problem

$$
\begin{cases}
(a_l^3 z_{l,p})_p + (a_l z_{l,q})_q &=& 0 & \text{in } \mathbb{S} \times ((p_{l-1} + p_l)/2, p_l), \\
(a_{l+1}^3 z_{l+1,p})_p + (a_{l+1} z_{l+1,q})_q &=& 0 & \text{in } \mathbb{S} \times (p_l, (p_l + p_{l+1})/2), \\
z_l &=& \phi_l & \text{on } p = (p_{l-1} + p_l)/2, \\
z_{l+1} &=& \phi_{l+1} & \text{on } p = (p_l + p_{l+1})/2, \\
z_l &=& z_{l+1} & \text{on } p = p_l, \\
z_{l,p} - z_{l+1,p} &=& \varphi_l & \text{on } p = p_l,
\end{cases}
$$

whereby $\varphi_l \in C^{1+\alpha}(\mathbb{S})$ (possibly $\varphi_l = 0$) and

$$
\phi_l := z_l\big|_{p=(p_{l-1}+p_l)/2}, \ \phi_{l+1} := z_{l+1}\big|_{p=(p_l+p_{l+1})/2} \ \text{belong to } C^{2+\alpha}(\mathbb{S}).
$$

Then, we can write

$$
(z_l, z_{l+1}) = (v_l, v_{l+1}) + (w_l, w_{l+1}),
$$

whereby $v_l \in C^{2+\alpha}(\mathbb{S} \times [(p_{l-1} + p_l)/2, p_l])$ solves the Dirichlet problem

$$
\begin{cases}
(a_l^3 v_{l,p})_p + (a_l v_{l,q})_q &=& 0 & \text{in } \mathbb{S} \times ((p_{l-1} + p_l)/2, p_l), \\
v_l &=& \phi_l & \text{on } p = (p_{l-1} + p_l)/2, \\
v_l &=& 0 & \text{on } p = p_l,
\end{cases}
$$

the function $v_{l+1} \in C^{2+\alpha}(\mathbb{S} \times [p_l, (p_l + p_{l+1})/2])$ is the solution of

$$
\begin{cases}
(a_{l+1}^3 v_{l+1,p})_p + (a_{l+1} v_{l+1,q})_q &=& 0 & \text{in } \mathbb{S} \times (p_l, (p_l + p_{l+1})/2), \\
v_{l+1} &=& \phi_{l+1} & \text{on } p = (p_l + p_{l+1})/2, \\
v_{l+1} &=& 0 & \text{on } p = p_l,
\end{cases}
$$

and (w_l, w_{l+1}) solves the diffraction problem

$$\begin{cases} (a_l^3 w_{l,p})_p + (a_l w_{l,q})_q &= 0 & \text{in } \mathbb{S} \times ((p_{l-1}+p_l)/2, p_l), \\ (a_{l+1}^3 w_{l+1,p})_p + (a_{l+1} w_{l+1,q})_q &= 0 & \text{in } \mathbb{S} \times (p_l, (p_l+p_{l+1})/2), \\ w_l &= 0 & \text{on } p = (p_{l-1}+p_l)/2, \\ w_{l+1} &= 0 & \text{on } p = (p_l+p_{l+1})/2, \\ w_l &= w_{l+1} & \text{on } p = p_l, \\ w_{l,p} - w_{l+1,p} &= \xi & \text{on } p = p_l, \end{cases}$$

with

$$\xi := \varphi_l - v_{l,p}\big|_{p=p_l} + v_{l+1,p}\big|_{p=p_l} \in C^{1+\alpha}(\mathbb{S}).$$

The arguments presented in the proof of Theorem 3.3 show that the mapping $[\xi \mapsto w_l|_{p=p_l}]$ is a Fourier multiplier and it belongs to $\mathcal{L}(C^{1+\alpha}(\mathbb{S}), C^{2+\alpha}(\mathbb{S}))$. But then (w_l, w_{l+1}) has the same regularity as (v_l, v_{l+1}). Thus, we have shown that $(z_1, \ldots, z_N) \in \Pi_{i=1}^N C^{2+\alpha}(\overline{\Omega}_i)$ and the desired Fredholm property follows at once. The proof of Theorem 2.1 follows now similarly as in the case $N = 2$ because all the arguments that we still need are trivial extensions of those presented when $N = 2$ (and therefore we omit them).

