

# VISUAL RIGHT-ANGLED ARTIN SUBGROUPS OF TWO-DIMENSIONAL RIGHT-ANGLED COXETER GROUPS

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ABSTRACT. There is a procedure, due to Dani and Levcovitz, for taking a finite simplicial graph  $\Gamma$  and a subgraph  $\Lambda$  of its complement, checking some conditions, and, if satisfied, producing a graph  $\Delta$  such that the right-angled Artin group with presentation graph  $\Delta$  is a finite index subgroup of the right-angled Coxeter group with presentation graph  $\Gamma$ . They do not tell us how to find  $\Lambda$ , given  $\Gamma$ .

We show, in the 2-dimensional case, that the existence of such a  $\Lambda$  is connected to the graph property of satellite-dismantlability of  $\Gamma$ , and we use this to give an algorithm for producing a suitable  $\Lambda$  or deciding that one does not exist.

## 1. INTRODUCTION

Every right-angled Artin group (RAAG) is a finite index subgroup of a right-angled Coxeter group (RACG), and given the presentation graph  $\Delta$  of the RAAG  $A_\Delta$  there is a simple graph operation that turns it into a graph  $\Gamma$  that is the presentation graph of the RACG supergroup  $W_\Gamma$  [8]. The converse is not true; RACGs are more varied, and there are invariants such as divergence that show that some RACGs are not even quasiisometric to a RAAG. So, what graph conditions on  $\Gamma$  imply that  $W_\Gamma$  is commensurable to a RAAG?

Consider the case that  $\Gamma$  is a square. Each pair of diagonal vertices generate an infinite dihedral group, and these two dihedral groups commute. Each of the dihedral groups has an index two, infinite cyclic subgroup, and these make an index four,  $\mathbb{Z}^2$  subgroup of  $W_\Gamma$ . This is the basic example of a finite index *visual* RAAG subgroup; it is ‘visual’ in the sense that we ‘see’ the RAAG generators as pairs of non-adjacent vertices of  $\Gamma$ , and they commute when the vertex pairs from  $\Gamma$  make a square.

This situation generalizes as follows: let  $\Lambda$  be a subgraph of the complement graph  $\Gamma^c$  of  $\Gamma$ ; that is,  $\Gamma^c$  has the same vertex set as  $\Gamma$ , and has an edge if and only if  $\Gamma$  does not. Edges of  $\Lambda$  give pairs of generators of  $W_\Gamma$  that generate an infinite dihedral subgroup. Let  $\Delta$  be the graph with one vertex for each edge of  $\Lambda$ , and an edge between two vertices if the corresponding subgroups commute, which is the case exactly when they span a square in  $\Gamma$ . There is a homomorphism from  $A_\Delta$  to  $W_\Gamma$  given by sending a generator of  $A_\Delta$  to the product in  $W_\Gamma$  of the two endpoints of the corresponding edge of  $\Lambda$ . In general, however, this homomorphism is not injective, nor does it have finite index image. Based on initial results of LaForge [12], Dani and Levcovitz [5] give conditions on  $\Lambda$  that determine whether the natural homomorphism is injective when  $\Lambda$  has at most two connected components. In particular, this is always sufficient [5, Lemma 4.7] for the 2-dimensional case, when  $\Gamma$  is triangle-free, and in this case they give necessary and sufficient conditions on  $\Lambda$  for  $A_\Delta$  to be a finite index subgroup of  $W_\Gamma$ .

We focus on the 2-dimensional case, and call a  $\Lambda < \Gamma^c$  satisfying their conditions a *finite index Dani-Levcovitz*  $\Lambda$  (FIDL- $\Lambda$ ). Checking that a given subgraph of  $\Gamma^c$  is a FIDL- $\Lambda$  is algorithmic, and since  $\Gamma$  is finite, one can simply enumerate subgraphs of  $\Gamma^c$  and check them all. This is slow, even for rather small examples. We are interested in a more efficient algorithm for starting from  $\Gamma$  alone and either producing a FIDL- $\Lambda$  or deciding that one does not exist.

We give such an algorithm as Global Search Algorithm 5.3. The key step, Theorem 4.7, is that a FIDL- $\Lambda$  exists if and only if  $\Gamma$  admits a satellite-dismantling sequence that reduces it to a square and satisfies some additional conditions that can be checked only from  $\Gamma$ . We apply this algorithm to a large number of examples via computer computations in a forthcoming work.

## 2. PRELIMINARIES

**2.1. Graphs.**  $A * B$  denotes the *join* of  $A$  and  $B$ ; that is, the complete bipartite graph with one part the elements of  $A$  and the other the elements of  $B$ . A graph is *complete* if for every pair of vertices there exists an edge between them. It is *incomplete* if there exist two vertices that are not joined by an edge. The empty graph and a graph consisting of a single vertex are complete. A *clique* is a complete

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*Key words and phrases.* Right-angled Coxeter group, right-angled Artin group, RACG, RAAG, visual subgroup.

This paper is an enhancement of a section of the thesis of the second author [9, Section 4.1], submitted in partial fulfilment of the requirements for the degree of Doktorin der Naturwissenschaft (Dr. rer. nat.) at the University of Vienna. A.E. is grateful to Pallavi Dani for drawing her attention to the questions addressed in this paper. Both authors supported by the Austrian Science Fund (FWF): P34214-N.

subgraph. A graph  $\Gamma$  is *separated by a clique* if there is a clique  $C$  such that  $\Gamma - C$  has more than one connected component. A disconnected graph is separated by the empty clique. The *link*  $\text{lk}(v)$  of a vertex  $v$  in a graph is the induced subgraph on its neighboring vertices. The *star*  $\text{st}(v)$  of a vertex  $v$  is  $\{v\} * \text{lk}(v)$ . A *loop* is an edge path that starts and ends at the same vertex, and a *cycle* is a loop that has no repeated vertices. A subgraph of  $\Gamma$  is *induced* if it contains all of the edges between its vertices. The *induced subgraph* of a set of vertices is the induced subgraph that they span. Given a cycle  $\gamma$ , an  $n$ -*chord*, or just *chord* when  $n = 1$ , is a path of length  $n$  between vertices  $x$  and  $y$  of  $\gamma$  such that both subsegments of  $\gamma$  between  $x$  and  $y$  have length greater than  $n$ .

Distinct vertices  $v$  and  $w$  are *twins* if  $\text{lk}(v) = \text{lk}(w)$ , and  $v$  is a *satellite of  $w$*  if  $\text{lk}(v) \subset \text{lk}(w)$ . A vertex is a *satellite* if it is a satellite of some vertex. A graph  $\Gamma$  is *satellite-dismantlable to a square* if there exists a sequence  $\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_n$  such that  $\Gamma_i - \Gamma_{i+1}$  is a single satellite and  $\Gamma_n$  is a square. This is reminiscent of the more common graph-theoretic notion of a *dismantlable graph*, in which a vertex  $v$  is *dominated by  $w$*  if  $\text{st}(v) \subset \text{st}(w)$ , and a graph is dismantlable if it is possible to reduce it to a single vertex by removing one dominated vertex at a time.

**2.2. RACGs and RAAGs.** See [7] and [4, Section 2.6] for background on RACGs and RAAGs.

**Definition 2.1.** Let  $\Omega$  be a finite, simplicial graph with vertex set  $V(\Omega)$ . The *right-angled Coxeter group (RACG)*  $W_\Omega$  is defined as the group given by the following presentation

$$W_\Omega = \langle s \in V(\Omega) \mid s^2 = 1 \text{ for all } s \in V(\Omega), st = ts \text{ if } (s, t) \in E(\Omega) \rangle.$$

The *right-angled Artin group (RAAG)*  $A_\Omega$  is defined as the group given by the following presentation

$$A_\Omega = \langle m \in V(\Omega) \mid mn = nm \text{ if } (m, n) \in E(\Omega) \rangle.$$

The graph  $\Omega$  is called the *presentation graph*<sup>1</sup>.

**Definition 2.2.** If  $W_\Gamma$  is a RACG and  $\Upsilon$  is an induced subgraph of  $\Gamma$ , then the subgroup of  $W_\Gamma$  generated by  $V(\Upsilon)$  is called a *special subgroup*. It is a RACG with presentation graph  $\Upsilon$ , and is denoted  $W_{V(\Upsilon)}$  or  $W_\Upsilon$ . The analogous statement and terminology also applies to RAAGs.

We will restrict to one-ended groups. A RACG is one-ended if and only if it is incomplete and has no separating clique [7, Theorem 8.7.2]. A RAAG is one-ended if it is connected and has at least two vertices. To further simplify the set-up, we consider only RACGs and RAAGs whose Davis and Salvetti complexes, respectively, are two-dimensional. This is satisfied for both RACGs and RAAGs if the presentation graph is triangle-free.

There is a quasiisometry invariant known as *divergence*. In particular, if a group has polynomial divergence then the degree of the polynomial is a quasiisometry invariant. RAAGs have at most quadratic divergence [1, Corollary 4.8], thus so does every group quasiisometric to a RAAG. By work of Dani and Thomas [6, Theorem 1.1] a one-ended, two-dimensional RACG has at most quadratic divergence if and only if its presentation graph has a property known as  $\mathcal{CFS}$  (component of full support/constructed from squares). This was generalized to higher dimension in [2, Definition 1.3].

