The status of the QI classification of RAAGs and RACGs

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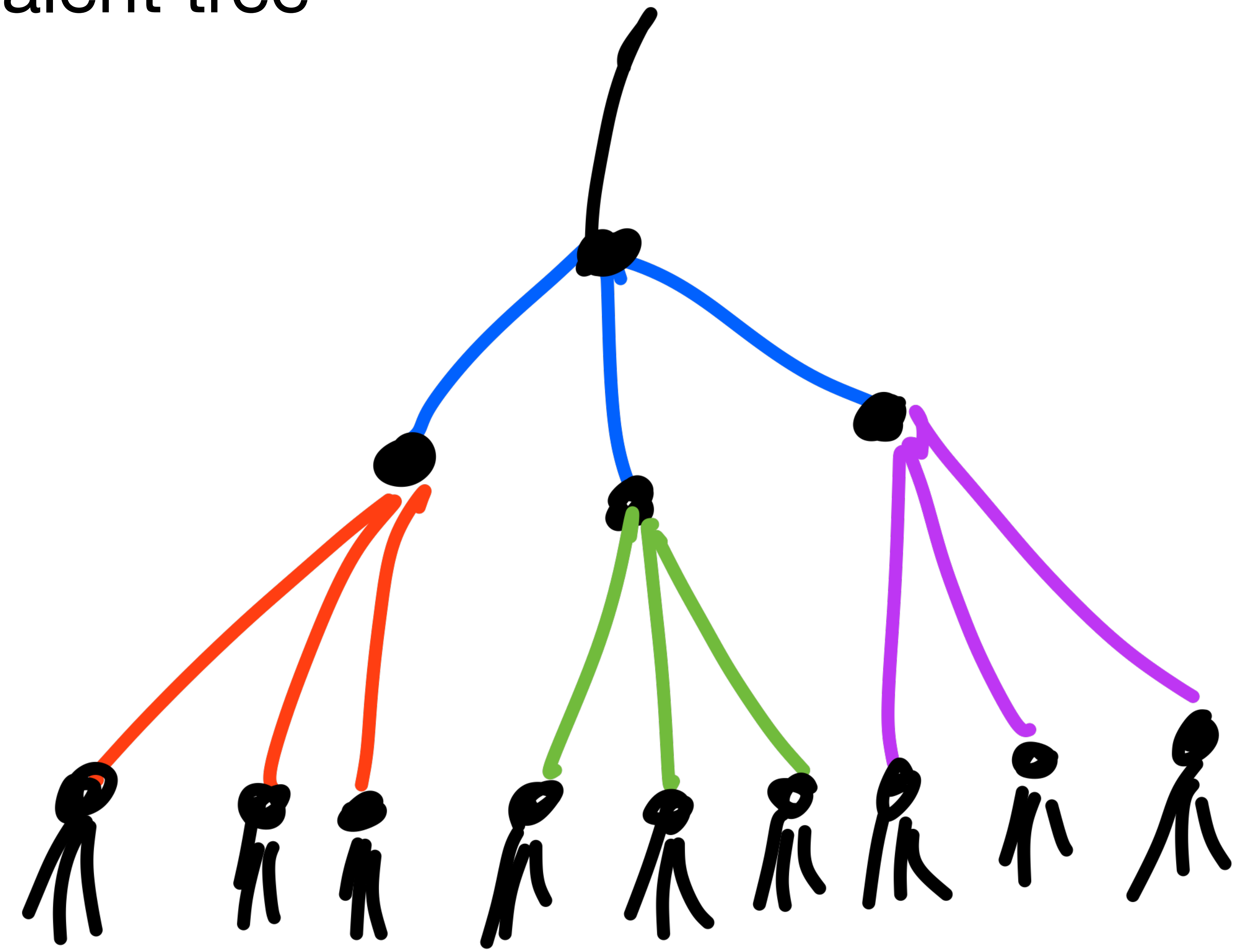
- Quasiisometries of finitely presented groups
- RAAGs
- RACGs

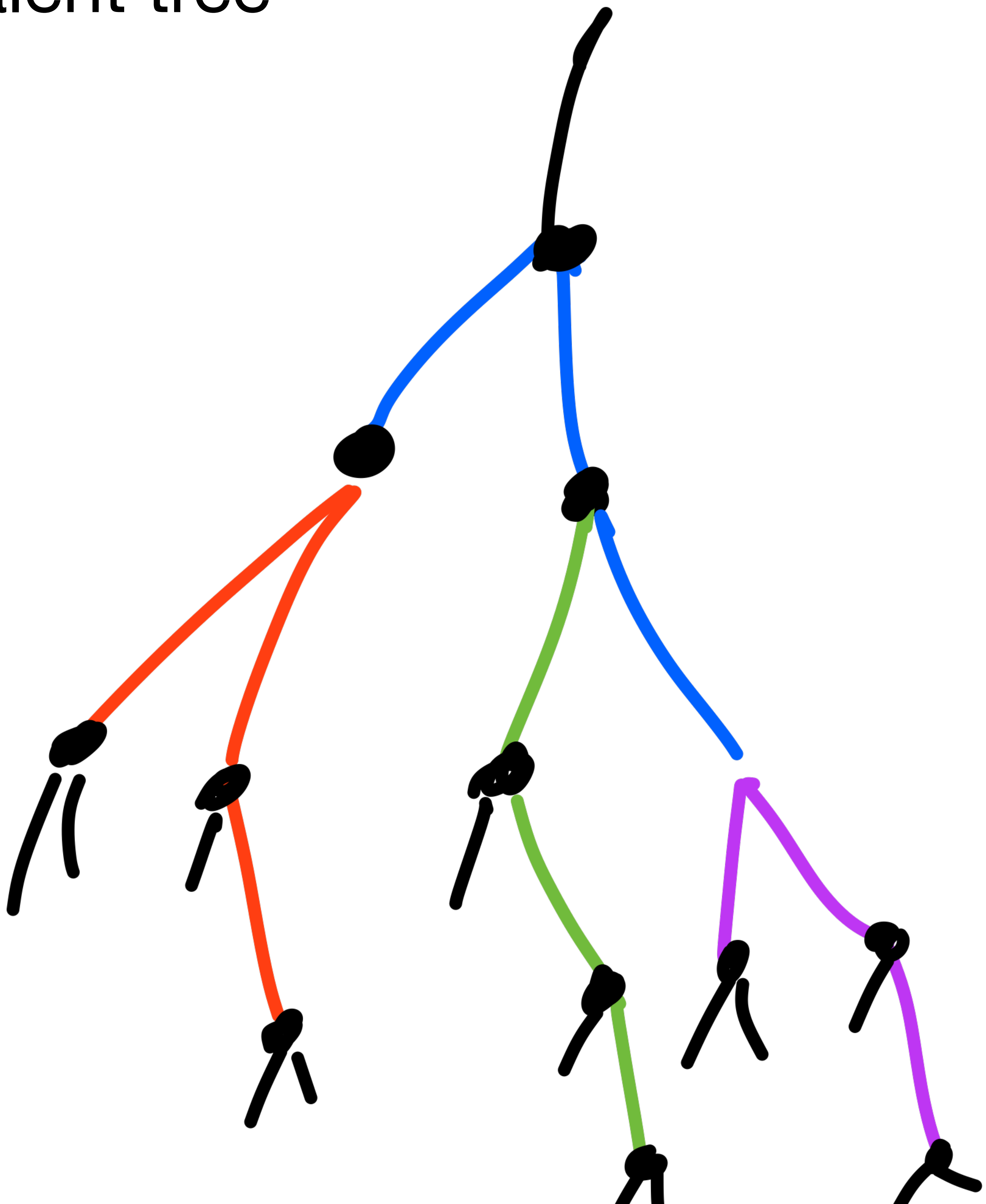
4-valent tree



A quasiisometry $T_4 \to T_3$.

- Number the down edges at each vertex 1, 2, 3.
- At each vertex, slide edge 3 across edge 2.
- At each vertex, if an edge slid in from above, slide edge 2 across edge 1.
- Colored trios stay connected; map is (3,0)-QI.

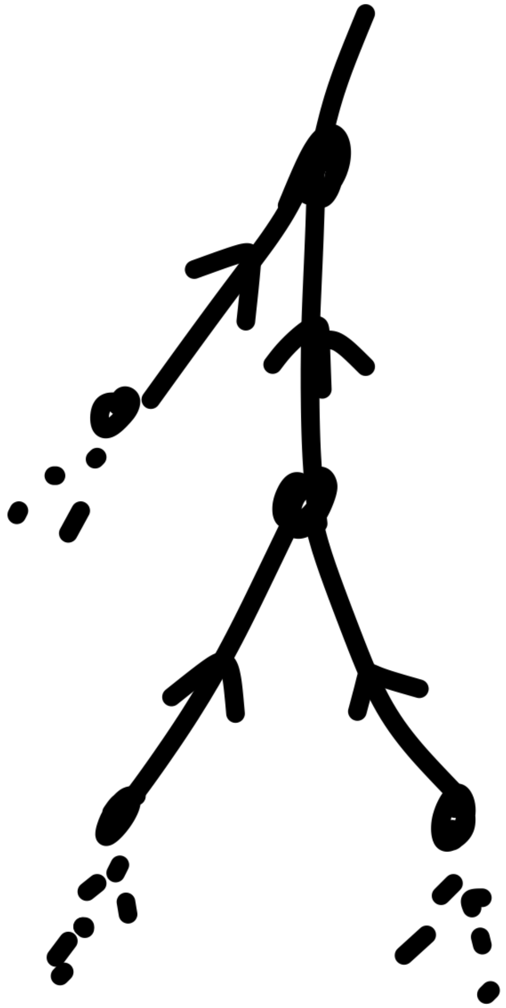
3-valent tree

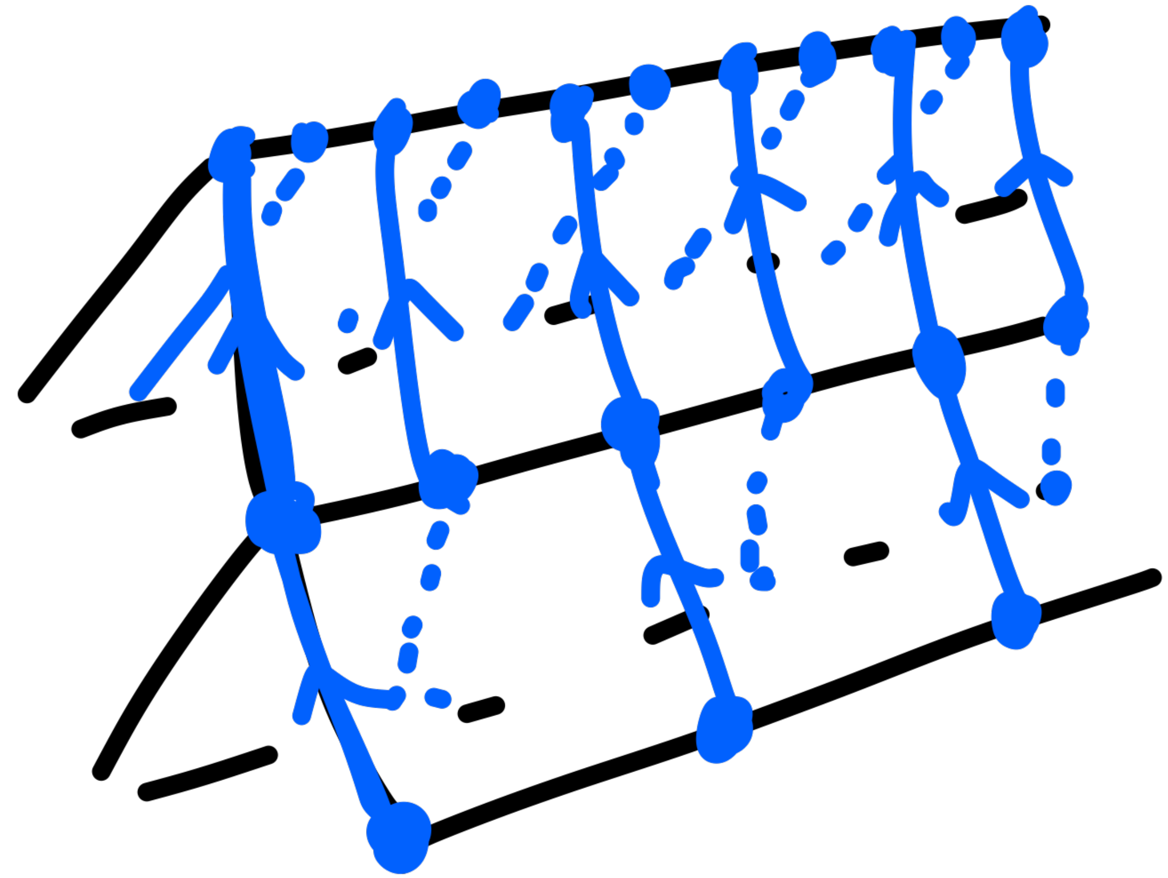


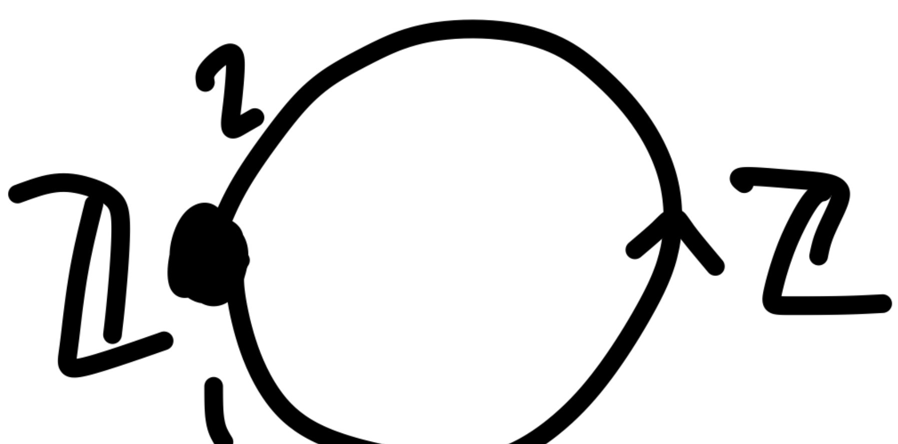
Corollary: Trees of valence at least 3 and bounded above are quasiisometric.