The dispersion relation. We end this paper by considering again the case $N = 2$. We determine an explicit relation, the so-called dispersion relation, between the wave properties: mean depth d, the average thickness d_1 (resp. d_2) of the layer of vorticity γ_1 (resp. γ_2), the wavelength $L = 2\pi/n$, $n \in \mathbb{N} \setminus \{0\}$, the vorticities γ_1 and γ_2, and the relative speed at the surface of the laminar flow $c - \mathbf{u}(0)$ which has to be satisfied in order to have bifurcation from the laminar flow solutions. Computing the dispersion relation for the case $N = 3$ is also possible, but the computations are much more involved.

More precisely, we look for conditions on the physical parameters which guarantee that the problem

$$\begin{cases} (a^3 v_n')' - n^2 a v_n &= 0 & p_0 < p < p_1, \\ (A^3 V_n')' - n^2 A V_n &= 0 & p_1 < p < 0, \\ (g + \sigma n^2) V_n(0) &= \lambda^{3/2} V_n'(0), \\ v_n(p_1) &= V_n(p_1), \\ v_n'(p_1) &= V_n'(p_1), \\ v_n(p_0) &= 0 \end{cases} \tag{5.9}$$

possesses a nontrivial solution $(v_n, V_n) \in C^\infty([p_0, p_1]) \times C^\infty([p_1, 0])$. Assuming first that $\gamma_1 \gamma_2 \neq 0$, we know that the general solutions v_n, V_n of the first two equations of (5.9) are given by (3.17), with constants $\beta, \delta, \theta, \vartheta$ which need to be chosen such that the last four equations of (5.9) are also satisfied. Thus, we are left with a linear system of four equations with four unknowns. Using algebraic manipulations, we find that the latter system possesses nontrivial solutions exactly when the following relation is satisfied

$$\frac{\gamma_2 - \gamma_1}{a(p_1)} \left(g + \sigma n^2 - \lambda^{1/2} \gamma_2 - \lambda n \coth(n\Theta_2) \right)$$

$$= n(\coth(n\Theta_1) + \coth(n\Theta_2)) \left(g + \sigma n^2 - \lambda^{1/2} \gamma_2 - \lambda n \coth(n(\Theta_1 + \Theta_2)) \right),$$

where Θ_1 and Θ_2 are given by (3.19). This formula is also true when $\gamma_1 \gamma_2 = 0$ (if $\gamma_1 = 0$ (resp. $\gamma_2 = 0$), then v_n (resp. V_n) solves an ordinary differential equation with constants coefficients and the computations are easier). By virtue of the formula (3.19), we also have that $a(p_1) = A(p_1) = \lambda^{1/2} + \gamma_2 \Theta_2$.

In order to obtain the desired dispersion relation, we need to give an interpretation to Θ_1 and Θ_2. Therefore, we recall that the height function h gives the height of a particle above the flat bed, which implies that

$$d = \frac{1}{2\pi} \int_{\mathbb{S}} h(q, 0)\, dq \qquad \text{and} \qquad d_1 = \frac{1}{2\pi} \int_{\mathbb{S}} h(q, p_1)\, dq,$$

where d_1 denotes the average height of the fluid layer bounded from below by the flat bed and from above by the interface $\eta_1 := h(\cdot, p_1) - d$ separating the two currents of different vorticities γ_1 and γ_2. The constant $d_2 := d - d_1 > 0$ is the average thickness of the fluid layer of vorticity γ_2. It follows then readily from the formula (3.4) that in fact