**Definition 2.3.** The *diagonal graph*  $\square(\Gamma)$  of  $\Gamma$  is the graph whose vertices are diagonals of induced squares in  $\Gamma$ , with  $\{a, b\}$  and  $\{c, d\}$  connected by an edge if  $\{a, b\} * \{c, d\}$  is an induced square in  $\Gamma$ .

The *support*  $\text{supp}(\{a, b\})$  of a vertex  $\{a, b\}$  of  $\square(\Gamma)$  is the pair of vertices  $\{a, b\}$  in  $\Gamma$ . The support of a subset of  $\square(\Gamma)$  is the union of the supports of its vertices.

The graph  $\Gamma$  is  $\mathcal{CFS}$  if  $\square(\Gamma)$  has a connected component whose support is all non-cone vertices of  $\Gamma$ . If  $\Gamma$  is triangle-free and not a star then it has no cone vertices, and we can simply say that some component of  $\square(\Gamma)$  has full support, ie, its support is all of  $\Gamma$ .

The graph  $\Gamma$  is *strongly  $\mathcal{CFS}$*  if it is  $\mathcal{CFS}$  and  $\square(\Gamma)$  is connected.

*Remark.* The usual definition of  $\mathcal{CFS}$  uses a graph  $\square(\Gamma)$  whose vertices are induced squares of  $\Gamma$ , with an edge between two vertices if they intersect in a diagonal. The graphs  $\square(\Gamma)$  and  $\square(\Gamma)$  carry the same information, but  $\square(\Gamma)$  is topologically simpler, since many squares intersecting in a common diagonal form a star in  $\square(\Gamma)$  but a clique in  $\square(\Gamma)$ . The diagonal graph is also more natural for our purposes because when we have  $\Gamma$  with a FIDL- $\Lambda$ , then the commuting graph  $\Delta$  of  $\Lambda$  sits as a subgraph in  $\square(\Gamma)$ .

<sup>1</sup>This is different from the conventions used to define the *Coxeter graph*, which is more commonly used for not-necessarily-right-angled Coxeter groups.

**Theorem 2.4** ([6]). *If  $\Gamma$  is an incomplete, triangle-free graph without separating cliques such that  $W_\Gamma$  is quasiisometric to a RAAG then  $\Gamma$  is CFS.*

**Lemma 2.5.** *If  $\Gamma$  is incomplete, triangle-free, and CFS then it has no separating clique.*

*Proof.* Take  $a, b \in \Gamma$ . If  $a$  and  $b$  are the diagonal of some square then they are not separated by a clique. Otherwise, there is a path  $\{p_0, q_0\}, \dots, \{p_n, q_n\}$  in the full support component of  $\square(\Gamma)$  with  $n > 0$ ,  $p_0 = a$ , and  $p_n = b$ . This path corresponds to a chain of squares  $\{p_i, q_i\} * \{p_{i+1}, q_{i+1}\}$  in  $\Gamma$ , with successive squares sharing a diagonal. The union of the squares is not separated by a clique.  $\square$

**2.3. Dani-Levcovitz conditions.** Let  $\Theta = \Theta(\Gamma, \Lambda)$  be the graph with vertex set  $\Gamma$ , with edges from both  $\Gamma$  and  $\Lambda < \Gamma^c$ . The edges coming from  $\Gamma$  are  $\Gamma$ -edges, and the edges coming from  $\Lambda$  are  $\Lambda$ -edges. Similarly, a path consisting only of  $\Gamma$ -edges is a  $\Gamma$ -path, etc. The  $\Lambda$ -hull,  $\text{hull}_\Lambda$ , of a subset of vertices of  $\Theta$  is the vertex set of their convex hull in  $\Lambda$ . A set of vertices is  $\Lambda$ -convex if it is equal to its  $\Lambda$ -hull.

Dani and Levcovitz [5] give *subgroup conditions*  $\mathcal{R}_1$ – $\mathcal{R}_4$  to determine that the RAAG  $A_\Delta$  on the commuting graph  $\Delta$  associated to  $\Lambda$  is a visual RAAG subgroup of  $W_\Gamma$ . They give *index conditions*  $\mathcal{F}_1$  and  $\mathcal{F}_2$  to ensure that the visual RAAG subgroup is of finite index in  $W_\Gamma$ . They show in the 2-dimensional case that it always suffices to find  $\Lambda$  with two components, and for two component  $\Lambda$  their conditions are necessary and sufficient.

The conditions are as follows, simplified by specializing to the case that  $\Gamma$  is an incomplete, triangle-free graph without separating cliques. Let  $\Lambda_r$  (red) and  $\Lambda_b$  (blue) be disjoint, connected subgraphs of  $\Gamma^c$ , with  $\Lambda = \Lambda_r \sqcup \Lambda_b$ .

$\mathcal{R}_1$ :  $\Lambda_r$  and  $\Lambda_b$  are trees.

$\mathcal{R}_2$ :  $\Lambda_r$  and  $\Lambda_b$  are induced subgraphs of  $\Theta$ .

$\mathcal{F}_1$ :  $\Lambda$  spans  $\Gamma$ .

These conditions are true if and only if  $\Gamma$  is bipartite, with a bicoloring  $r/b$  (every vertex is colored either  $r$  or  $b$ , and adjacent vertices have different colors), and  $\Lambda_r$  and  $\Lambda_b$  are trees in  $\Gamma^c$  spanning the  $r$  and  $b$  parts, respectively. We will not state  $\mathcal{F}_2$ . In our case it is always satisfied if  $\mathcal{R}_2$  and  $\mathcal{F}_1$  are [5, Remark 4.3]. Assuming these conditions, we can state the remaining two conditions in simplified form:

$\mathcal{R}_3$ : If  $\{a, b\} * \{c, d\}$  is a square in  $\Gamma$  then  $\text{hull}_\Lambda\{a, b\} * \text{hull}_\Lambda\{c, d\} \subset \Gamma$ .

$\mathcal{R}_4$ : If  $a \leftrightarrow b$  is an edge in a cycle  $\gamma$  then there is a square  $\{a, a'\} * \{b, b'\}$  with  $a', b' \in \text{hull}_\Lambda(\gamma)$ .

Notice that the assumption that  $\Gamma$  is incomplete with no separating clique implies  $W_\Gamma$  is 1-ended, so  $A_\Delta$  is 1-ended, so  $\Delta$  is connected and has more than one vertex. Thus, every edge of  $\Lambda$  is a diagonal of a square in  $\Gamma$ , since otherwise it would have nothing to commute with, so would give an isolated vertex in  $\Delta$ . Thus, we may identify  $\Delta$  with a subgraph of  $\square(\Gamma)$ .

One nice application of these conditions in [5] is to connect them to conditions given by Nguyen and Tran [15] on deciding when a *planar* graph  $\Gamma$  defines a RACG that is quasiisometric to a RAAG. The conclusion is that, for planar  $\Gamma$ ,  $W_\Gamma$  being quasiisometric to a RAAG implies graph conditions that imply  $\mathcal{R}_1$ – $\mathcal{R}_4$  and  $\mathcal{F}_1$  and  $\mathcal{F}_2$ , so  $W_\Gamma$  actually has a finite index visual RAAG subgroup, which happens always to be defined by a tree  $\Delta$ . Dani and Levcovitz also give two families of non-planar graphs to which their conditions apply and yield  $\Delta$  that are not trees. We mention one of these families here:

*Example 2.6.* A *bicycle wheel* is a graph consisting of adjacent vertices  $x$  and  $y$ , the ‘hub’, a circle of even length  $2n \geq 6$  given by  $c_1, d_1, c_2, \dots, d_n$ , the ‘rim’, and edges from each  $c_i$  to  $x$ , and from each  $d_i$  to  $y$ , the ‘spokes’.



FIGURE 2.1. A bicycle wheel

A bicycle wheel admits a 2-component FIDL- $\Lambda$  consisting of the opposite of the spokes: one star consisting of an edge from  $x$  to each  $d_i$  and another consisting of an edge from  $y$  to each  $c_i$ . The commuting graph  $\Delta$  is a circle of the same length as the rim.

In Figure 2.2 the case  $n = 3$  is also recognizable as the 1-skeleton of a 3-cube with one space diagonal:

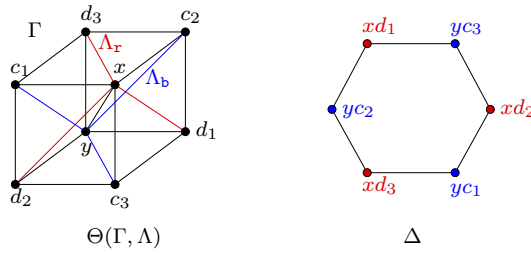


FIGURE 2.2

**2.4. Splittings of RACGs.** We have mentioned that  $W_\Gamma$  is one-ended when it is incomplete with no separating clique. This corresponds to not having a splitting as an amalgamated product over a finite group. The next simplest splittings are over two-ended, or virtually  $\mathbb{Z}$ , groups.