Corollary: Finite rank nonAbelian free groups are quasiisometric.

Bass-Serre Complex







Stallings: A group splits nontrivially over a finite group if and only if it has more than one end.

Algebraic splitting over finite $\iff$ geometric splitting over finite

Dunwoody: Finitely presented groups have maximal graph of groups decomposition over finite groups.

Vertex groups either finite or 1-ended, edge groups finite.

Papasoglu-Whyte: Set of QI types of 1-ended factors is a complete QI invariant for fp groups with infinitely many ends.

QI classification of (some) 1-ended fp groups

1-ended groups don't split over finite groups.

What about splitting over 2-ended groups?
Need analogues of:

Stallings (Geometric invariance of splittings.)

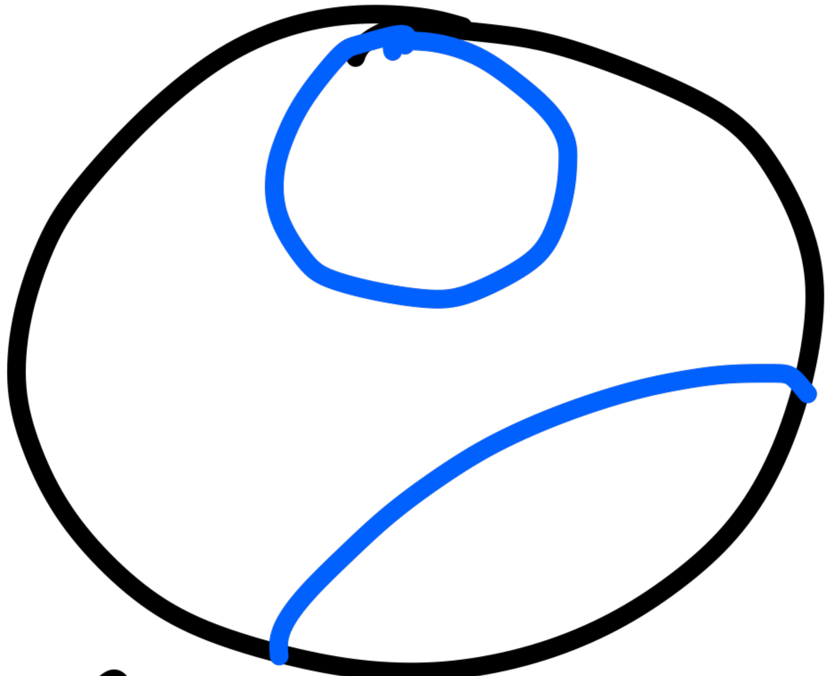
Dunwoody (Stop after finitely many splittings.)

Papasoglu-Whyte (QI classification in terms of resulting vertex groups.)

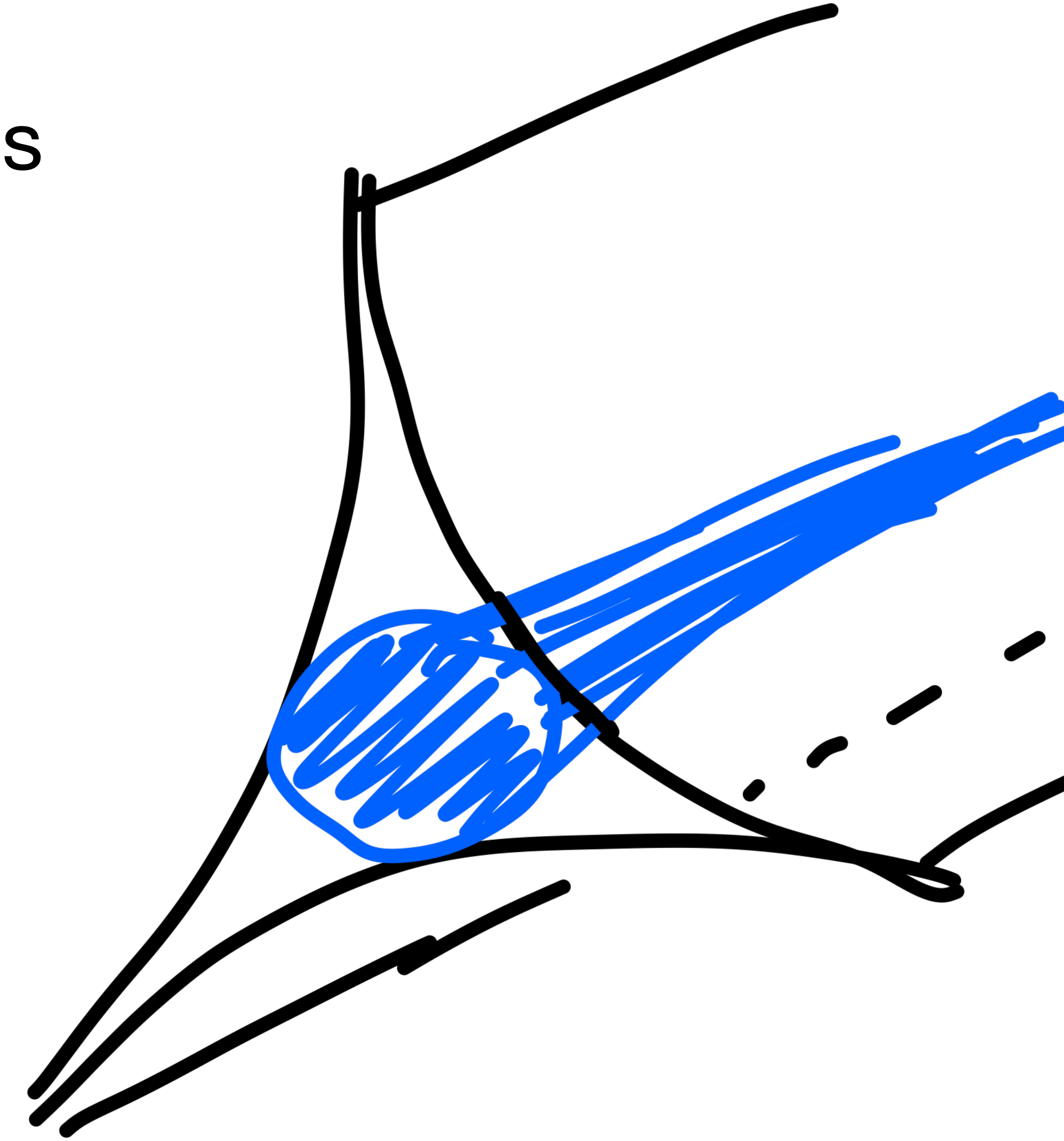
Papasoglu: If G is fp and not commensurable to a surface group then it splits over a 2-ended subgroup if and only if it has a separating quasiline.

quasiline: A path connected subset L with induced length metric d_L such that (L, d_L) is QI to $\mathbb{R}$ and there exist unbounded nondecreasing ρ_0, ρ_1 such that $\rho_0(d_L(a,b)) \leq d(a,b) \leq \rho_1(d_L(a,b))$ for all $a, b \in L$.