$$\Theta_1 = d_1 \qquad \text{and} \qquad \Theta_2 = d_2.$$

Consequently, we obtain the following dispersion relation

$$
\begin{aligned}
&\frac{\gamma_2 - \gamma_1}{\lambda^{1/2} + \gamma_2 d_2} \left(g + \sigma n^2 - \lambda^{1/2}\gamma_2 - \lambda n \coth(nd_2) \right) \\
&= n(\coth(nd_1) + \coth(nd_2)) \left(g + \sigma n^2 - \lambda^{1/2}\gamma_2 - \lambda n \coth(nd) \right),
\end{aligned}
\tag{5.10}
$$

or equivalently

$$
\begin{aligned}
\lambda^{3/2} &+ \frac{1}{n}\left[\gamma_2\left(d_2 + \frac{\sinh(nd_2)\cosh(nd_1)}{\cosh(nd)} \right) + \gamma_1 \frac{\sinh(nd_1)\cosh(nd_2)}{\cosh(nd)} \right]\lambda \\
&+ \tanh(nd)\left[\frac{\gamma_2^2 d_2 - (g + \sigma n^2)}{n} + \gamma_2(\gamma_1 - \gamma_2)\frac{\sinh(nd_1)\sinh(nd_2)}{n^2\sinh(nd)} \right]\lambda^{1/2} \\
&+ \frac{(g + \sigma n^2)\tanh(nd)}{n^2}\left[\frac{(\gamma_2 - \gamma_1)\sinh(nd_1)\sinh(nd_2)}{\sinh(nd)} - \gamma_2 d_2 n \right] = 0,
\end{aligned}
\tag{5.11}
$$

where λ is related to the relative speed at the surface of the laminar flow by the equation $\sqrt{\lambda} = c - \mathbf{u}(0)$. In general, there is local bifurcation if the equation (5.11) has a positive root for $\sqrt{\lambda}$.

It is easy to infer from (5.10) that if $\gamma_1 = \gamma_2 =: \gamma$, then we obtain the dispersion relation for capillary-gravity waves on a linearly sheared current, cf. [47], as we have

$$c - \mathbf{u}(0) = -\frac{\gamma}{2}\frac{\tanh(nd)}{n} + \sqrt{\frac{\gamma^2}{4}\left(\frac{\tanh(nd)}{n} \right)^2 + (g + \sigma n^2)\frac{\tanh(n)}{n}}.$$

Moreover, setting $\sigma = 0$ in (5.10) we obtain the dispersion relation for gravity water waves obtained in [9] when $\gamma_1 = 0$, respectively in [6] for $\gamma_2 = 0$ (see also [19]). In the case when $\gamma_1 = \gamma_2 = 0$, we obtain from (5.10) the dispersion relation for irrotational capillary-gravity water waves

$$c - \mathbf{u}(0) = \sqrt{(g + \sigma n^2)\frac{\tanh(n)}{n}},$$

cf. [21, 31].

References

[1] S. Agmon, A. Douglis, and L. Nirenberg. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I. *Comm. Pure Appl. Math.*, 12:623–727, 1959.

[2] H. Amann. *Gewöhnliche Differentialgleichungen*. de Gruyter Lehrbuch. [de Gruyter Textbook]. Walter de Gruyter & Co., Berlin, 1983.

[3] H. Chen, W.-X. Li, and L.-J. Wang. Regularity of traveling free surface water waves with vorticity. *J. Nonlinear Sci.*, 23:1111–1142, 2013.

[4] A. Constantin. The trajectories of particles in Stokes waves. *Invent. Math.*, 166(3):523–535, 2006.

[5] A. Constantin. *Nonlinear Water Waves with Applications to Wave-Current Interactions and Tsunamis*, volume 81 of *CBMS-NSF Conference Series in Applied Mathematics*. SIAM, Philadelphia, 2011.