A *JSJ decomposition* of a finitely presented group is a certain maximal graph of groups decomposition (see [11] for the precise definition) with two-ended edge groups and vertex groups in three categories: two-ended, *hanging* or *rigid*. A hanging vertex group is essentially the fundamental group of a surface with boundary and a rigid vertex group does not split any further with respect to its incident edge groups. JSJ decompositions are not unique, but there is a way to encode all of them simultaneously in a *JSJ graph of cylinders*. It is the canonical representative for the deformation space of JSJ decompositions of the group and it can be used to deduce quasiisometry invariants, by [3]. The idea is that some of the two-ended edge and vertex groups in a JSJ decomposition may be commensurable, and these can be grouped together to form *cylinders*, and from this a new decomposition is derived. For RACGs, all of this is visible in the presentation graph: Mihalik and Tschantz [14] show a one-ended, two-dimensional RACG  $W_\Gamma$  admits a splitting over a two-ended subgroup if and only if  $\Gamma$  has a *cut*  $\binom{a}{b}$ :

**Definition 2.7.** If  $\Gamma$  is an incomplete, triangle-free graph without separating cliques, a *cut*  $\binom{a}{b}$  means one of the following, both of which have the property that the element  $ab \in W_\Gamma$  generates an infinite cyclic subgroup that is finite index in  $W\binom{a}{b}$ .

- A *cut pair*  $\binom{a}{b} = \{a, b\}$ : a pair of non-adjacent vertices such that  $\Gamma - \{a, b\}$  is not connected.
- A *2-path cut triple*  $\binom{a}{b} = \{a, b, c\}$ : a triple of vertices with  $c \in \text{lk}(a) \cap \text{lk}(b)$  such that  $\Gamma - \{a, b\}$  is connected but  $\Gamma - \{a, b, c\}$  is not.

Definition 2.7 implies that every component of  $\Gamma - \binom{a}{b}$  contains a neighbor of each vertex in  $\binom{a}{b}$ .

A cut pair  $\{a, b\}$  is *crossed* by another, disjoint, cut pair  $\{c, d\}$  if  $a$  and  $b$  lie in different connected components of  $\Gamma - \{c, d\}$ . A cut  $\binom{a}{b} = \{a, b, c\}$  is *crossed* by a cut  $\binom{d}{e} = \{d, e, f\}$  if  $c$  is equal to  $f$  and  $a$  and  $b$  lie in different connected components of  $\Gamma - \{d, e, c\}$ . A cut that is not crossed by any other cut is *uncrossed*. Crossing cuts are responsible for hanging vertices in the JSJ decomposition, but RAAGs do not have these [13], so they cannot appear in groups quasiisometric to RAAGs.

**Theorem 2.8** ([10, Theorem 3.29]). *Let  $\Gamma$  be an incomplete, triangle free graph without separating cliques. If the JSJ graph of cylinders of  $W_\Gamma$  has no hanging vertices, it consist of:*

- For every pair  $\{a, b\}$  such that there is an uncrossed cut  $\binom{a}{b}$ , there is a cylinder vertex with vertex group  $W_{\{a, b\} \cup (\text{lk}(a) \cap \text{lk}(b))}$ .
- For every set  $B$  of essential (valence at least 3) vertices in  $\Gamma$  satisfying the following conditions, there is a rigid vertex with vertex group  $W_B$ :
  - (B1) No cut separates  $B$ .
  - (B2) The set  $B$  is maximal among all sets satisfying (B1).
  - (B3)  $|B| \geq 4$ .

Furthermore a pair of vertices is connected by an edge if and only if the pair consists of a cylinder vertex and a rigid vertex whose vertex groups intersect in a subgroup containing the two-ended cut defining the cylinder. The edge group is the intersection of its vertex groups.

### 3. FIDL- $\Lambda$ CONVEXITY

**Lemma 3.1.** *Let  $\Gamma$  be an incomplete, triangle-free graph with no separating clique that admits a FIDL- $\Lambda$ . Let  $\binom{v}{v'}$  be a cut of  $\Gamma$ . Every component of  $\Gamma - \binom{v}{v'}$  contains a common neighbor of  $v$  and  $v'$ .*

*Proof.* Pick a component of  $\Gamma - \binom{v}{v'}$ ; let  $b$  be a vertex in that component, and let  $a$  be a vertex from a different component. By Theorem 2.4 we can choose a geodesic  $\{p_0, q_0\}, \dots, \{p_n, q_n\}$  in  $\square(\Gamma)$  such that  $a \in \{p_0, q_0\}$  and  $b \in \{p_n, q_n\}$ . Let  $i_0$  be the least index such that  $\{p_{i_0}, q_{i_0}\}$  contains a vertex, say  $p_{i_0}$ , in the same component of  $\Gamma - \binom{v}{v'}$  as  $b$ .

If  $i_0 = 0$  then  $q_{i_0} = a$  and there is a square with one diagonal containing vertices in different components of the cut. This is only possible if the cut  $\binom{v}{v'}$  is a cut pair  $\{v, v'\}$  and the square is  $\{a, p_0\} * \{v, v'\}$ , in which case  $p_0$  is a common neighbor of  $v$  and  $v'$  in the  $b$ -component.

If  $i_0 > 0$  then  $\{p_{i_0-1}, q_{i_0-1}\} * \{p_{i_0}, q_{i_0}\}$  is a square with  $p_{i_0}$  in the  $b$ -component and with  $p_{i_0-1}$  and  $q_{i_0-1}$  non-adjacent vertices that are both adjacent to  $p_{i_0}$ , but neither of which are in the same component of  $\Gamma - \binom{v}{v'}$  as  $p_{i_0}$ . Then  $\{p_{i_0-1}, q_{i_0-1}\} = \{v, v'\}$ .  $\square$

**Lemma 3.2.** *Let  $\Gamma$  be an incomplete, triangle-free graph with no separating clique that admits a FIDL- $\Lambda$ . Let  $\binom{v}{v'}$  be a cut of  $\Gamma$ . Then  $\Lambda$  contains an edge between  $v$  and  $v'$ .*

*Proof.* By Lemma 3.1, it is possible to choose vertices  $a$  and  $b$  that are common neighbors of  $v$  and  $v'$  and contained in different components of  $\Gamma - \binom{v}{v'}$ . Every common neighbor of  $a$  and  $b$  must lie in the cut, so the triangle-free condition implies the only common neighbors of  $a$  and  $b$  are  $v$  and  $v'$ .

Now,  $\{a, b\} * \{v, v'\}$  is a square, so  $\mathcal{R}_3$  implies  $\{a, b\} * \text{hull}_\Lambda\{v, v'\} \subset \Gamma$ , so  $\text{hull}_\Lambda\{v, v'\} = \{v, v'\}$ , so there is a  $\Lambda$ -edge between  $v$  and  $v'$ .  $\square$

**Lemma 3.3.** *Let  $\Gamma$  be an incomplete, triangle-free graph with no separating clique that admits a FIDL- $\Lambda$ . The link of every vertex is  $\Lambda$ -convex.*

*Proof.* Suppose not. Then there exist  $a, b, v$  with  $a, b \in \text{lk}(v)$  such that  $\text{lk}(v) \cap \text{hull}_\Lambda\{a, b\} = \{a, b\}$  and  $a$  and  $b$  are not adjacent in  $\Lambda$ . Assume  $v \in \Lambda_{\mathbf{r}}$  and  $a, b \in \Lambda_{\mathbf{b}}$ . Since  $\Lambda_{\mathbf{b}}$  is a tree, there is a unique  $\Lambda_{\mathbf{b}}$ -geodesic  $a = c_0, c_1, \dots, c_n = b$ ,  $n \geq 2$ , such that each pair  $\{c_i, c_{i+1}\}$  is the diagonal of a square  $\{c_i, c_{i+1}\} * \{d_i, d'_i\}$  of  $\Gamma$ . None of the  $d_i$  and  $d'_i$  equal  $v$ , since the  $c_i$  for  $i \neq 0, n$  are not in  $\text{lk}(v)$ .

We build a cycle  $\gamma$  as follows: start with  $b, v, a = c_0$ , and  $d_0$ . Next add  $c_{j_0}$  where  $j_0 \geq 1$  is the largest index such that  $c_{j_0}$  is adjacent to  $d_0$ . Then add  $d_{j_0}$ . Continue, where, having most recently added  $d_i$  we next add  $c_j$  such that  $j$  is the maximal index with  $d_i$  adjacent to  $c_j$ . If  $j < n$  then add  $d_j$  and repeat. The point is that while  $d_j$  is adjacent to  $c_{j+1}$ , the previous  $d_i$  that occur in  $\gamma$  are not adjacent to any  $c_k$  for  $k > j$ , so we guarantee that no  $d$  is repeated in  $\gamma$ .