separating quasilines

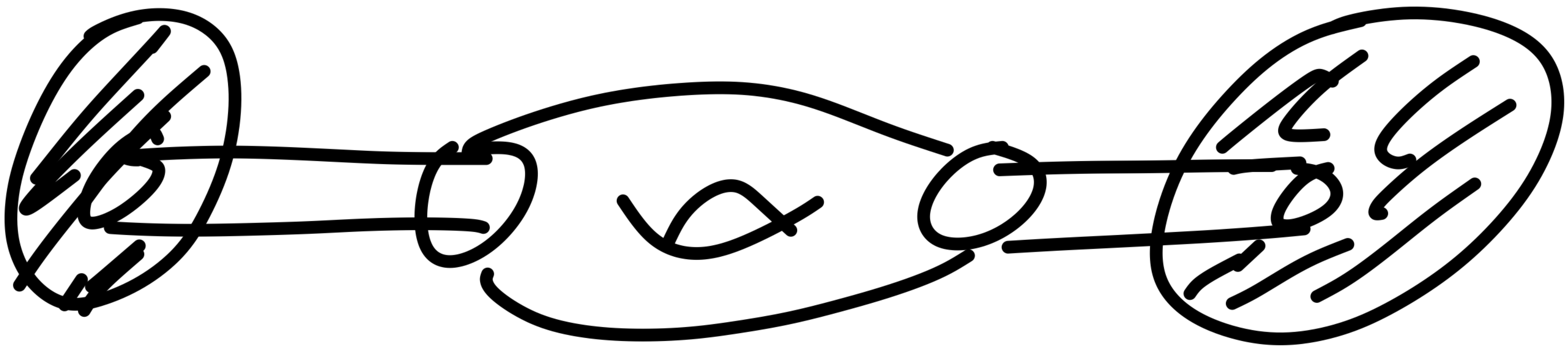


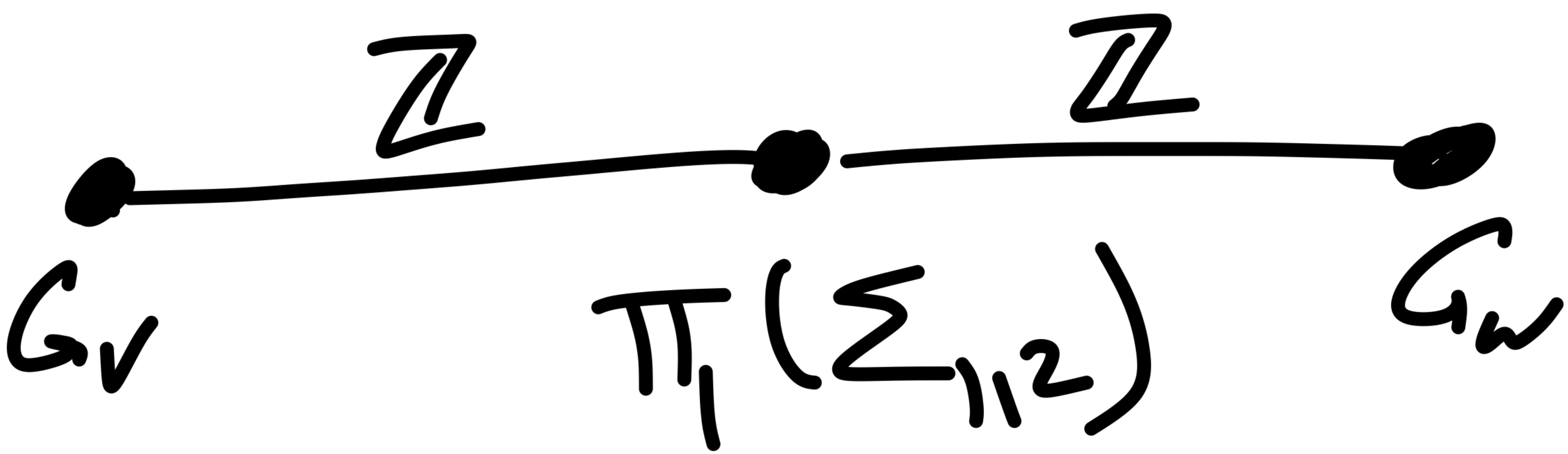
$\mathbb{H}^2$



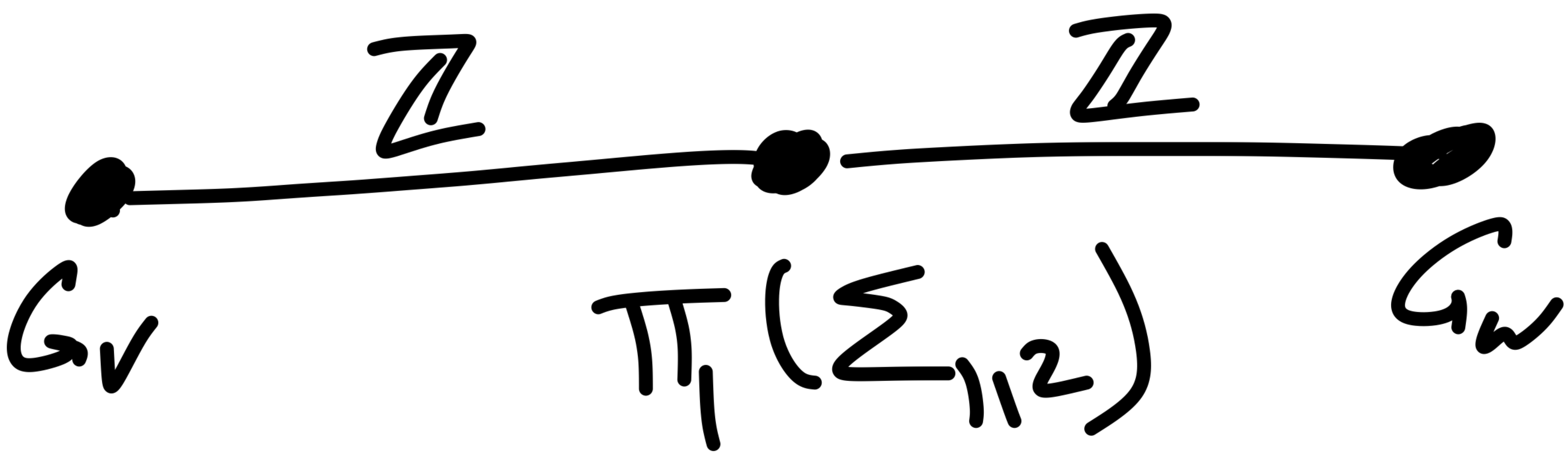
JSJ Decomposition: Splitting of G as a reduced graph of groups such that:

- Edge groups are 2-ended.
- Vertex groups are either 2-ended, hanging, or rigid.
- Maximality condition for hanging vertices.



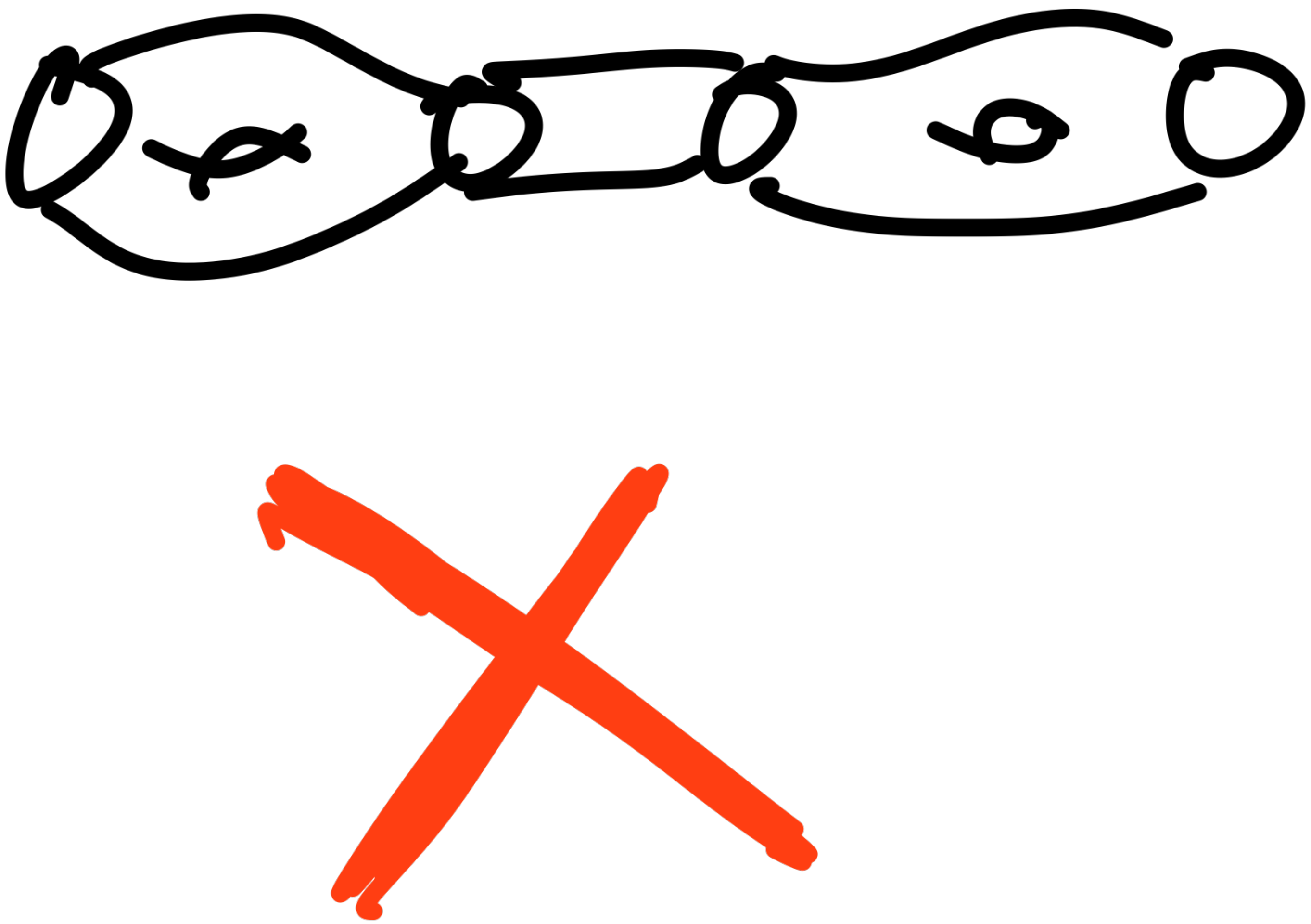


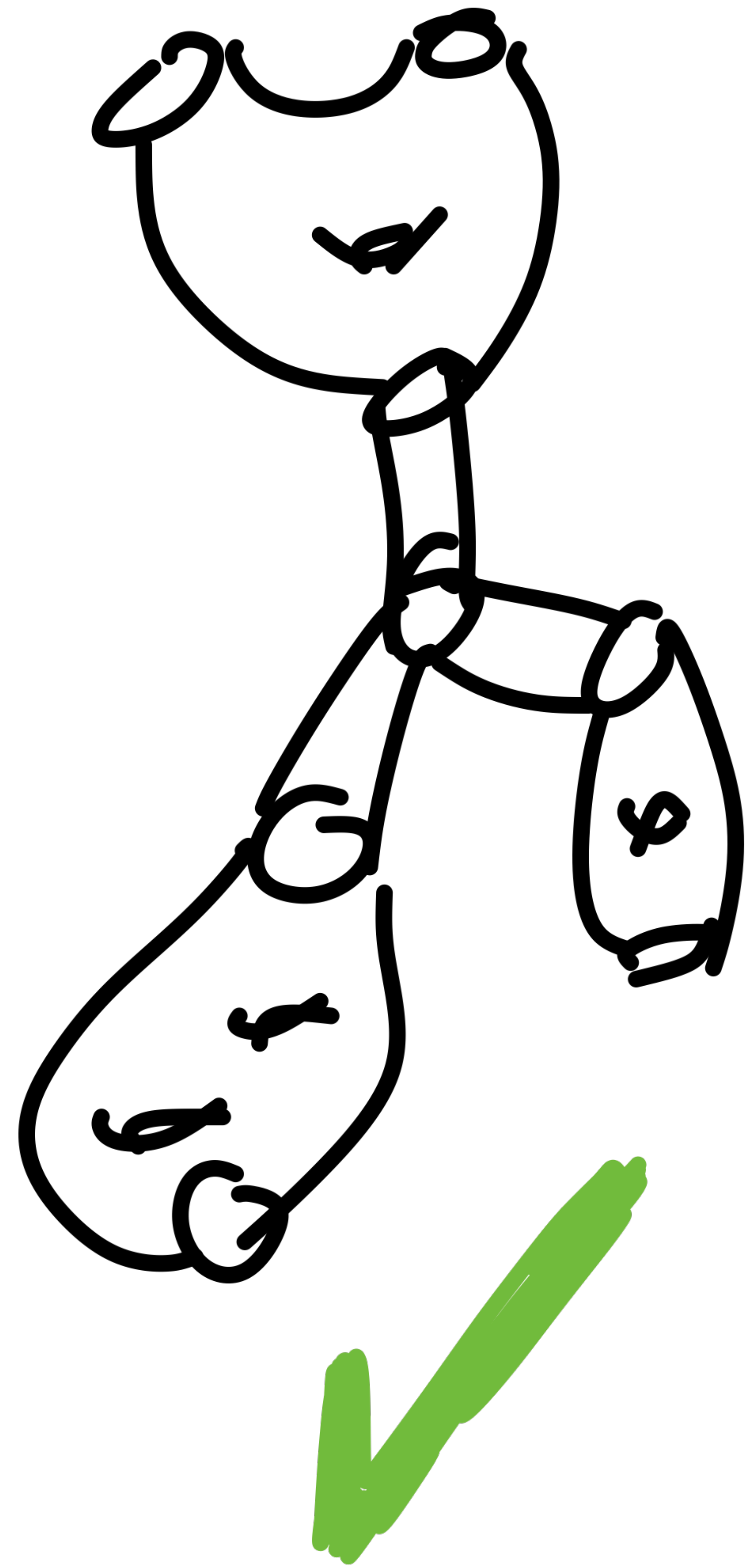
$\mathbb{Z}$
$\mathbb{Z}$
G_v
$\pi_1(\Sigma_{1,2})$
G_w



$\mathbb{Z}$
$\mathbb{Z}$
G_v
$\pi_1(\Sigma_{1,2})$
G_w

Maximality for hanging vertices





Good news: Finitely presented groups have JSJ decompositions!

(Dunwoody and Sageev, Fujiwara and Papasoglu, Guirardel and Levitt, Rips and Sela)

Bad news: In general they are not unique.

However: There is a canonical object we can extract.

JSJ tree of cylinders

after Guirardel and Levitt

Let Γ be a JSJ decomposition of G, and T its Bass-Serre tree.

cylinder: Equivalence class of edges of T by commensurability of stabilizer.

Fact: cylinders are subtrees of T.

JSJ tree of cylinders

after Guirardel and Levitt

T_C built from two sets of vertices:

- cylinders of T
- vertices of T contained in more than one cylinder

Edges determined by incidence in T.