[6] A. Constantin. Dispersion relations for periodic traveling water waves in flows with discontinuous vorticity. *Commun. Pure Appl. Anal.*, 11(4):1397–1406, 2012.

[7] A. Constantin and J. Escher. Analyticity of periodic traveling free surface water waves with vorticity. *Ann. of Math.*, 173:559–568, 2011.

[8] A. Constantin and W. Strauss. Exact steady periodic water waves with vorticity. *Comm. Pure Appl. Math.*, 57(4):481–527, 2004.

[9] A. Constantin and W. Strauss. Periodic traveling gravity water waves with discontinuous vorticity. *Arch. Ration. Mech. Anal.*, 202(1):133–175, 2011.

[10] A. Constantin and E. Varvaruca. Steady periodic water waves with constant vorticity: regularity and local bifurcation. *Arch. Ration. Mech. Anal.*, 199(1):33–67, 2011.

[11] A. Constantin and G. Villari. Particle trajectories in linear water waves. *J. Math. Fluid Mech.*, 10(1):1–18, 2008.

[12] M. G. Crandall and P. H. Rabinowitz. Bifurcation from simple eigenvalues. *J. Functional Analysis*, 8:321–340, 1971.

[13] J. Escher, A.-V. Matioc, and B.-V. Matioc. A generalized Rayleigh-Taylor condition for the Muskat problem. *Nonlinearity*, 25:73–92, 2012.

[14] J. Escher and B.-V. Matioc. A moving boundary problem for periodic Stokesian Hele-Shaw flows. *Interfaces Free Bound.*, 11(1):119–137, 2009.

[15] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer Verlag, 2001.

[16] D. Henry. Particle trajectories in linear periodic capillary and capillary-gravity deep-water waves. *J. Nonlinear Math. Phys*, 14:1–7, 2007.

[17] D. Henry. Analyticity of the streamlines for periodic travelling free surface capillary-gravity water waves with vorticity. *SIAM J. Math. Anal*, 42(6):3103–3111, 2010.

[18] D. Henry. Analyticity of the free surface for periodic travelling capillary-gravity water waves with vorticity. *J. Math. Fluid Mech.*, 14(2):249–254, 2012.

[19] D. Henry. Dispersion relations for steady periodic water waves with an isolated layer of vorticity at the surface. *Nonlinear Anal. Real World Appl.*, 14:1034–1043, 2013.

[20] D. Henry and B.-V. Matioc. On the regularity of steady periodic stratified water waves. *Commun. Pure Appl. Anal.*, 11(4):1453–1464, 2012.

[21] R. S. Johnson. *A Modern Introduction to the Mathematical Theory of Water Waves*, volume 19. Cambridge Univ Pr, 1997.

[22] M. Jones. Small amplitude capillary-gravity waves in a channel of finite depth. *Glasgow Math. J.*, 31(2):141–160, 1989.

[23] M. Jones and J. Toland. The bifurcation and secondary bifurcation of capillary-gravity waves. *Proc. Roy. Soc. London Ser. A*, 399(1817):391–417, 1985.

[24] M. Jones and J. Toland. Symmetry and the bifurcation of capillary-gravity waves. *Arch. Rational Mech. Anal.*, 96(1):29–53, 1986.

[25] I. G. Jonsson. *Wave-current interactions*, volume 9. Wiley, New York, 1990.

[26] W. Kinnersley. Exact large amplitude capillary waves on sheets of fluid. *J. Fluid Mech.*, 77(2):229–241, 1976.

[27] J. Ko and W. Strauss. Large-amplitude steady rotational water waves. *Eur. J Mech. B Fluids*, 27:96–109, 2007.

[28] J. Ko and W. Strauss. Effect of vorticity on steady water waves. *J. Fluid Mech.*, 608:197–215, 2008.