Thus,  $\gamma$  is a cycle in  $\Gamma$  with  $\text{hull}_{\Lambda_{\mathbf{b}}}(\gamma) = \text{hull}_\Lambda\{a, b\} = \{c_0, \dots, c_n\}$ . Condition  $\mathcal{R}_4$  implies there exists a square  $\{a, a'\} * \{v, v'\}$  with  $a', v' \in \text{hull}_\Lambda \gamma$ . However,  $a$  and  $b$  are the only vertices of  $\text{lk}(v) \cap \text{hull}_\Lambda \gamma$ , so  $a' = b$ . Condition  $\mathcal{R}_3$  implies  $\{v\} * \text{hull}_\Lambda\{a, b\} \subset \Gamma$ , contradicting  $\text{hull}_\Lambda\{a, b\} \not\subset \text{lk}(v)$ .  $\square$

**Corollary 3.4.** *Let  $\binom{v}{v'}$  be a cut of  $\Gamma$ , and let  $\Gamma'$  be a connected component of  $\Gamma - \binom{v}{v'}$ . Let  $\bar{\Gamma}' := \Gamma' \cup \binom{v}{v'}$ . The intersection of each component of  $\Lambda$  with  $\bar{\Gamma}'$  is  $\Lambda$ -convex.*

*Proof.* Since there are no separating cliques, for every pair of vertices  $a, b \in \bar{\Gamma}'$ , there exists a  $\Gamma$ -path  $c_0 = a, \dots, c_n = b$  such that  $c_i \in \Gamma'$  for all  $i \neq 0, n$ . If  $a$  and  $b$  are in the same component of  $\Lambda$  then  $n$  is even. For each odd  $i$ ,  $\text{lk}(c_i)$  is  $\Lambda$ -convex and contains  $c_{i-1}$  and  $c_{i+1}$ . Since  $c_i \in \Gamma'$  for odd  $i$ ,  $\text{lk}(c_i) \subset \bar{\Gamma}'$ , so we can piece together a  $\Lambda$ -path from  $a$  to  $b$  that stays in  $\bar{\Gamma}'$ .  $\square$

**Corollary 3.5.** *For any two vertices  $a, b \in \Gamma$ ,  $\text{lk}(a) \cap \text{lk}(b)$  is  $\Lambda$ -convex.*

*Proof.* If  $|\text{lk}(a) \cap \text{lk}(b)| < 2$  there is nothing to prove, so assume  $c$  and  $d$  are distinct common neighbors of  $a$  and  $b$ . By Lemma 3.3, there is a  $\Lambda$ -geodesic from  $c$  to  $d$  in  $\text{lk}(a)$  and a  $\Lambda$ -geodesic from  $c$  to  $d$  in  $\text{lk}(b)$ , but  $\Lambda$  is a forest, so these are the same path, and are therefore in  $\text{lk}(a) \cap \text{lk}(b)$ .  $\square$

Here are some consequences of these convexity results:

**Proposition 3.6.** *Let  $\Gamma$  be an incomplete, triangle-free graph with no separating clique that admits a FIDL- $\Lambda$ . Then every cycle of  $\Gamma$  has even length, every cycle of length greater than 6 has a 1 or 2-chord, and an induced cycle of length 6 occurs only as the rim of a bicycle wheel subgraph of  $\Gamma$ .*

*Proof.*  $\Gamma$  is bipartite, since its vertices are 2-colored  $\mathbf{r}/\mathbf{b}$  according to which component  $\Lambda_{\mathbf{r}}$  or  $\Lambda_{\mathbf{b}}$  of  $\Lambda$  they belong. Thus,  $\Gamma$  has no odd cycles. Suppose  $\gamma := c_0, c_1, \dots, c_{n-1}$  is a cycle of length  $n > 4$  with no 1 or 2-chords. We always take subscripts modulo  $n$ , without further comment.

Let  $[x, y]_\Lambda$  denote the unique  $\Lambda$ -geodesic joining vertices  $x$  and  $y$  if they are in the same  $\Lambda$  component.

Construct a  $\Lambda$ -loop at  $c_0$  by taking  $[c_0, c_2]_\Lambda + [c_2, c_4]_\Lambda + \dots + [c_{n-2}, c_n]_\Lambda$ . This is a loop in a tree, so it is degenerate. In particular, each edge is crossed an even number of times. By Lemma 3.3,

$[c_{2m}, c_{2m+2}]_\Lambda \subset \text{lk}(c_{2m+1})$ . Since the cycle has no 1 or 2-chords, for odd  $j > i$  we have that  $\text{lk}(c_i)$  and  $\text{lk}(c_j)$  intersect only if  $j = i + 2$  or  $i = 1$  and  $j = n - 1$ , so only  $[c_{2m-2}, c_{2m}]_\Lambda$  and  $[c_{2m+2}, c_{2m+4}]_\Lambda$  potentially share edges with  $[c_{2m}, c_{2m+2}]_\Lambda$ . Since a geodesic uses an edge once or not at all, to have all of the edges of  $[c_{2m}, c_{2m+2}]_\Lambda$  crossed evenly in total by the loop we need that each edge of  $[c_{2m}, c_{2m+2}]_\Lambda$  is also contained in exactly one of  $[c_{2m-2}, c_{2m}]_\Lambda$  and  $[c_{2m+2}, c_{2m+4}]_\Lambda$ . Thus, there is  $x \in [c_{2m}, c_{2m+2}]_\Lambda$  such that  $[c_{2m}, x]_\Lambda = [c_{2m-2}, c_{2m}]_\Lambda \cap [c_{2m}, c_{2m+2}]_\Lambda$  and  $[x, c_{2m+2}]_\Lambda = [c_{2m}, c_{2m+2}]_\Lambda \cap [c_{2m+2}, c_{2m+4}]_\Lambda$ . Such an  $x$  is in  $\text{lk}(c_{2m-1}) \cap \text{lk}(c_{2m+1}) \cap \text{lk}(c_{2m+3})$ . But then  $d_\Gamma(c_{2m-1}, c_{2m+3}) = 2$ , so, since  $\gamma$  has no 2-chords,  $d_\gamma(c_{2m-1}, c_{2m+3}) = 2$ , so  $n = 6$ .

The same argument, reversing evens and odds, shows there exists  $y \in \text{lk}(c_0) \cap \text{lk}(c_2) \cap \text{lk}(c_4)$ .

Consider the cycle  $\gamma' := y, c_0, c_1, x, c_3, c_4$ . By condition  $\mathcal{R}_4$ , there is a square  $\{c_0, v\} * \{c_1, w\}$  with  $v, w \in \text{hull}_\Lambda(\gamma') = \{x, c_0, c_4\} \sqcup \{y, c_1, c_3\}$ , so  $v \in \{x, c_4\}$  and  $w \in \{y, c_3\}$ . But since  $\gamma$  has no chords,  $c_1$  is not adjacent to  $c_4$  and  $c_3$  is not adjacent to  $c_0$ , so  $v = x$  and  $w = y$ , implying  $x$  and  $y$  are adjacent. Thus,  $\gamma$  is the rim of a bicycle wheel with hub  $\{x, y\}$ .  $\square$

**Proposition 3.7.** *Let  $\Gamma$  be an incomplete, triangle-free graph with no separating clique. Suppose  $\Gamma$  has an uncrossed cut  $\binom{a}{b}$ . Let  $\Gamma_i$  be the components of  $\Gamma - \binom{a}{b}$ , and let  $\bar{\Gamma}_i := \Gamma_i \cup \binom{a}{b}$ . Then  $\Gamma$  admits a FIDL- $\Lambda$  if and only if each  $\bar{\Gamma}_i$  admits a FIDL- $\Lambda_i$  that contains an edge  $a \leftrightarrow b$ .*

*Proof.* The ‘only if’ direction follows easily from Lemma 3.2 and Corollary 3.4.

For the converse, suppose  $\Lambda_{\Gamma_i} := \Lambda_{\Gamma_i, \mathbf{r}} \cup \Lambda_{\Gamma_i, \mathbf{b}}$  contains an edge  $a \leftrightarrow b$  in  $\Lambda_{\Gamma_i, \mathbf{r}}$  for all  $i$ . This is the only intersection of any of the  $\mathbf{r}$  trees, since  $\binom{a}{b} = \cap_i \bar{\Gamma}_i$ , so  $\Lambda_{\mathbf{r}} := \cup_i \Lambda_{\Gamma_i, \mathbf{r}}$  is a tree.

For  $\Lambda_{\mathbf{b}}$  there are two cases. If  $\binom{a}{b} = \{a, b\}$  is a cut pair then the  $\Lambda_{\Gamma_i, \mathbf{b}}$  are disjoint. For each  $i$ , choose  $c_i \in \Gamma_i \cap \text{lk}(a) \cap \text{lk}(b)$ , which exists by Lemma 3.1. Define  $\Lambda_{\mathbf{b}}$  as the union of the  $\Lambda_{\Gamma_i, \mathbf{b}}$ , together with a tree spanning the  $c_i$ . If  $\binom{a}{b} = \{a, b, c\}$  is a 2-path cut triple then  $c$  is a vertex in all  $\Lambda_{\Gamma_i, \mathbf{b}}$ , so  $\Lambda_{\mathbf{b}} := \cup_i \Lambda_{\Gamma_i, \mathbf{b}}$  is a tree. In this case, set each  $c_i := c$ , so that we can write a common argument.