JSJ tree of cylinders

after Guirardel and Levitt

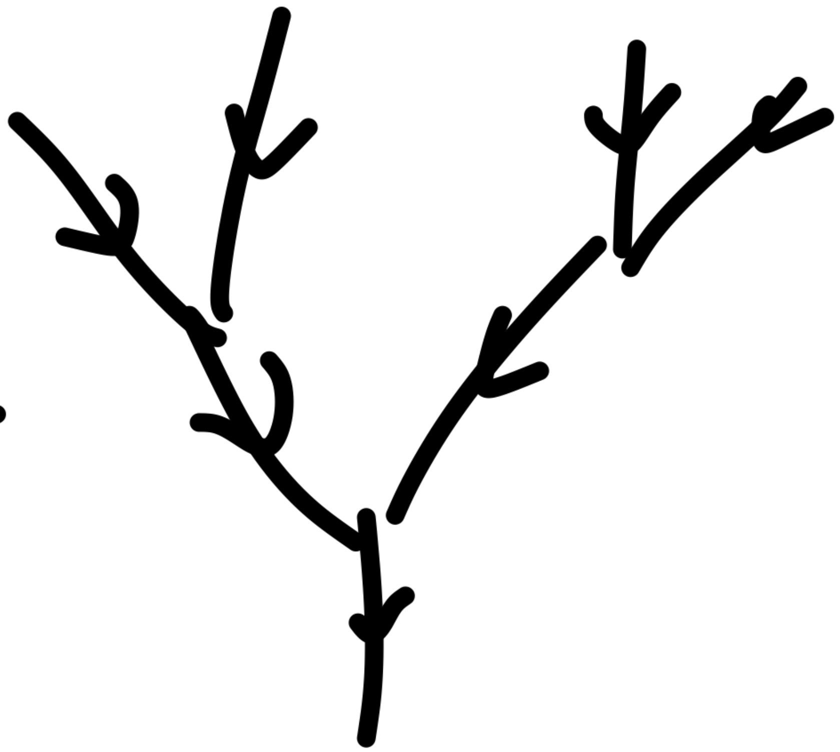
Properties of T_C:

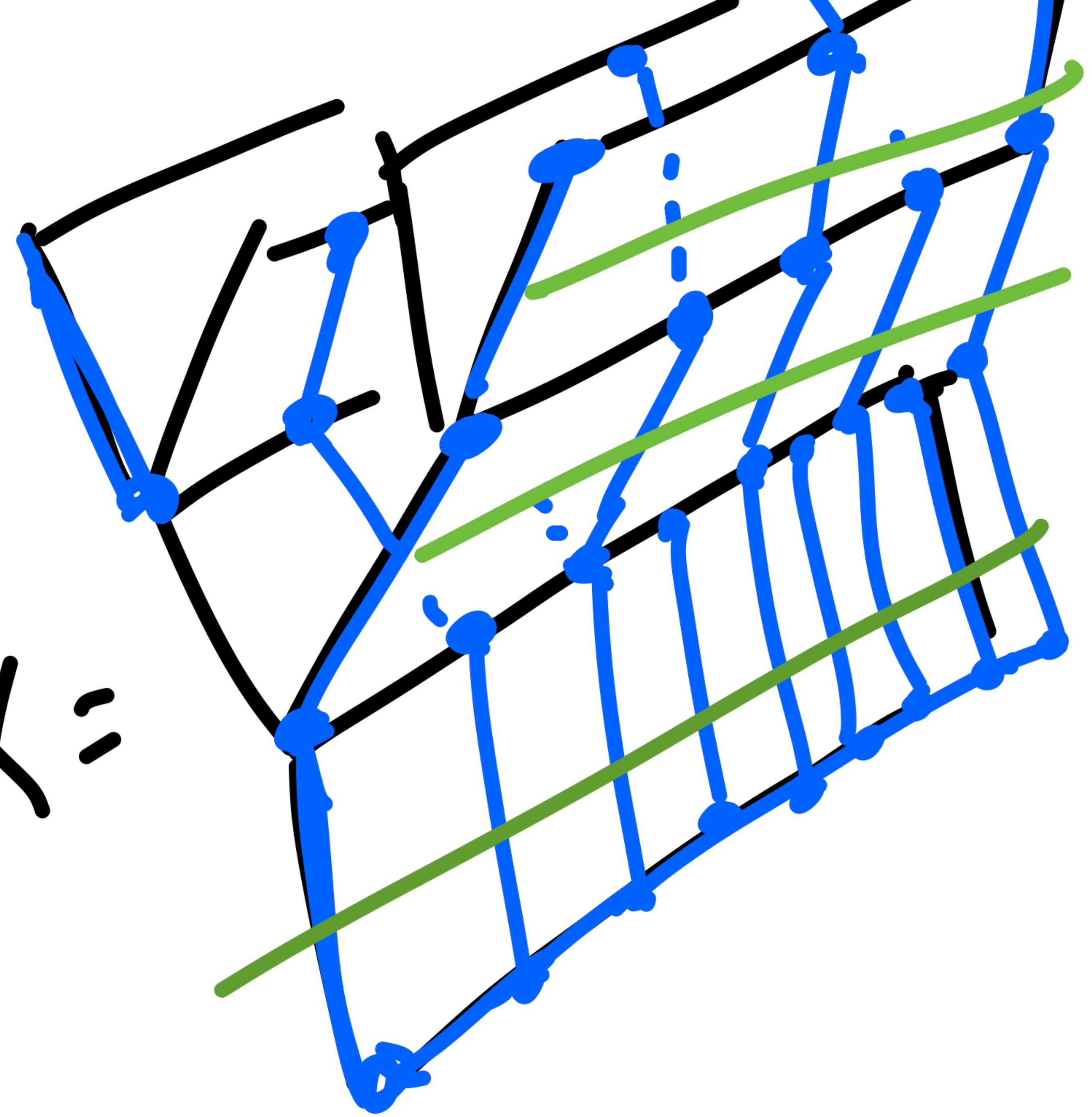
- Bipartite tree: cylidrical and noncylindrical vertices.
- Noncylindrical vertices are either hanging or rigid vertices of T.
- G action induced from $G \curvearrowright T$.
- Independent of choice of JSJ decomposition.

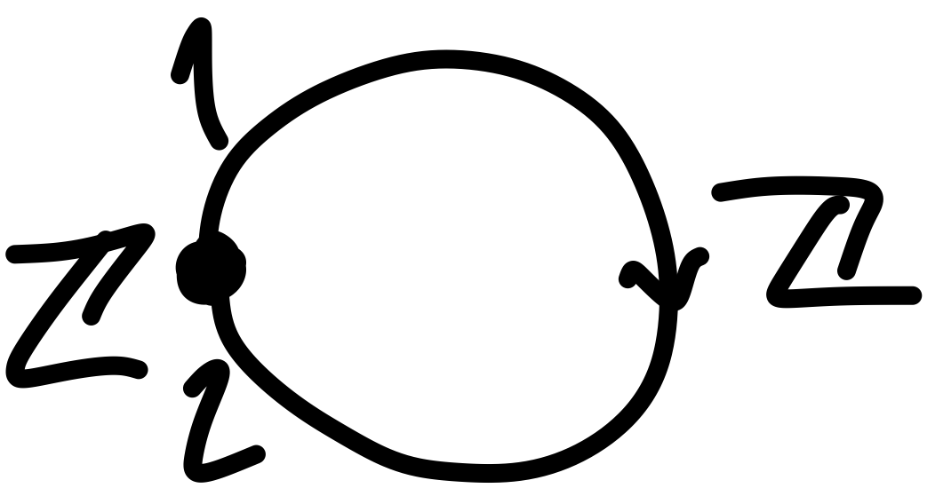
Let $\Gamma_C := G\backslash T_C$ = *graph of cylinders*

If cylinders in T are finite, which is true, in particular, when G is hyperbolic, then cylinder stabilizers are 2-ended and Γ_C is a JSJ decomposition of G.

(= Bowditch canonical JSJ)

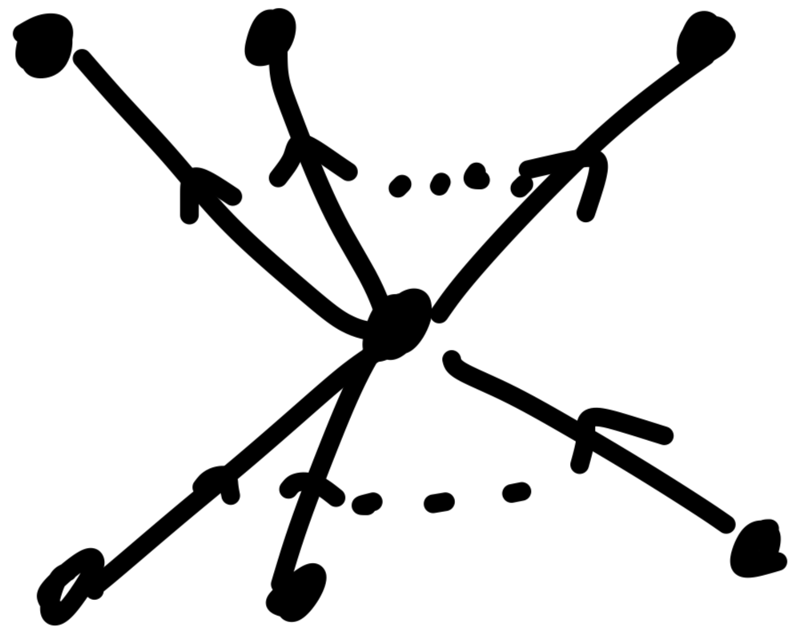
$T_\Gamma =$ 

$X =$ 

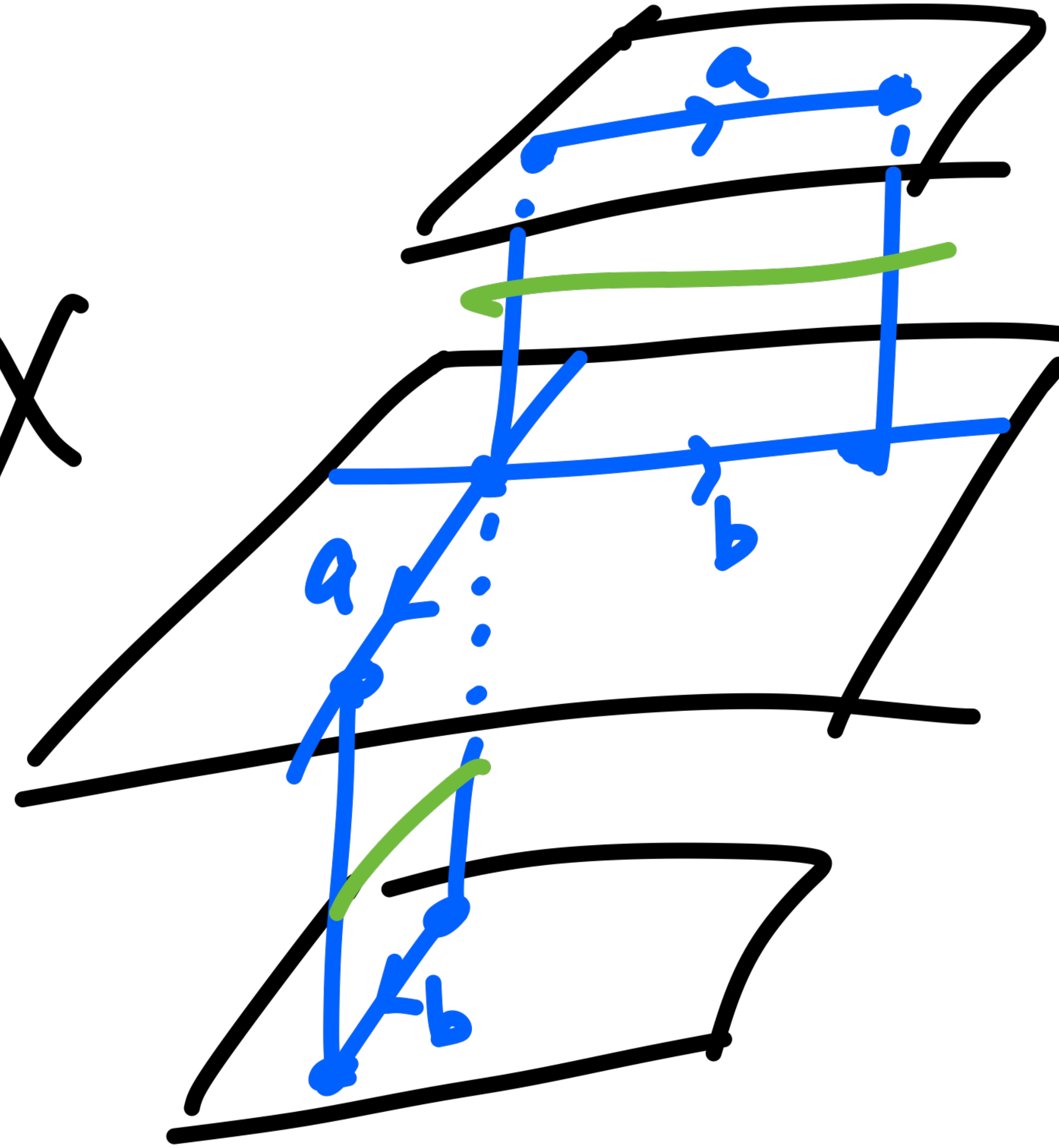
$\Gamma =$ 

$\Gamma_c = T_c = \bullet\, BS(1,2)$

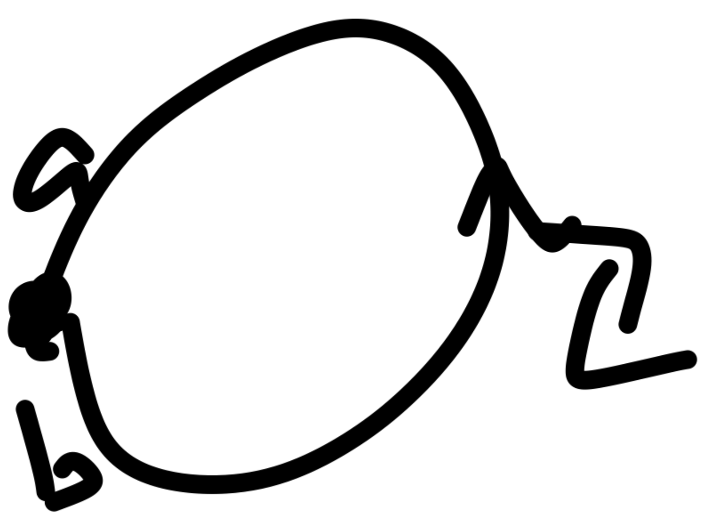
T_Γ



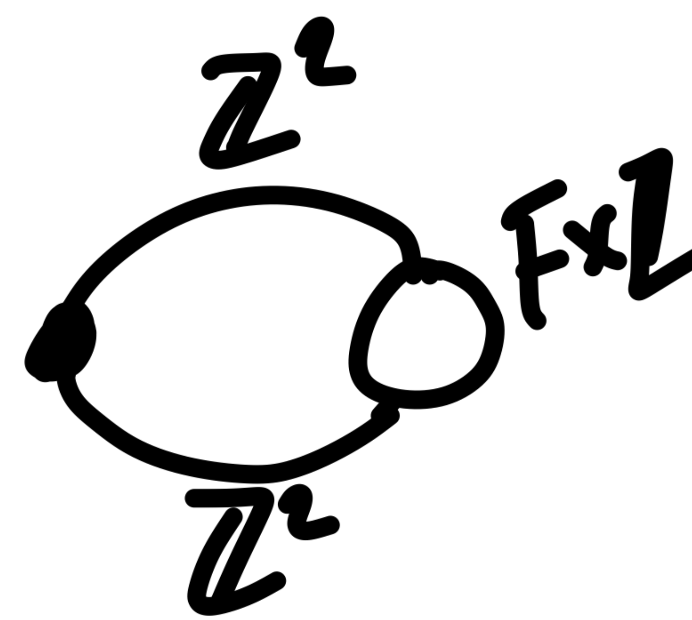
X



$\Gamma = \mathbb{Z}^2$



$\Gamma_C = \mathbb{Z}^2$



Quasiisometries take separating quasilines to within bounded Hausdorff distance of separating quasilines, so consequences of Papasoglu's theorem:

- QI $\phi : G \to G'$ between fp groups induces isomorphism ϕ_C between trees of cylinders.

- ϕ_C preserves vertex type: cylinder, hanging, rigid.

- $\phi|_{G_v} : G_v \to G'_{\phi_C(v)}$ is a quasiisometry for each $v \in T_C$.

When does such an isomorphism between trees of cylinders exist?

Structure invariants for decorated trees

Are two trees with cocompact isometry groups isomorphic?

Answer: They are if and only if they have same *structure invariant*.

- ‘Decorate’ each vertex with its valence.
- Fix ordering of set of ornaments that appear in previous step. Refine decoration by counting neighbors of each type.
- Repeat.
- By cocompactness, process eventually stabilizes.
- Structure invariant is matrix whose i, j entry is number of neighbors with ornament j for each vertex with ornament i, up to block reordering rows and columns.

How to build tree isomorphism $\phi : T \to T'$?

Piece by piece.

- Define $\phi(v) = v'$ for some $v \in T$ and $v' \in T'$ bearing same ornaments.

- Extend ϕ to decoration preserving bijection between neighbors of v and neighbors of v'. Structure invariant guarantees this is possible.

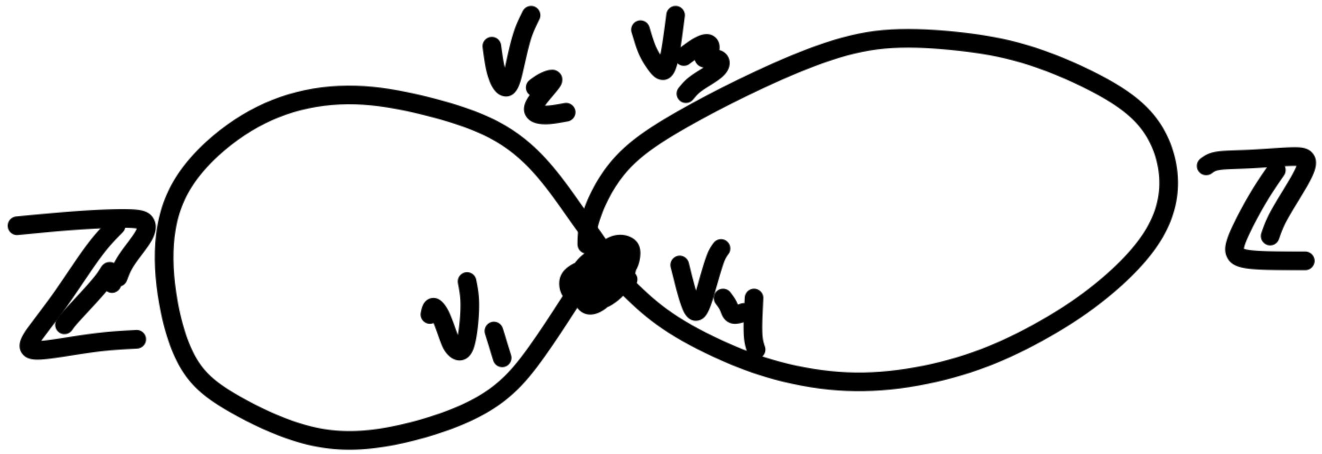
- Repeat.

Observe that we can start with a cocompactly decorated tree, and the resulting structure invariant classifies the set of tree isomorphism that preserve the original decoration.

For instance, decorate the tree of cylinders of a fp 1-ended group by (type, QI type)...

In fact, we can do better. Use not just QI type of vertex, but *relative QI type.*

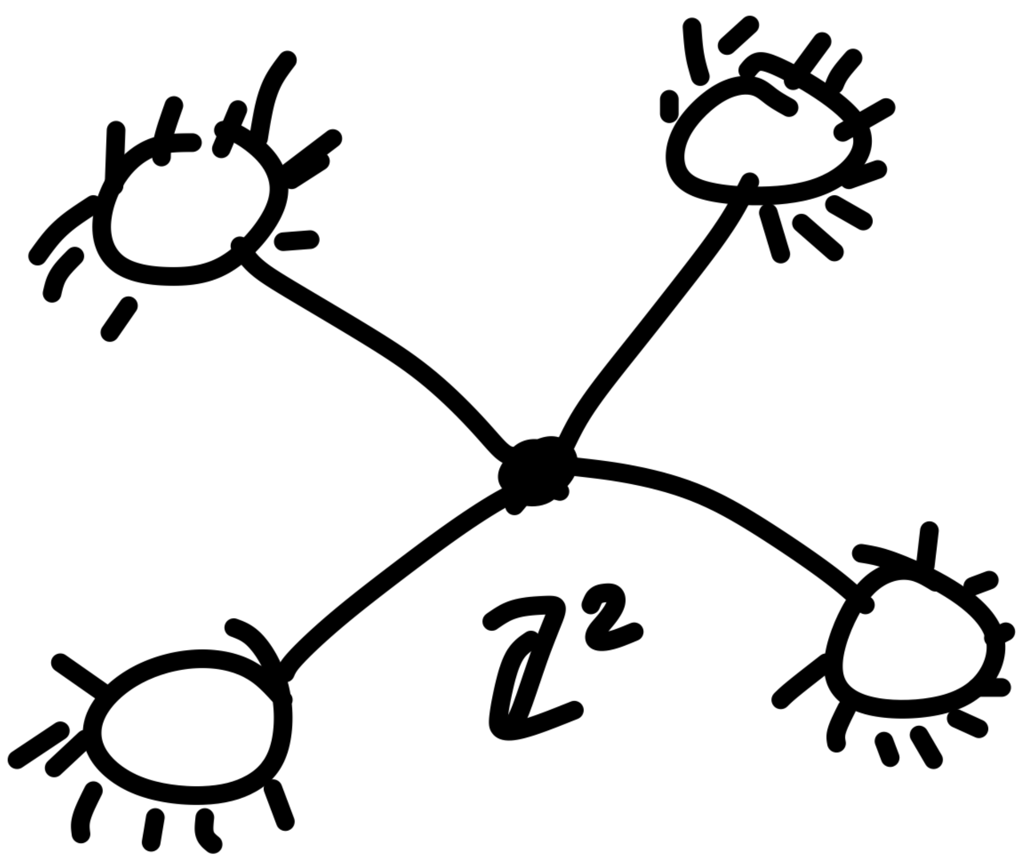
Recall that QI must take vertex group quasiisometrically to vertex group, but must also preserve pattern of incident separating quasilines.

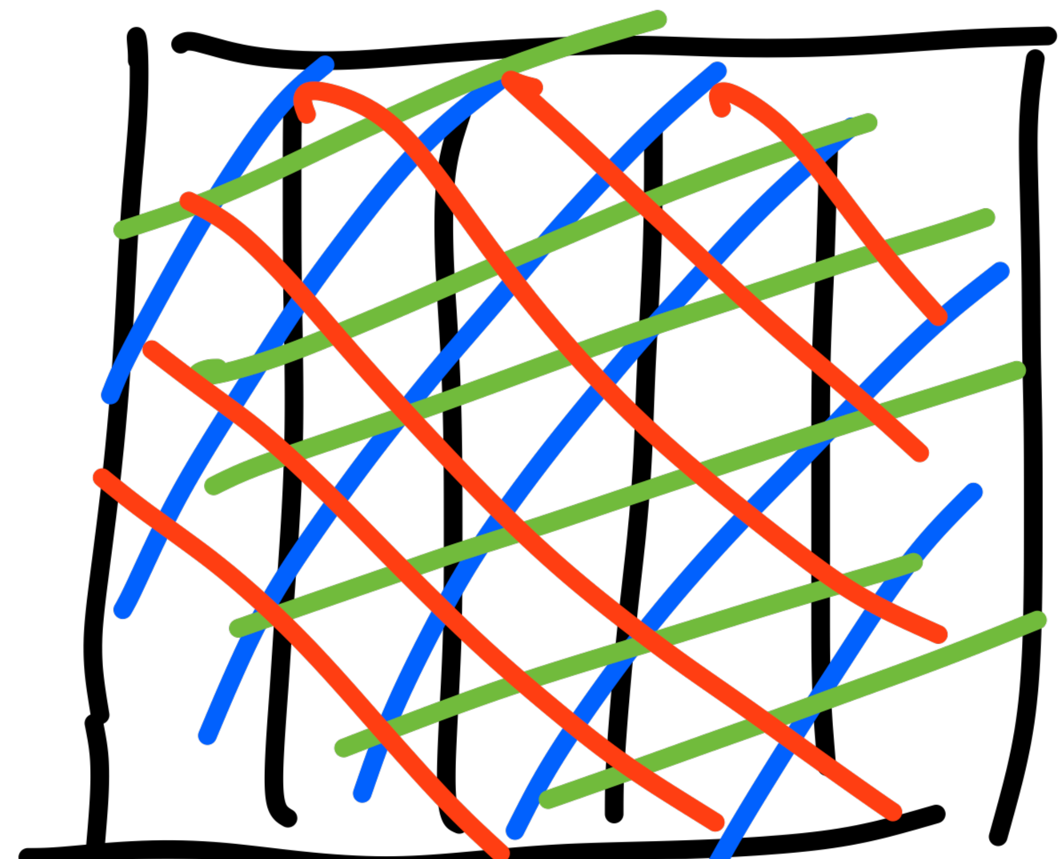


$$\mathbb{Z}^2 = \langle a, b \rangle$$

$$V_i = a^{p_i} b^{q_i}$$

w/ distinct slopes





QI type of the rigid vertex is $\mathbb{Z}^2$.

Mosher-Sageev-Whyte: Relative QI type of rigid vertex is ($\mathbb{Z}^2$, cross ratio of 4 slopes).

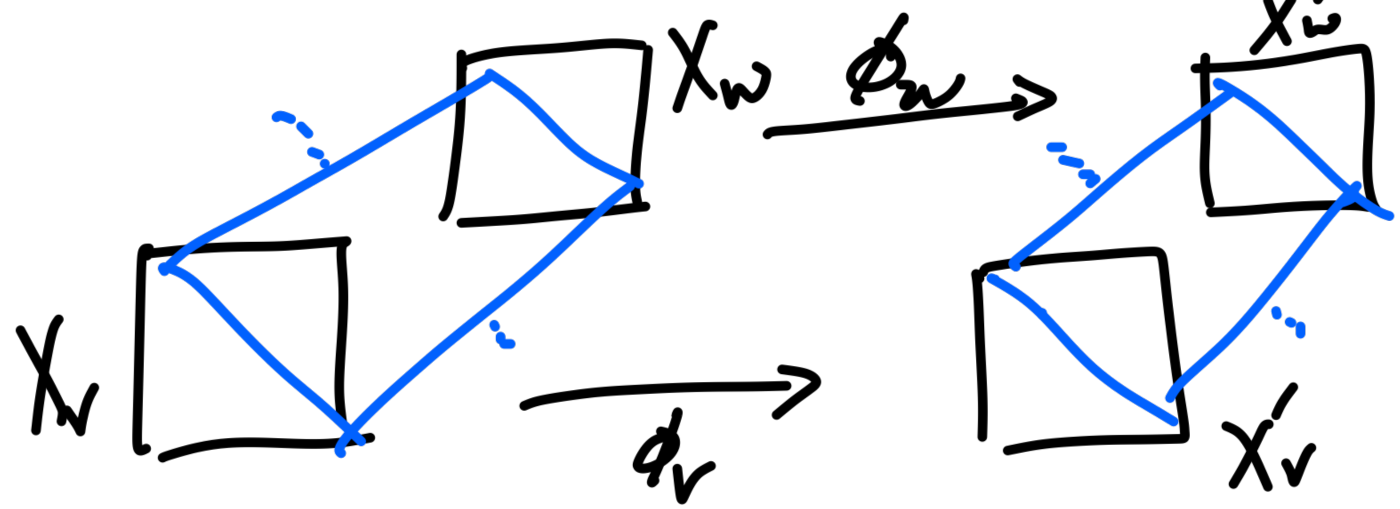
Relative QI type gives finer invariant than QI type alone.