[29] O. A. Ladyzhenskaya and N. N. Uraltseva. *Linear and quasilinear elliptic equations.* Translated from the Russian by Scripta Technica, Inc. Translation editor: Leon Ehrenpreis. Academic Press, New York, 1968.

[30] J. LeCrone. Elliptic operators and maximal regularity on periodic little-Hölder spaces. *J. Evol. Equ.,* 12(2):295–325, 2012.

[31] J. Lighthill. *Waves in Fluids.* Cambridge Univ Pr, 2001.

[32] Y. Luo and N. S. Trudinger. Linear second order elliptic equations with Venttsel boundary conditions. *Proc. Roy. Soc. Edinburgh Sect. A,* 118(3-4):193–207, 1991.

[33] C. I. Martin. Local bifurcation and regularity for steady periodic capillary–gravity water waves with constant vorticity. *Nonlinear Anal. Real World Appl.,* (14):131–149, 2013.

[34] C. I. Martin and B.-V. Matioc. Existence of Wilton ripples for water waves with constant vorticity and capillary effects. *SIAM J. Appl. Math.,* 73(4):1582–1595, 2013.

[35] C. I. Martin and B.-V. Matioc. Steady periodic water waves with unbounded vorticity: equivalent formulations and existence results. 2013. arXiv:1311.6935.

[36] A.-V. Matioc. On particle trajectories in linear water waves. *Nonlinear Anal. Real World Appl.,* 11(5):4275–4284, 2010.

[37] A.-V. Matioc. Steady internal water waves with a critical layer bounded by the wave surface. *J. Nonlinear Math. Phys.,* (19(1)):1250008, 21p, 2012.

[38] B.-V. Matioc. Analyticity of the streamlines for periodic traveling water waves with bounded vorticity. *Int. Math. Res. Not.,* 17:3858–3871, 2011.

[39] B.-V. Matioc. Global bifurcation for water waves with capillary effects and constant vorticity. *Monatsh. Math.,* 2013. DOI 10.1007/s00605-013-0583-1.

[40] B.-V. Matioc and A.-V. Matioc. Capillary-gravity water waves with discontinuous vorticity: existence and regularity results. *Comm. Math. Phys.,* 2013. arXiv:1311.6593.

[41] K. Okuda. Internal flow structure of short wind waves. *Journal of the Oceanographical Society of Japan,* 38:28–42, 1982.

[42] C. Pattiaratchi and M. Collins. Sediment transport under waves and tidal currents: A case study from the northern Bristol Channel, U.K. *Marine Geology,* 56(1-4):27–40, 1984.

[43] O. M. Phillips and M. L. Banner. Wave breaking in the presence of wind drift and swell. *J. Fluid Mech.,* 66:625–640, 1974.

[44] J. Reeder and M. Shinbrot. On Wilton ripples. II. Rigorous results. *Arch. Rational Mech. Anal.,* 77(4):321–347, 1981.

[45] G. Thomas and G. Klopman. *Wave-current interactions in the nearshore region.* WIT, Southampton, United Kingdom, 1997.

[46] G. P. Thomas. Wave-current interractions: An experimental and numerical study. *J. Fluid Mech.,* 216:505–536, 1990.

[47] E. Wahlén. Steady periodic capillary-gravity waves with vorticity. *SIAM J. Math. Anal.,* 38(3):921–943 (electronic), 2006.

INSTITUT FÜR MATHEMATIK, UNIVERSITÄT WIEN, NORDBERGSTRASSE 15, 1090 WIEN, AUSTRIA
E-mail address: `calin.martin@univie.ac.at`

INSTITUT FÜR ANGEWANDTE MATHEMATIK, LEIBNIZ UNIVERSITÄT HANNOVER, WELFENGARTEN 1, 30167 HANNOVER, GERMANY.
E-mail address: `matioc@ifam.uni-hannover.de`