Let  $\Lambda := \Lambda_{\mathbf{r}} \sqcup \Lambda_{\mathbf{b}}$ . Condition  $\mathcal{R}_1$  we arranged by constructing trees. Conditions  $\mathcal{R}_2$ , and  $\mathcal{F}_1$  are immediate. We check conditions  $\mathcal{R}_3$  and  $\mathcal{R}_4$ . The only interesting cases are squares and cycles that include vertices from different components of  $\Gamma - \binom{a}{b}$ .

Suppose  $x_1 \in \Gamma_1$  and  $x_2 \in \Gamma_2$  are the diagonals of a square. Since  $\binom{a}{b}$  is a cut, the square is  $\{x_1, x_2\} * \{a, b\}$ . Since the only  $\Lambda_{\mathbf{b}}$  connection between  $\Gamma_1$  and  $\Gamma_2$  is through  $c_1$  and  $c_2$ , we have:

$$\text{hull}_\Lambda \{a, b\} * \text{hull}_\Lambda \{x_1, x_2\} = \{a, b\} * (\text{hull}_{\Lambda_{\Gamma_1, \mathbf{b}}} \{x_1, c_1\} \cup \text{hull}_{\Lambda_{\Gamma_2, \mathbf{b}}} \{c_2, x_2\} \cup \text{hull}_{\Lambda_{\mathbf{b}}} \{c_1, c_2\})$$

This join is indeed contained in  $\Gamma$ , because  $x_i, c_i \in \text{lk}_{\bar{\Gamma}_i}(a) \cap \text{lk}_{\bar{\Gamma}_i}(b)$ , and by Corollary 3.5,  $\text{lk}_{\bar{\Gamma}_i}(a) \cap \text{lk}_{\bar{\Gamma}_i}(b)$  is  $\Lambda_{\bar{\Gamma}_i}$ -convex, so all of  $\text{hull}_{\Lambda_{\Gamma_i, \mathbf{b}}} \{x_i, c_i\}$  are common neighbors of  $a$  and  $b$ , whereas  $\text{hull}_{\Lambda_{\mathbf{b}}} \{c_1, c_2\} \subset \text{lk}(a) \cap \text{lk}(b)$  by construction. Thus,  $\mathcal{R}_3$  is satisfied.

If  $\mathcal{R}_4$  is violated there is a shortest cycle  $\gamma$  containing an edge  $e$  that is not included in a square with vertices in  $\text{hull}_\Lambda(\gamma)$ . We may assume such a cycle is induced, since if it has vertices that are adjacent in  $\Gamma$  and not in  $\gamma$  we could use that edge to cut  $\gamma$  into two strictly shorter cycles whose union contains all the edges of  $\gamma$ . In particular,  $e$  is in one of these shorter cycles,  $\gamma'$ . But  $\text{hull}_\Lambda \gamma' \subset \text{hull}_\Lambda \gamma$ , so  $e$  is not included in a square with vertices in  $\text{hull}_\Lambda \gamma'$ , contradicting that  $\gamma$  was a shortest counterexample.

So, consider an induced cycle  $\gamma$  that contains, without loss of generality, vertices from  $\Gamma_1$  and  $\Gamma_2$ . Since the cycle crosses the cut, it contains  $a$  and  $b$ . Since it is induced, either  $\gamma \cap \bar{\Gamma}_1 = a \leftrightarrow c_1 \leftrightarrow b$ , or  $c_1 \notin \gamma$ . In the first case, both edges  $a \leftrightarrow c_1$  and  $c_1 \leftrightarrow b$  are in the square  $\{c_1, c_2\} * \{a, b\}$ , and  $c_2 \in \text{hull}_\Lambda(\gamma)$  since the only  $\Lambda_{\mathbf{b}}$ -connection between  $\Gamma_1$  and  $\Gamma_2$  is through  $c_2$ . On the other hand, if  $c_1 \notin \gamma$  then there is a cycle  $\gamma'$  made from  $\gamma \cap \bar{\Gamma}_1$  and the segment  $a \leftrightarrow c_1 \leftrightarrow b$  that is completely contained in  $\bar{\Gamma}_1$ , so its edges all participate in squares with vertices in  $\text{hull}_{\Lambda_{\Gamma_1}}(\gamma')$ , which is contained in  $\text{hull}_{\Lambda_\Gamma}(\gamma)$ , since the vertex  $c_1 = \gamma' - \gamma$  is in the  $\Lambda_{\mathbf{b}}$ -hull of  $\gamma$ , being the only  $\Lambda_{\mathbf{b}}$ -connection between  $\bar{\Gamma}_1$  and  $\bar{\Gamma}_2$ . This shows that all  $\bar{\Gamma}_1$ -edges of  $\gamma$  are contained in a square with vertices in  $\text{hull}_\Lambda(\gamma)$ .  $\square$

#### 4. SATELLITE-DISMANTLABILITY AND THE CONING ALGORITHM

In this section we take a graph with a FIDL- $\Lambda$  apart and then put it back together again.

**Theorem 4.1.** *Let  $\Gamma$  be an incomplete, triangle-free graph with no separating clique that admits a FIDL- $\Lambda$ . Then  $\Gamma$  is satellite-dismantlable to a square through a sequence of graphs  $\Gamma_0 = \Gamma \supset \Gamma_1 \supset \dots \supset \Gamma_n$  such that for all  $i$ ,  $\Gamma_i$  is incomplete, triangle-free with no separating clique and has  $\Lambda \cap \Gamma_i$  as a FIDL- $\Lambda$ , with the satellite vertex  $\Gamma_i - \Gamma_{i+1}$  being a leaf in  $\Lambda \cap \Gamma_i$ .*

*Proof.* Our goal is to find a leaf  $v$  of  $\Lambda$  such that  $v$  is a satellite in  $\Gamma$  and  $\Gamma_1 := \Gamma - \{v\}$  has the desired properties. The proof is easy if  $\Gamma$  is a suspension, so assume not.

Identify  $\Delta$  with its image in  $\square(\Gamma)$ . This graph is connected, since  $W_\Gamma$  and  $A_\Delta$  are one-ended. We claim that every vertex of  $\square(\Gamma)$  is adjacent to a vertex of  $\Delta$ . Suppose that  $\{a, b\} \in \square(\Gamma) - \Delta$ . By definition,  $\{a, b\}$  is a diagonal of some square, so  $a$  and  $b$  have at least two common neighbors. By Corollary 3.5,  $\text{lk}(a) \cap \text{lk}(b)$  is  $\Lambda$ -convex, so it contains a subtree of  $\Lambda$  with at least one edge. Any such edge gives a vertex of  $\Delta$  adjacent to  $\{a, b\}$  in  $\square(\Gamma)$ . Note that this shows  $\Gamma$  is strongly CFS.

The vertex  $v$  is a leaf of  $\Lambda$  if and only if it has a unique neighbor  $v'$  in  $\Lambda$ . Equivalently,  $\{v, v'\}$  is the unique vertex of  $\Delta$  containing  $v$ . We claim that  $v$  is a satellite of  $v'$ . Let  $w \in \text{lk}(v)$ . Since  $\{v, v'\}$  is the diagonal of some square,  $|\text{lk}(v) \cap \text{lk}(v')| \geq 2$ , so there exists  $c \in (\text{lk}(v) \cap \text{lk}(v')) - \{w\}$ . Since the edge  $v \leftrightarrow c$  is not a separating clique, there is a path from  $v'$  to  $w$  that does not go through  $v$  or  $c$ , so there is a cycle  $\gamma$  that goes from  $v$  to  $c$  to  $v'$  to  $w$  and then back across the edge between  $v$  and  $w$ . By  $\mathcal{R}_4$  there is a square  $\{v, x\} * \{w, u\}$  with  $x, u \in \text{hull}_\Lambda(\gamma)$ . But then  $\mathcal{R}_3$  implies  $\{w\} * \text{hull}_\Lambda\{v, x\} \subset \Gamma$ . Since  $v$  was a  $\Lambda$ -leaf,  $v' \in \text{hull}_\Lambda\{v, x\}$ , so  $w$  is adjacent to  $v'$ . Thus,  $\text{lk}(v) \subset \text{lk}(v')$ .

The claim that  $\Lambda - \{v\}$  is a FIDL- $\Lambda$  is automatic, as all of the defining properties are inherited from  $\Lambda$ . The key for this is that since only a leaf of  $\Lambda$  was deleted, convex sets in  $\Lambda - \{v\}$  are still convex in  $\Lambda$ .

We must show it is possible to choose a leaf of  $\Lambda$  such that  $\Gamma - \{v\}$  is not separated by a clique. First, let  $v$  be a  $\Lambda$ -leaf and  $\{v, v'\}$  the unique vertex of  $\Delta$  containing  $v$ , and suppose that  $\{v, v'\}$  is not a cut vertex of  $\Delta$ . Then  $\square(\Gamma) - \{v, v'\}$  consists of leaves of  $\square(\Gamma)$  that were connected only to  $\{v, v'\}$ , plus one additional component  $\Omega$  containing  $\Delta - \{v, v'\}$ . The reason for this is that  $\Gamma$  being triangle-free implies  $\square(\Gamma)$  is triangle-free, so if  $\{a, b\}$  is adjacent to  $\{v, v'\}$ , but is not a leaf of  $\square(\Gamma)$ , then it is also adjacent to some other vertex  $\{c, d\}$ , which is adjacent to a vertex of  $\Delta$ , but not to  $\{v, v'\}$ .