The resulting structure invariant gives a QI invariant of the group, but still not a complete one.

Idea: Piece together QIs of vertex spaces to get QI of whole group. Maps on neighboring vertex spaces must agree on their intersection.

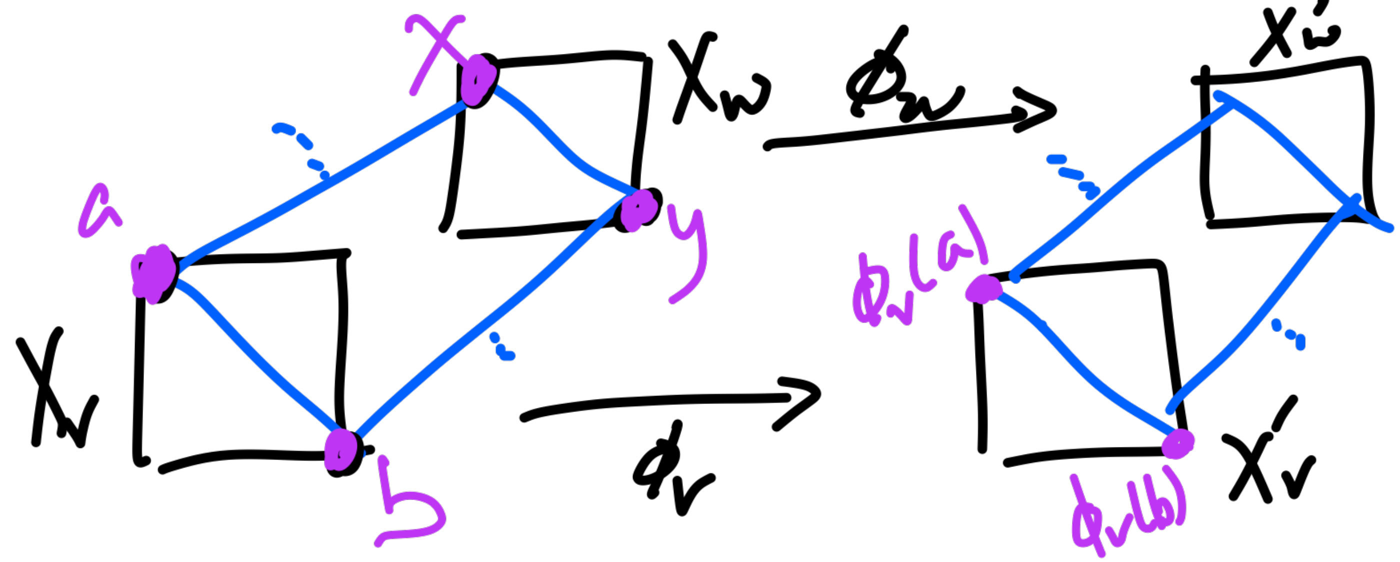
Deciding if this is possible is easiest if vertex spaces are either very flexible or very rigid, so that at each step we either have lots of freedom to make maps match up, or a clear obstruction to doing so.

Obstruction to extending maps across edge spaces:



Glue two planes X_v, X_w together by attaching infinite strip. Same for X'_v, X'_w. Assume in all cases gluing map is equivariant w/respect cocompact action. Given isometry ϕ_v, is there an isometry ϕ_w such that they piece together to give QI of the whole thing?

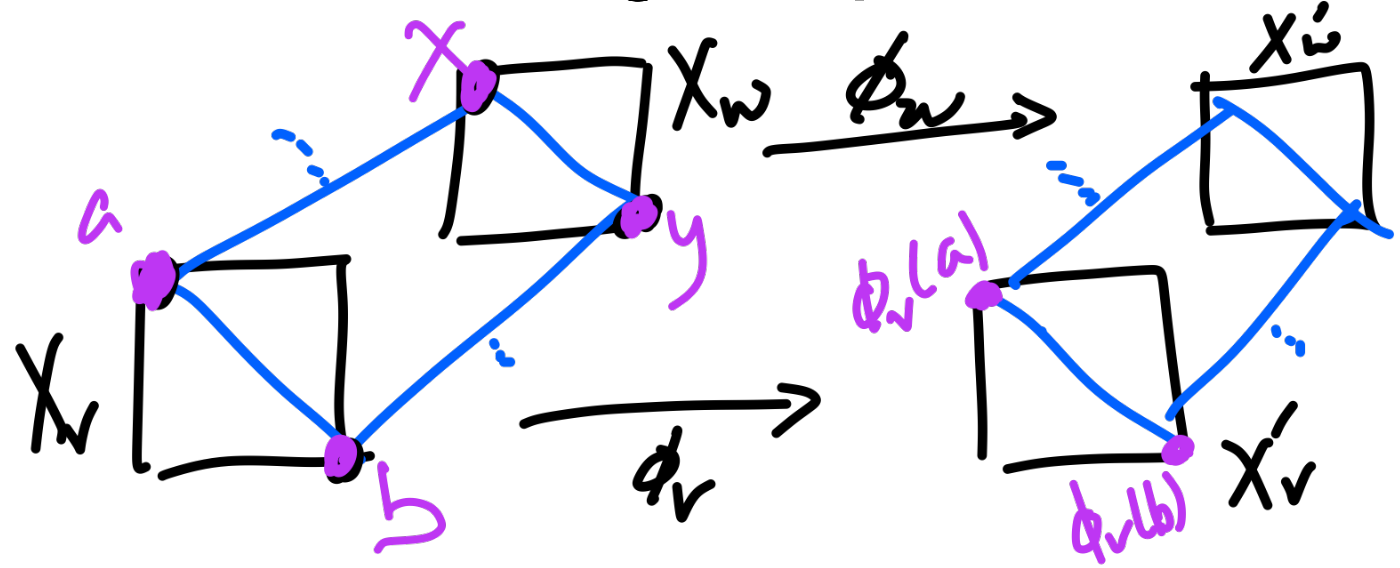
Obstruction to extending maps across edge spaces:



Consider $a, b \in X_v$ far apart along attaching line. Let $x, y \in X_w$ be respective closest points.

Claim QI if and only if closest points to $\phi_v(a), \phi_v(b)$ are close to $\phi_w(x), \phi_w(y)$, and this is true if and only if 'stretch factor' across $\mathbb{R} \times [0,1]$ is same for X's and X''s.

Obstruction to extending maps across edge spaces:



Is such stretch factor QI well defined?

We will add hypothesis to ensure that it is.

C-Martin: Under the following hypothesis there is an initial QI invariant decoration on tree of cylinders and a refinement process such that the resulting structure invariant is a **complete** QI invariant.

- Cylinders are finite (graph of cylinders is a JSJ).
- We have detailed knowledge of QIs of vertex groups.
- Rigid vertices groups have ‘relative rigidity’ property.

Compare

Papasoglu-Whyte	C-Martin
QI types of vertices	detailed knowledge of relative QI types of vertices
arrangement irrelevant	arrangement matters; encoded by structure invariant
QIs on neighboring vertices independent	QIs on neighboring vertices must agree on intersection

Relative rigidity property:

For every rigid vertex group G_v with peripheral structure P_v induced by incident separating quasilines, there exists a quasiisometry $\mu_v : G_v \to X_v$ such that:

- $\mu_v(\mathrm{QIsom}(G_v, P_v))$ is a uniform subgroup of $\mathrm{CIsom}(X_v, \phi(P_v))$
- If $g \in G_v$ is an infinite order element fixing an element of P_v then $i \to \mu_v(g^i)$ is a coarse similitude.

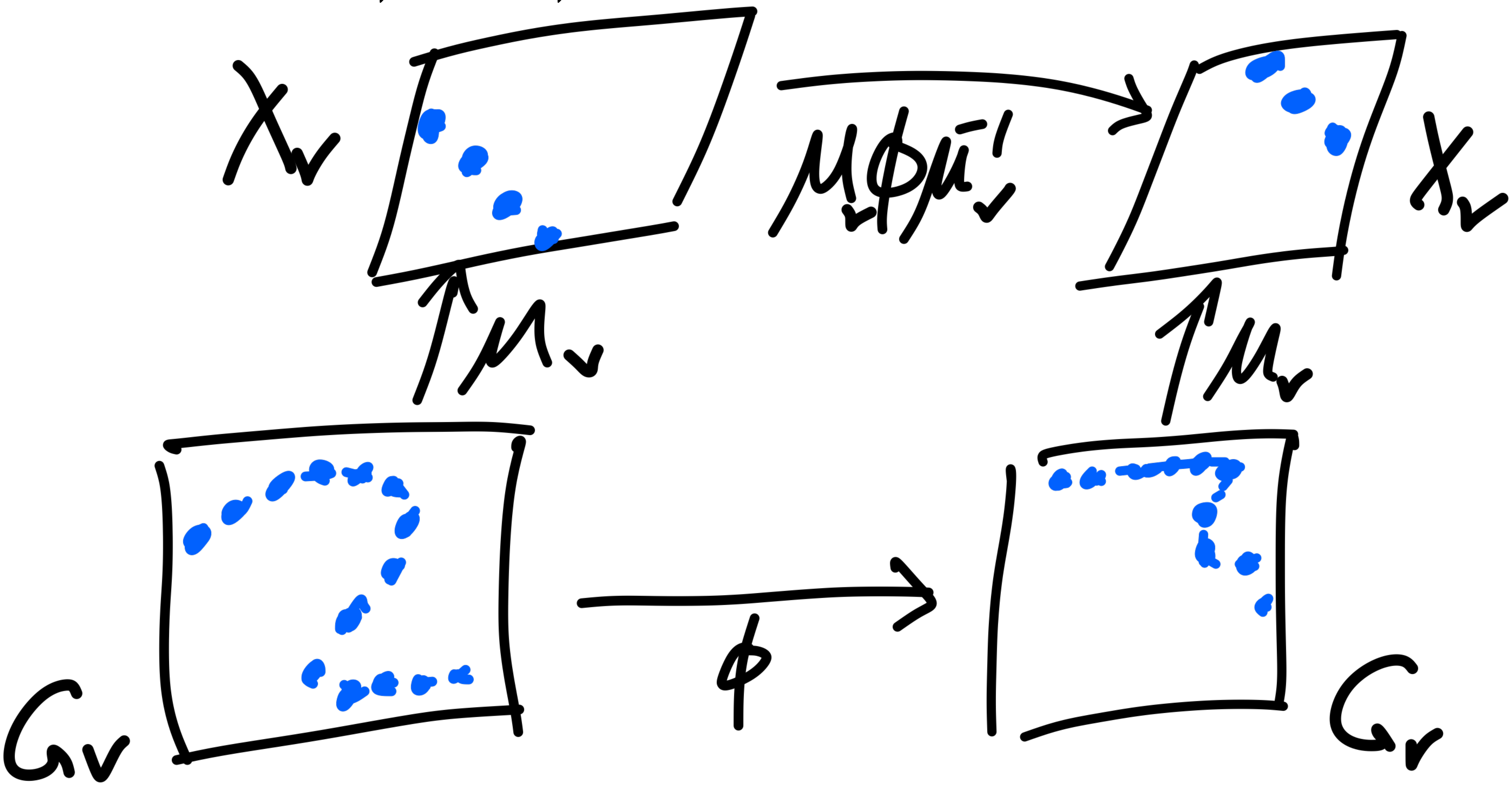
μ_v straightens out $\langle g \rangle$ to act along a geodesic in X_v with translation length M.