We further note that  $\Omega$  has support  $\Gamma - \{v\}$ . Since  $\Gamma$  is not a suspension,  $v'$  is not a  $\Lambda$ -leaf, so it is contained in at least one other vertex of  $\Delta$ , so  $v' \in \text{supp}(\Omega)$ . For other vertices, we only worry about those appearing in a leaf  $\{a, b\}$  of  $\square(\Gamma)$  connected to  $\{v, v'\}$ . Since  $\{v, v'\}$  is not a cut vertex of  $\Delta$ ,  $\{a, b\} \notin \Delta$ . But  $\Lambda$  spans  $\Gamma$ , so there is some vertex of  $\Delta$  with  $a$  in its support, and likewise for  $b$ .

Now consider  $\square(\Gamma - \{v\})$ . If  $\{a, b\}$  is a leaf of  $\square(\Gamma)$  connected only to  $\{v, v'\}$  then the only square of  $\Gamma$  with  $\{a, b\}$  as a diagonal is  $\{a, b\} * \{v, v'\}$ , so  $\{a, b\}$  is not the diagonal of a square in  $\Gamma - \{v\}$  and does not appear as a vertex of  $\square(\Gamma - \{v\})$ . Since  $\{v, v'\}$  was the unique vertex of  $\square(\Gamma)$  containing  $v$ ,  $\square(\Gamma - \{v\}) = \Omega$ , which is connected, by hypothesis, and has full support in  $\Gamma - \{v\}$ . Thus,  $\Gamma - \{v\}$  is strongly CFS, and Lemma 2.5 says  $\Gamma - \{v\}$  has no separating clique.

Now we argue that there always exists a  $\Lambda$ -leaf not giving a cut vertex of  $\Delta$ , so we can find it and apply the previous argument. Suppose the first chosen  $\Lambda$ -leaf  $v$  does give a cut vertex  $\{v, v'\}$  of  $\Delta$ . This implies that  $\Gamma$  has a cut  $\binom{v}{v'}$ , since  $W_\Gamma$  and  $A_\Delta$  split over a two-ended subgroup commensurable to  $\langle vv' \rangle$ . Let  $v_0 := v$  and  $v'_0 := v'$ . By Corollary 3.4, for each component  $\Gamma'$  of  $\Gamma - \binom{v_0}{v'_0}$ , the intersection of each component of  $\Lambda$  with  $\bar{\Gamma}'$  is  $\Lambda$ -convex, so  $\Gamma'$  contains at least one  $\Lambda$ -leaf  $v_1$ . If  $v_1$  uniquely appears as  $\{v_1, v'_1\}$  in  $\Delta$  and  $\{v_1, v'_1\}$  is not a cut vertex of  $\Delta$ , we are done. Otherwise,  $\binom{v_1}{v'_1}$  is a cut of  $\Gamma$ . Since  $\Gamma$  has no crossing cuts,  $\binom{v_0}{v'_0} - \binom{v_1}{v'_1}$  is a non-empty set contained in a single component of  $\Gamma - \binom{v_1}{v'_1}$ . Choose a component  $\Gamma''$  of  $\Gamma - \binom{v_1}{v'_1}$  not containing  $\binom{v_0}{v'_0} - \binom{v_1}{v'_1}$  and repeat, always choosing a complementary component of the most recent cut that does not contain the previous cuts, so that the size of the components strictly decreases at each step. If  $v_{i+1}$  is the lone vertex in its component of  $\Gamma - \binom{v_i}{v'_i}$  then it cannot be part of a cut, since the cut would cross  $\binom{v_i}{v'_i}$ , so eventually this process produces a  $\Lambda$ -leaf that does not appear in a cut vertex of  $\Delta$ .  $\square$

**Corollary 4.2.** *If  $\Gamma$  is an incomplete, triangle-free graph with no separating clique and no satellite vertex then  $\Gamma$  does not admit a FIDL- $\Lambda$ .*

**Corollary 4.3.** *If  $\Gamma$  is an incomplete, triangle-free graph with no separating clique and it admits a FIDL- $\Lambda$  then it is strongly CFS.*

*Coning Algorithm 4.4.* We perform the following inductive procedure to build a graph  $\Gamma$  with an associated graph  $\Lambda \leq \Gamma^c$  with two connected components:

- (1) The initial graph  $\Gamma_0$  is a square, the associated graph  $\Lambda_0 = \Gamma_0^c$  is the complement graph of  $\Gamma_0$ .
- (2) Build a sequence of pairs  $((\Gamma_0, \Lambda_0), (\Gamma_1, \Lambda_1), \dots, (\Gamma_n, \Lambda_n))$  by applying the induction step: Given  $(\Gamma_i, \Lambda_i)$ , pick a vertex  $v_i$  and a set  $N_i \subseteq \text{lk}_{\Gamma_i}(v_i)$  of at least two vertices that is  $\Lambda_i$ -convex. Define  $\Gamma_{i+1}$  by coning-off  $N_i$  to  $x_{i+1}$ , and define  $\Lambda_{i+1}$  by adding an edge from  $v_i$  to  $x_{i+1}$  to  $\Lambda_i$ .

Note that at each step,  $x_{i+1}$  is a satellite of  $v_i$  in  $\Gamma_{i+1}$ .

**Theorem 4.5.** *If the pair  $(\Gamma, \Lambda)$  can be constructed by the Coning Algorithm 4.4, then  $\Gamma$  is incomplete, triangle-free and has no separating cliques, and  $\Lambda$  is a FIDL- $\Lambda$  for  $\Gamma$ .*

*Proof.* The proof is by induction on  $n$ .  $\Theta(\Gamma_0, \Lambda_0)$  satisfies all the conditions. Assume  $\Theta(\Gamma_i, \Lambda_i)$  does too. We show that  $\Theta(\Gamma_{i+1}, \Lambda_{i+1})$  satisfies  $\mathcal{R}_3$  and  $\mathcal{R}_4$ . The other conditions are easy to verify.

Since  $\Theta(\Gamma_i, \Lambda_i)$  satisfies condition  $\mathcal{R}_3$ , we only need to check squares in  $\Gamma_{i+1}$  containing the new vertex  $x_{i+1}$ . By construction of  $\Lambda_{i+1}$ , any such square is of the form  $\{x_{i+1}, l\} * \{n, n'\}$  with  $n, n' \in N_i$ , and, since  $N_i = \text{lk}_{\Gamma_{i+1}}(x_{i+1})$  is  $\Lambda_i$ -convex,  $\{x_{i+1}\} * \text{hull}_{\Lambda_{i+1}}\{n, n'\} \subset \Gamma_{i+1}$ .

The vertex  $x_{i+1}$  is only connected to  $v_i$  in  $\Lambda_{i+1}$ . Hence,  $\text{hull}_{\Lambda_{i+1}}\{x_{i+1}, l\} - \{x_{i+1}\} = \text{hull}_{\Lambda_{i+1}}\{v_i, l\} = \text{hull}_{\Lambda_i}\{v_i, l\}$ . Since  $\Gamma_i$  is triangle-free,  $v_i$  and  $l$  are not adjacent, so  $\{v_i, l\} * \{n, n'\} \subseteq \Gamma_i$  is a square. Condition  $\mathcal{R}_3$  for  $\Theta(\Gamma_i, \Lambda_i)$  implies  $\text{hull}_{\Lambda_i}\{v_i, l\} * \text{hull}_{\Lambda_i}\{n, n'\} \subset \Gamma_i$ . Condition  $\mathcal{R}_3$  is satisfied, since:

$$\text{hull}_{\Lambda_{i+1}}\{x_{i+1}, l\} * \text{hull}_{\Lambda_{i+1}}\{n, n'\} = \{x_{i+1}\} * \text{hull}_{\Lambda_i}\{n, n'\} \cup \text{hull}_{\Lambda_i}\{v_i, l\} * \text{hull}_{\Lambda_i}\{n, n'\} \subset \Gamma_{i+1}$$

Since  $\Theta(\Gamma_i, \Lambda_i)$  satisfies condition  $\mathcal{R}_4$ , we only need to check cycles containing  $x_{i+1}$ . Let  $\gamma$  be a cycle containing  $x_{i+1}$ . Since  $\text{lk}_{\Gamma_{i+1}}(x_{i+1}) = N_i$ ,  $\gamma$  is of the form  $\gamma = (x_{i+1}, n, l_1, \dots, l_k, n')$  for  $n, n' \in N_i$ . The edges incident to  $x_{i+1}$  are contained in the square  $\{x_{i+1}, v_i\} * \{n, n'\}$ , all of whose vertices are in  $\gamma$ , except possibly  $v_i$ . But  $x_{i+1}$  is a leaf of  $\Lambda_{i+1}$  connected only to  $v_i$ , so  $v_i$  is certainly in  $\text{hull}_{\Lambda_{i+1}}(\gamma)$ . For the remaining edges of  $\gamma$ , replace  $\gamma$  by  $\gamma' = (v_i, n, l_1, \dots, l_k, n')$ , which is a loop in  $\Gamma_i$  that can be split into at most two cycles in  $\Gamma_i$ . Condition  $\mathcal{R}_4$  for  $\Theta(\Gamma_i, \Lambda_i)$  implies each of these edges belongs to a square with vertices in the  $\Lambda_i$ -hull of its cycle, which is a subset of the  $\Lambda_{i+1}$ -hull of  $\gamma$ .  $\square$

*Example 4.6.* The graph  $\Gamma$  of Example 2.6 that is the 1-skeleton of a 3-cube with one space diagonal has a FIDL- $\Lambda$ . The pair  $(\Gamma, \Lambda)$  can be constructed by the Coning Algorithm 4.4. A coning sequence  $(x_i, \Gamma_i, \Lambda_i, v_i, N_i)$  is illustrated in Figure 4.1.

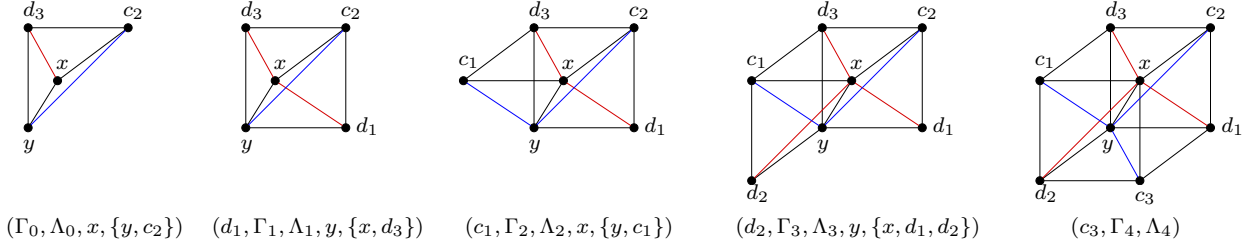


FIGURE 4.1. Example of a Coning Sequence

**Theorem 4.7.** *Let  $\Gamma$  be an incomplete, triangle-free graph without separating cliques.  $\Gamma$  admits a FIDL- $\Lambda$  if and only if it admits a satellite-dismantling sequence  $\Gamma = \Gamma_n \supset \Gamma_{n-1} \supset \dots \supset \Gamma_0$  such that  $\Gamma_0$  is a square, each  $\Gamma_i$  is incomplete and triangle-free with no separating clique, and the following condition is satisfied: For  $0 \leq i < n$  let  $\{x_{i+1}\} := \Gamma_{i+1} - \Gamma_i$  and  $N_i := \text{lk}_{\Gamma_{i+1}}(x_{i+1})$ .*

( $\dagger$ )  $\forall i < n, \exists v_i \in V_i := \{v \in \Gamma_i \mid N_i \subset \text{lk}_{\Gamma_i}(v)\}, \forall j > i, \text{ if } x_{i+1} \in N_j \text{ and } N_j \cap \Gamma_i \neq \emptyset \text{ then } v_i \in N_j$

*Proof.* Suppose we have a satellite-dismantling sequence for  $\Gamma$  satisfying the given conditions. Let  $\Lambda_0 := \Gamma_0^c$ , and construct  $\Lambda_{i+1}$  from  $\Lambda_i$  by choosing  $v_i$  satisfying ( $\dagger$ ), and adding a  $\Lambda$ -edge from  $v_i$  to  $x_{i+1}$ . To apply Theorem 4.5 we must verify that  $N_j$  is  $\Lambda_j$ -convex, for each  $j$ . Given  $j$ , let  $k_j \geq 0$  be the minimum index such that  $N_j \cap \Gamma_{k_j} \neq \emptyset$ . The vertices of  $N_j$  are pairwise at distance 2 from one another, so they all have the same color; assume it is  $\mathbf{b}$ . If  $n \in N_j$  then either  $n \in \Gamma_{k_j}$  or  $n = x_{i+1}$  for some  $k_j \leq i < j$ . In the second case, by ( $\dagger$ ),  $n$  is  $\Lambda_j$ -adjacent to  $v_i \in N_j$ , which is in a lower stratum. In this way, ( $\dagger$ ) implies that every vertex in  $N_j$  can be  $\Lambda_j$ -connected through vertices in  $N_j$  to a vertex in  $N_j \cap \Gamma_{k_j}$ . If  $k_j > 0$  then  $\Gamma_{k_j} - \Gamma_{k_j-1}$  is a single vertex, and if  $k_j = 0$  then the two vertices  $\Lambda_{0,\mathbf{b}}$  are connected by an edge. Thus,  $N_j$  is  $\Lambda_j$ -convex.

Conversely, if  $\Gamma$  admits a FIDL- $\Lambda$  then, by Theorem 4.1, it admits a satellite-dismantling sequence leading to a square through graphs with the desired properties. For each satellite  $x_{i+1}$  there is a unique  $\Lambda_{i+1}$ -edge at  $x_{i+1}$  connecting it to a  $v_i \in V_i$ . We check that this  $v_i$  satisfies ( $\dagger$ ).

Suppose for some  $j$  there is an  $i$  such that  $x_{i+1} \in N_j$  and there exists  $y \in \Gamma_i \cap N_j$ . Assume  $N_j \subset \Lambda_{\mathbf{r}}$ . Lemma 3.3 says  $N_j := \text{lk}_{\Gamma_{j+1}}(x_{j+1})$  is  $\Lambda_{j+1}$ -convex, but  $\text{hull}_{\Lambda_j}(N_j) = \text{hull}_{\Lambda_{j+1}}(N_j)$ , so  $N_j$  is  $\Lambda_j$ -convex. Now we have two points  $x_{i+1}$  and  $y$  that are contained in two different subtrees of  $\Lambda_{j,\mathbf{r}}$ : one is  $N_j \cap \Lambda_{j,\mathbf{r}}$



and the other is  $\Lambda_{i+1,r}$ . Thus, the  $\Lambda_{j,r}$ -geodesic from  $x_{i+1}$  to  $y$ , the  $N_j \cap \Lambda_{j,r}$ -geodesic from  $x_{i+1}$  to  $y$  and the  $\Lambda_{i+1,r}$ -geodesic from  $x_{i+1}$  to  $y$  coincide. We know that the  $\Lambda_{i+1,r}$ -geodesic from  $x_{i+1}$  to  $y$  goes through  $v_i$ , since  $x_{i+1}$  is a leaf of  $\Lambda_{i+1,r}$  connected only to  $v_i$ . Thus  $v_i \in N_j$ , and  $(\dagger)$  is satisfied.  $\square$

Here are a family of examples to illustrate Theorem 4.7.

**Proposition 4.8.** *Let  $T$  be a finite tree, with at least one edge, whose vertices are labelled by natural numbers  $n(v)$  such that 1 occurs only on leaves and if  $T$  is a single edge then both labels are greater than 1. Let  $\Gamma$  be the graph that has, for each  $v \in T$ ,  $n(v)$ -many vertices  $(v, 0), \dots, (v, n(v) - 1)$ . Connect  $(v, i)$  to  $(w, j)$  by an edge if and only if  $v$  is adjacent to  $w$  in  $T$ . Then  $\Gamma$  admits a FIDL- $\Lambda$ .*

*Proof.* We describe a satellite-dismantling and then say why it satisfies Theorem 4.7.

If  $v \in T$  with  $n(v) > 1$  then for  $i + 1 \leq n(v) - 1$ ,  $(v, i + 1)$  is a satellite of  $(v, i)$ , coning off  $\{(w, j) \mid w \in \text{lk}_T(v), 0 \leq j < n(w)\}$ . Pick any such  $v$  and iteratively remove  $(v, n(v) - 1), \dots, (v, 2)$ . Repeat for all  $v$  with  $n(v) > 2$ . This reduces us to the case that  $n$  is bounded by 2.

If  $\Gamma$  is not a square and there is a leaf  $v$  of  $T$  attached to  $w$  with  $n(v) = 1$  then there exists  $x \in T$  at distance 2 from  $v$ . In  $\Gamma$ ,  $(v, 0)$  is a satellite of  $(x, 0)$  coning-off  $\{(w, 0), (w, 1)\}$ . In this way, remove all  $T$ -leaves with  $n$ -value 1, until either we reach a square or have smaller  $\Gamma$  and  $T$  with  $n \equiv 2$ .

Suppose  $n \equiv 2$  and  $\Gamma$  is not a square, so  $T$  is not a single edge. If  $v$  is a leaf of  $T$  attached to  $w$  then  $(v, 1)$  is a satellite of  $(v, 0)$  coning-off  $\{(w, 0), (w, 1)\}$ . Thus, we reduce  $n$  to 1 on each of the leaves of  $T$ , and then rerun the case in the previous paragraph.