$i \to \mu_v(g^i)$ is a coarse similitude:

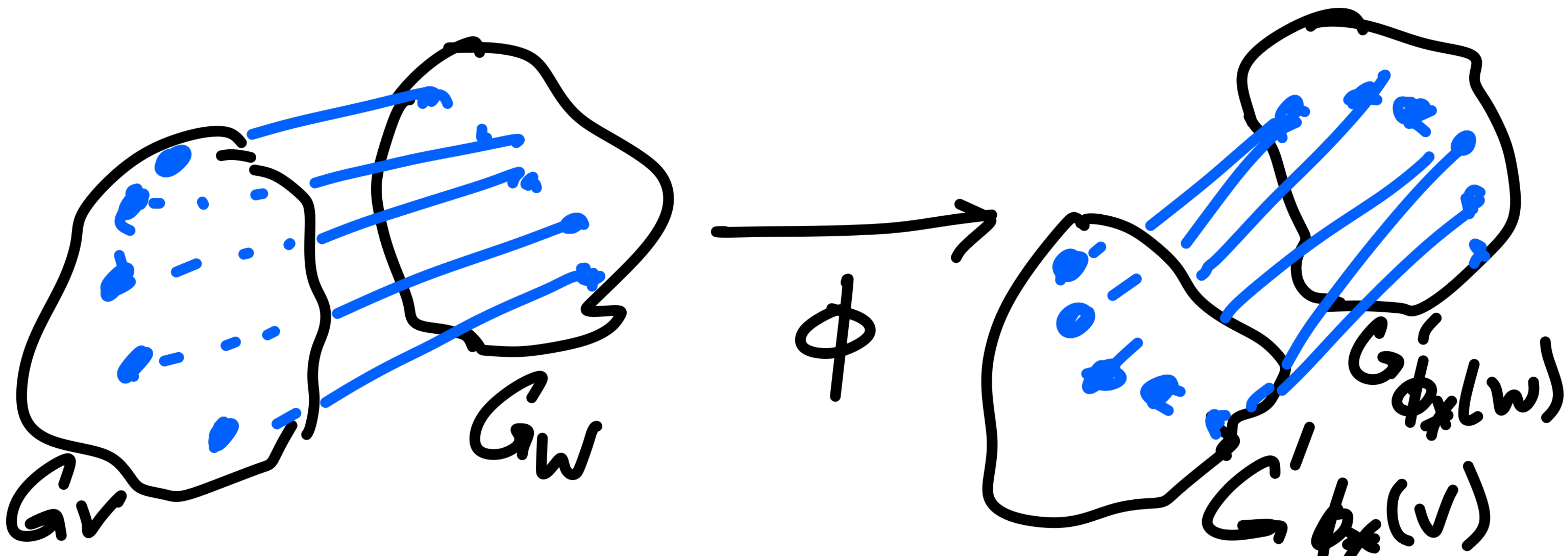
There exist $M > 0, C \geq 0$ such that

$$M\,|\,i - j\,| - C \leq d(\mu_v(i), \mu_v(j)) \leq M\,|\,i - j\,| + C$$

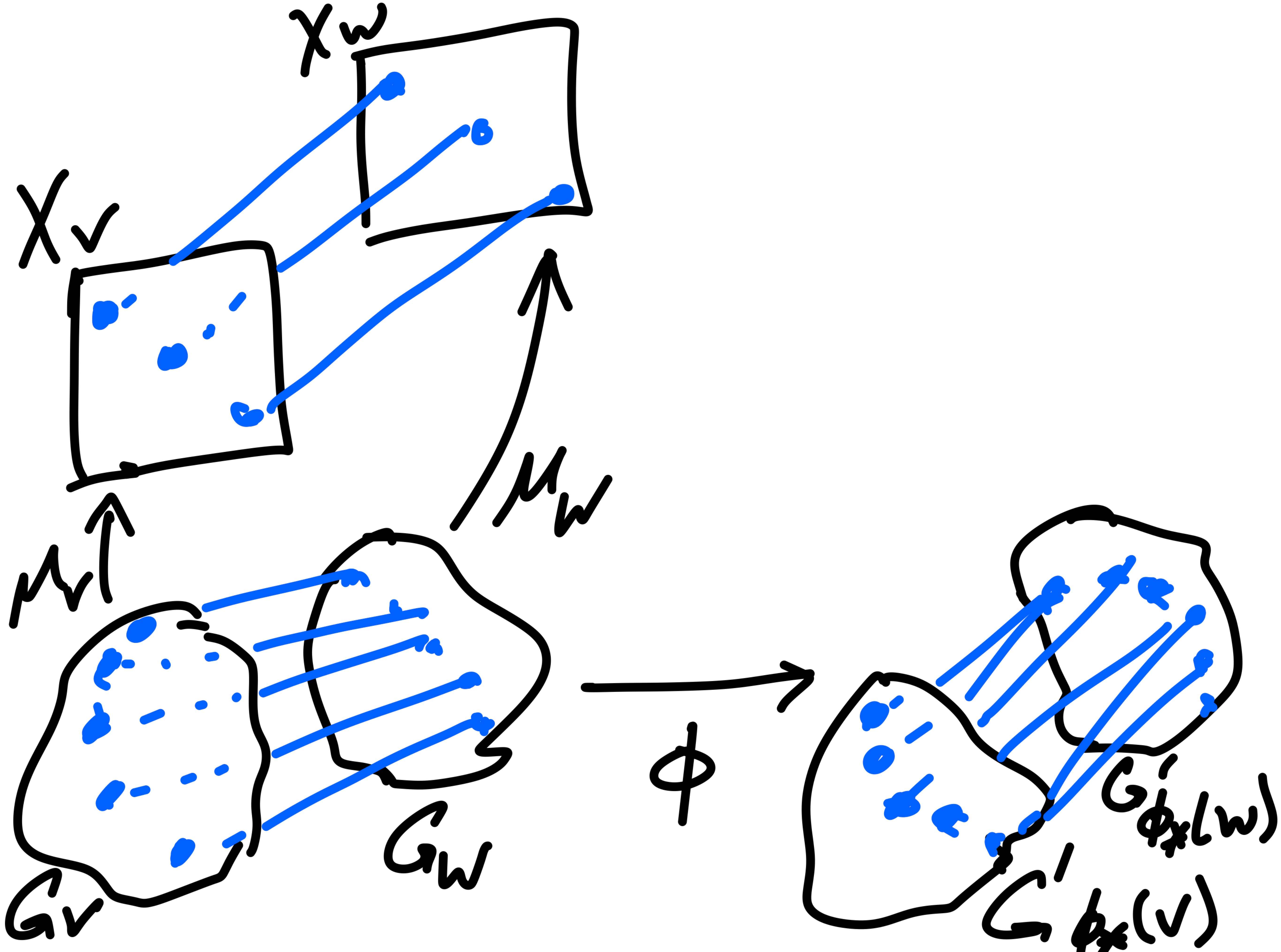
$\mu_v(\mathrm{QIsom}(G_v, P_v))$ is a uniform subgroup of $\mathrm{CIsom}(X_v, \phi(P_v))$



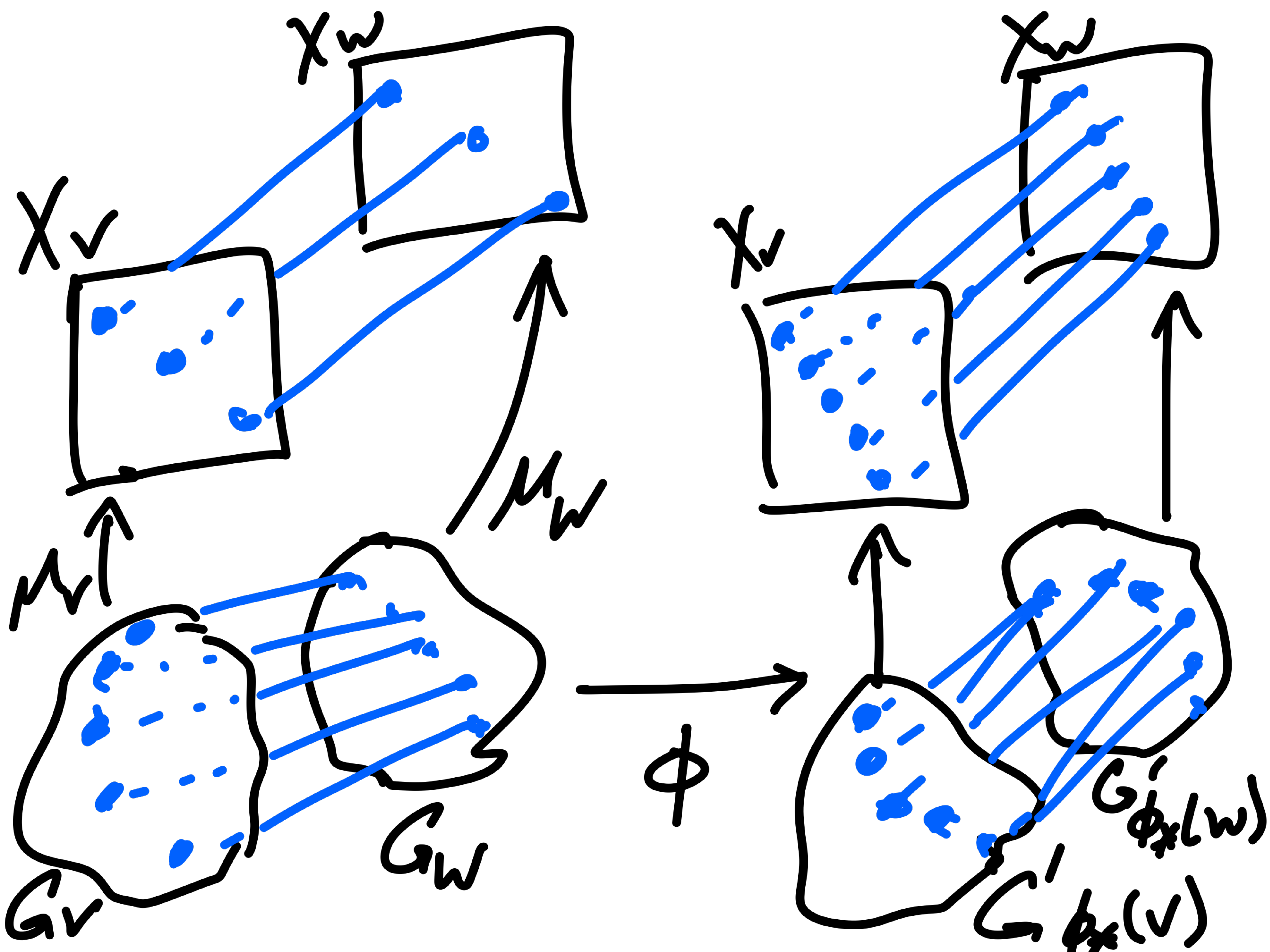
In particular the mult. similitude constant is invariant.



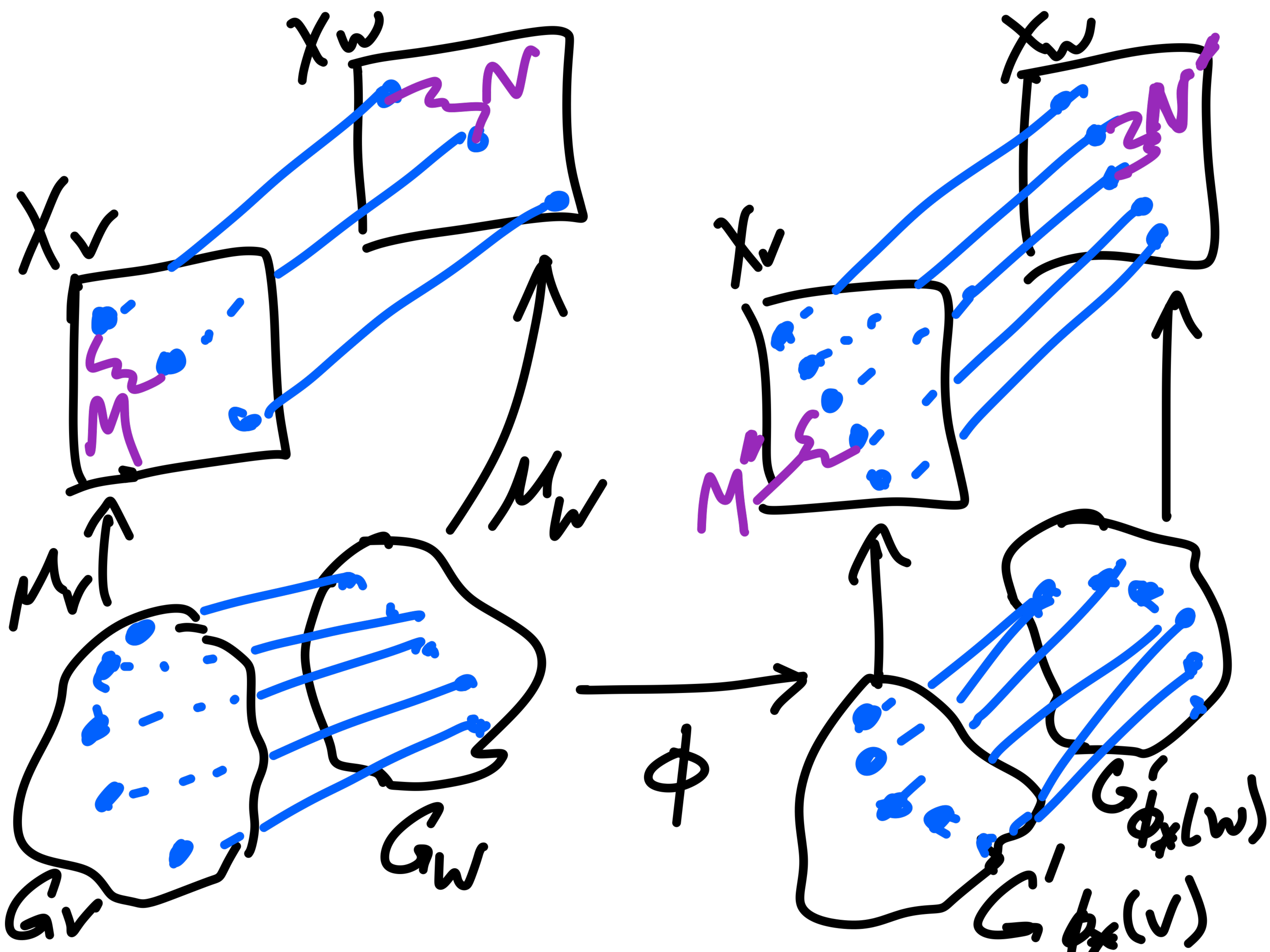
G_V
G_W
ϕ
$G'_{\phi_*(w)}$
$G'_{\phi_*(v)}$