Iterating gives a satellite-dismantling reducing  $\Gamma$  to a square. Furthermore, in the dismantling we specified for each satellite which vertex we were considering it as the satellite of, which allows an explicit construction of  $\Lambda$ . It is observed that the coned-off set is  $\Lambda$ -convex at each step, so this is a FIDL.  $\square$

*Remark.* The proof relies on the fact that  $T$  is a tree because after reducing to  $n \leq 2$  the dismantling proceeds by removing  $T$ -leaves. Proposition 4.8 cannot be true for arbitrary graphs  $\Delta$ , as if  $\Delta$  has a long isometrically embedded loop then so does  $\Gamma$ , which prevents the existence of a FIDL- $\Lambda$ , by Proposition 3.6. However, if  $\Delta$  is an arbitrary connected graph and  $n \equiv 2$  then the  $\Gamma$  constructed from  $\Delta$  as above is commensurable to a RAAG, by [8].

*Example 4.9.* Figure 4.2 works out an example of a tree as in Proposition 4.8 in detail.

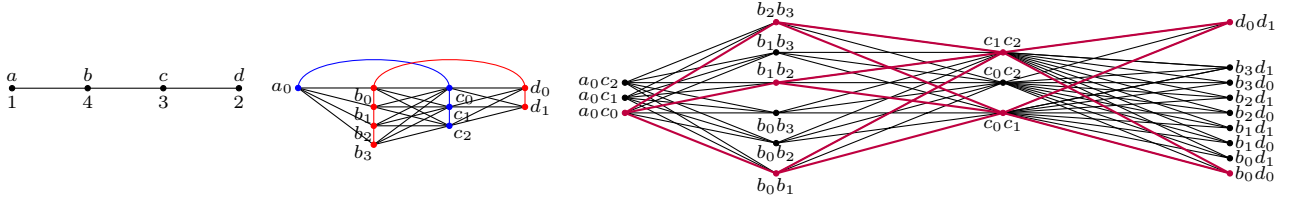


FIGURE 4.2. An example  $T$ ,  $\Theta(\Gamma, \Lambda)$ , and  $\Delta$  (purple) sitting in  $\square(\Gamma)$ .

## 5. A SEARCH ALGORITHM

**Definition 5.1.** A FIDL- $\Lambda$  for  $\Gamma$  relative to a collection of pairs of vertices  $\{\{p_0, q_0\}, \dots, \{p_n, q_n\}\}$  of  $\Gamma$  is a FIDL- $\Lambda$  that contains each of the pairs  $\{p_i, q_i\}$  as an edge of  $\Lambda$ .

*Relative Search Algorithm 5.2.* Given an incomplete, triangle-free graph  $\Gamma$  without separating cliques, find a FIDL- $\Lambda$  relative to  $\{\{p_0, q_0\}, \dots, \{p_n, q_n\}\}$  or decide that one does not exist as follows:

- (1) If  $\Gamma$  is not strongly  $\mathcal{CFS}$  stop; No FIDL- $\Lambda$  exists.
- (2) If there exists  $x_0, x_1, \dots, x_{n-1}$  such that for each  $i < n$  there exists  $j$  with  $\{x_i, x_{i+1}\} = \{p_j, q_j\}$ , then stop; No relative FIDL- $\Lambda$  exists.
- (3) Enumerate satellite-dismantling sequences reducing  $\Gamma$  to a square through graphs without separating cliques. If none exist, stop; No FIDL- $\Lambda$  exists.
- (4) For each such satellite-dismantling sequence, check, in the notation of Theorem 4.7, if Condition  $(\dagger)$  is satisfied. Moreover, if we assume for each  $k$  that for all  $i$ ,  $q_k \in \Gamma_i \implies p_k \in \Gamma_i$ , then we require for each  $k$  that either  $\{p_k, q_k\}$  is a diagonal of the square  $\Gamma_0$  or that for  $i$  such that  $q_k = x_{i+1}$ , we have that  $v_i = p_k$  is a choice for  $v_i \in V_i$  satisfying Condition  $(\dagger)$ .
- (5) If a suitable satellite-dismantling sequence is found, then Theorem 4.7 provides the relative FIDL- $\Lambda$ . Otherwise, no relative FIDL- $\Lambda$  exists.

*Proof.* Item (1) is Corollary 4.3. If the given  $\{\{p_0, q_0\}, \dots, \{p_n, q_n\}\}$  contains a cycle as described in (2), then the resulting  $\Lambda$  contains a cycle, violating  $\mathcal{R}_1$ . Item (3) is Theorem 4.1. Item (4) describes how it is possible to achieve the  $\{p_k, q_k\}$  as edges of FIDL- $\Lambda$  from the argument of Theorem 4.7.  $\square$

*Global Search Algorithm 5.3.* Given an incomplete, triangle-free graph  $\Gamma$  without separating cliques, find a FIDL- $\Lambda$  or decide that one does not exist as follows:

- (1) If  $\Gamma$  is not strongly  $\mathcal{CFS}$ , stop; No FIDL- $\Lambda$  exists.
- (2) Compute the JSJ graph of cylinders for  $W_\Gamma$  in terms of  $\Gamma$  as described in [10, Theorem 3.29] (recall Theorem 2.8). If it has a hanging vertex, stop;  $W_\Gamma$  is not even quasiisometric to a RAAG. If it has a rigid vertex such that the corresponding subgraph is not strongly  $\mathcal{CFS}$ , stop; No FIDL- $\Lambda$  exists.
- (3) For each subgraph  $\Gamma'$  corresponding to a rigid vertex of the graph of cylinders, use Relative Search Algorithm 5.2 to find a FIDL- $\Lambda$  for  $\Gamma'$  relative to the pairs  $\{p, q\} \subset \Gamma'$  such that  $\Gamma$  has a cut  $\binom{p}{q}$ . If this search fails for any rigid vertex, stop; No FIDL- $\Lambda$  exists.
- (4) If all rigid vertices have relative FIDL- $\Lambda$  then they can be assembled into a FIDL- $\Lambda$  of  $\Gamma$ .

*Proof.* Item (1) is Corollary 4.3.


[9, Proposition 4.43], shows that if  $\Gamma'$  is the subgraph corresponding to a rigid vertex, then it has no separating cliques and is  $\mathcal{CFS}$ . Iterating the cutting direction of Proposition 3.7 implies that if  $\Gamma$  has a FIDL- $\Lambda$  then its restriction to  $\Gamma'$  is a FIDL- $\Lambda$  for  $\Gamma'$  relative to the cuts. Together with Corollary 4.3, this explains the necessity of Items (2) and (3).

Torsion-generated groups do not surject  $\mathbb{Z}$ , so the underlying graph of the graph of cylinders is a tree. Iterating the assembly direction of Proposition 3.7 over the cuts combines the relative FIDL- $\Lambda$ 's of the subgraphs of the rigid vertices into a FIDL- $\Lambda$  for  $\Gamma$ , establishing Item (4).  $\square$

## REFERENCES


- [1] J. Behrstock and R. Charney, *Divergence and quasimorphisms of right-angled Artin groups*, Math. Ann. **352** (2012), no. 2, 339–356.
- [2] J. Behrstock, V. Falgas-Ravry, M. F. Hagen, and T. Susse, *Global structural properties of random graphs*, Int. Math. Res. Not. IMRN (2018), no. 5, 1411–1441.
- [3] C. H. Cashen and A. Martin, *Quasi-isometries between groups with two-ended splittings*, Math. Proc. Cambridge Philos. Soc. **162** (2017), no. 2, 249–291.
- [4] R. Charney, *An introduction to right-angled Artin groups*, Geom. Dedicata **125** (2007), 141–158.
- [5] P. Dani and I. Levcovitz, *Right-angled Artin subgroups of right-angled Coxeter and Artin groups*, Algebr. Geom. Topol. **24** (2024), no. 2, 755–802.
- [6] P. Dani and A. Thomas, *Divergence in right-angled Coxeter groups*, Trans. Amer. Math. Soc. **367** (2015), no. 5, 3549–3577.
- [7] M. W. Davis, *The geometry and topology of Coxeter groups*, London Mathematical Society Monographs Series, vol. 32, Princeton University Press, Princeton, NJ, 2008.
- [8] M. W. Davis and T. Januszkiewicz, *Right-angled Artin groups are commensurable with right-angled Coxeter groups*, J. Pure Appl. Algebra **153** (2000), no. 3, 229–235.
- [9] A. Edletzberger, *Quasi-Isometries for two-dimensional Right-Angled Coxeter Groups*, Phd Thesis, U. Vienna, 2024.
- [10] A. Edletzberger, *Quasi-isometries for certain right-angled Coxeter groups*, Groups Geom. Dyn. (in press).
- [11] V. Guirardel and G. Levitt, *JSJ decompositions of groups*, Astérisque (2017), no. 395, vii+165.
- [12] G. LaForge, *Visible Artin subgroups of right-angled Coxeter groups*, Ph.D. thesis, Tufts, 2017.
- [13] A. Margolis, *Quasi-isometry classification of right-angled Artin groups that split over cyclic subgroups*, Groups Geom. Dyn. **14** (2020), no. 4, 1351–1417.
- [14] M. Mihalik and S. Tschant, *Visual decompositions of Coxeter groups*, Groups Geom. Dyn. **3** (2009), no. 1, 173–198.
- [15] H. T. Nguyen and H. C. Tran, *On the coarse geometry of certain right-angled Coxeter groups*, Algebr. Geom. Topol. **19** (2019), no. 6, 3075–3118.

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