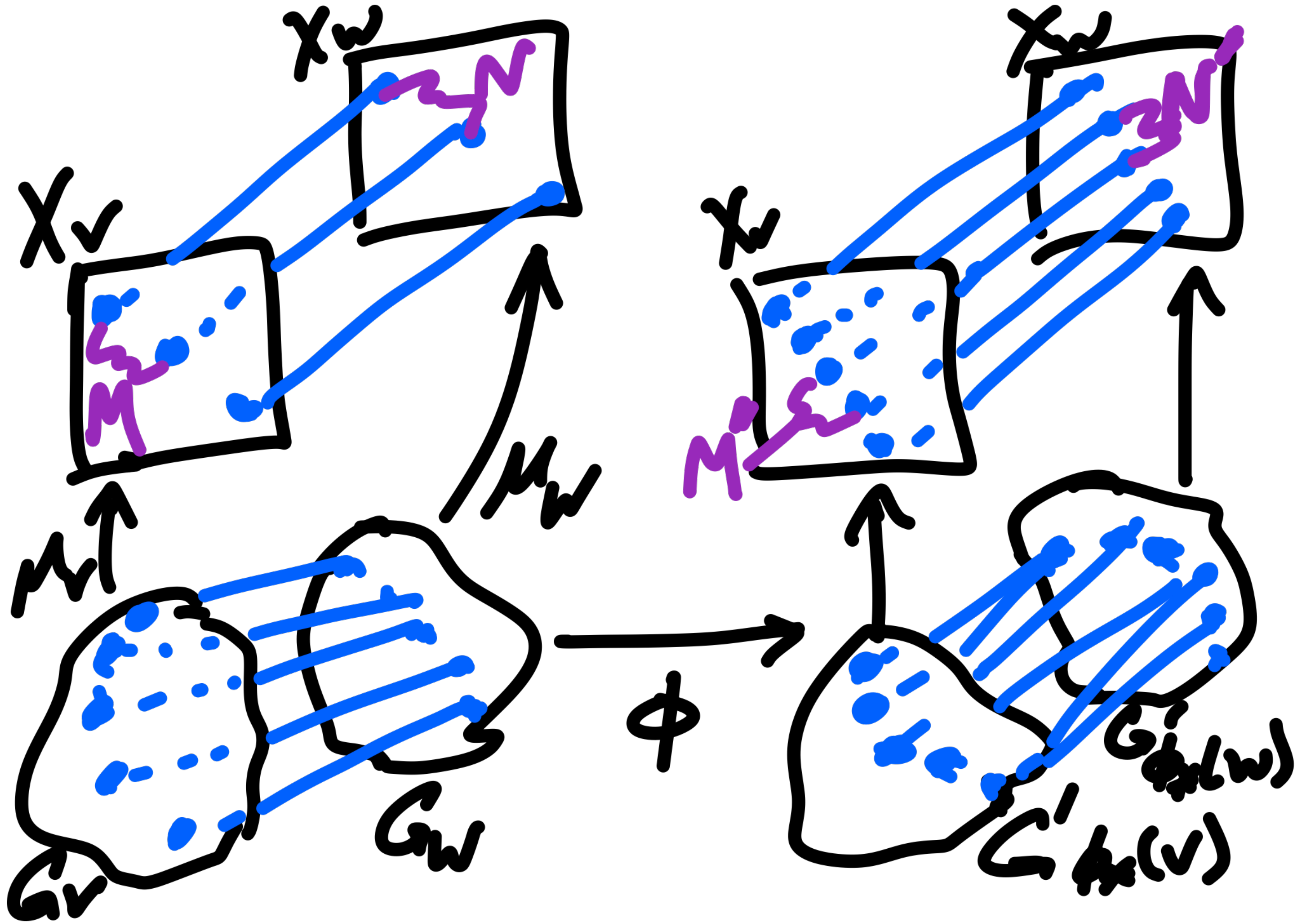
X_w
X_v
μ_v
μ_w
G_w
G_v
ϕ
$G'_{\phi_*(w)}$
$G'_{\phi_*(v)}$



X_w
X_v
μ_w
μ_v
G_w
G_v
ϕ
X_w
X_v
$G'_{\phi_*(w)}$
$G'_{\phi_*(v)}$



X_w
N
X_v
M
μ_w
μ_v
G_w
G_v
ϕ
X_w
N'
X_v
M'
$G'_{\phi_*(w)}$
$G'_{\phi_*(v)}$



$$\frac{M}{N} = \frac{M'}{N'}$$

This is good! QI invariant stretch factor between adjacent rigid vertex spaces.

What groups have relative rigidity property?

- Groups QI to a space X such that $\mathrm{Isom}(X)$ surjects onto $\mathrm{QI}(X)$. The peripheral structure plays no role here.
  - irreducible symmetric spaces other than real or complex hyperbolic space; thick Euclidean buildings; and products of such (Eskin-Farb, Kleiner-Leeb)
  - the 'topologically rigid' hyperbolic groups of Kapovich and Kleiner
  - certain Fuchsian buildings
  - mapping class groups of non-sporadic hyperbolic surfaces (Behrstock, Kleiner, Minsky, Mosher)

What groups have relative rigidity property?

- X such that $\mathrm{QIsom}(X) = \mathrm{CIsom}(X)$. Again, the peripheral structure plays no role in this case. Xie gives an example of a certain solvable Lie group with this property.

- X is real or complex hyperbolic space of dimension at least 3. (Schwartz)

- $X = \mathbb{H}^2$ (Kapovich-Kleiner, Markovich)

- G is virtually free. In this case the space X depends on P. (C-Macura)

- hyperbolic groups?

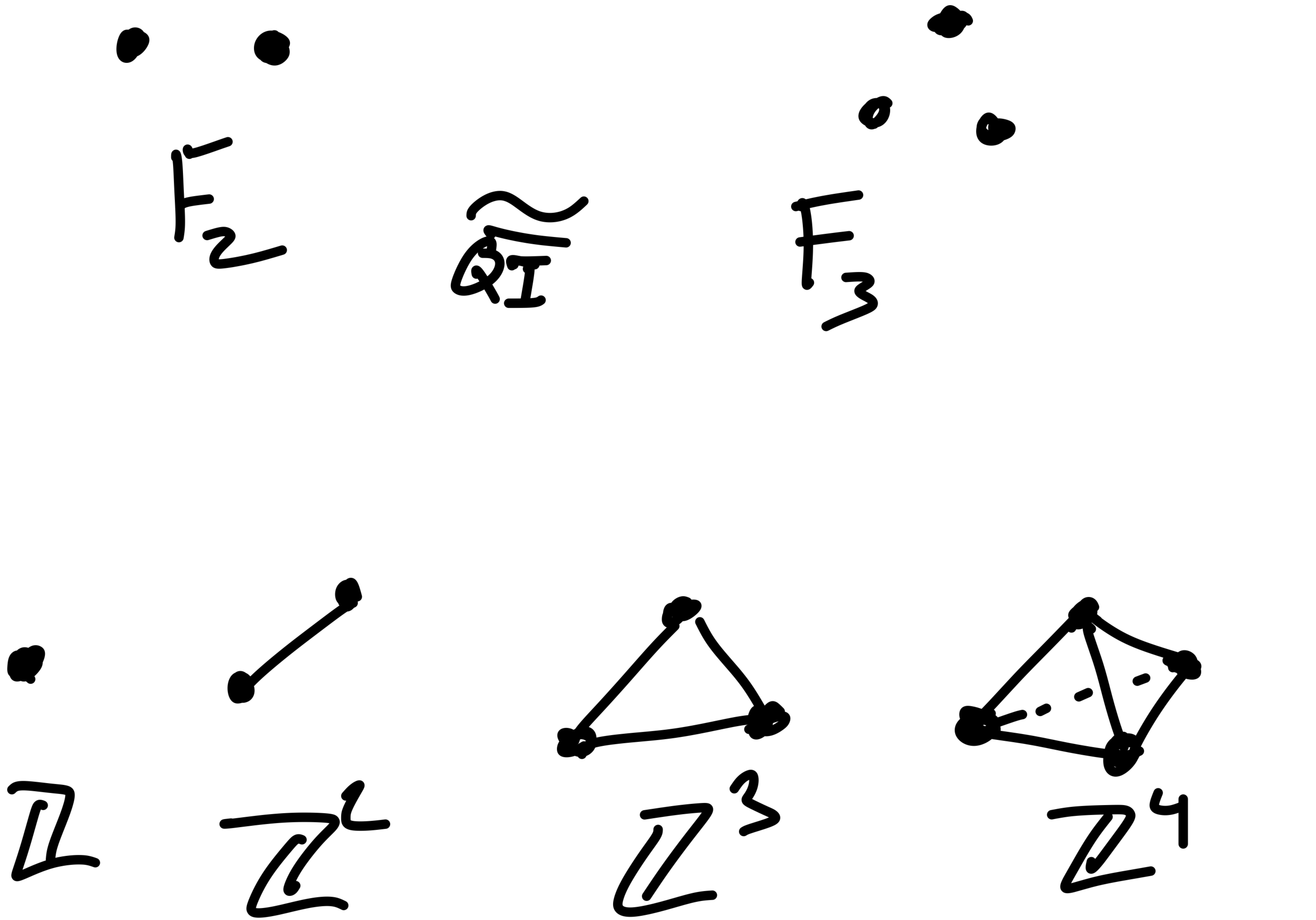
RAAGs

Γ finite simple graph.

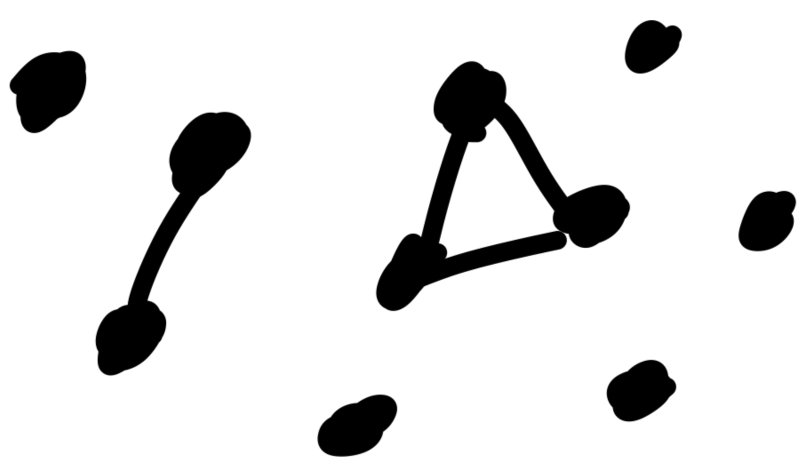
$A(\Gamma) := \langle v \in V\Gamma \,|\, [v, w]$ when v-$w\rangle$

Droms: $A(\Gamma) \cong A(\Gamma') \iff \Gamma = \Gamma'$

QI?


F_2
$\widetilde{\overline{QI}}$
F_3
$\mathbb{Z}$
$\mathbb{Z}^2$
$\mathbb{Z}^3$
$\mathbb{Z}^4$

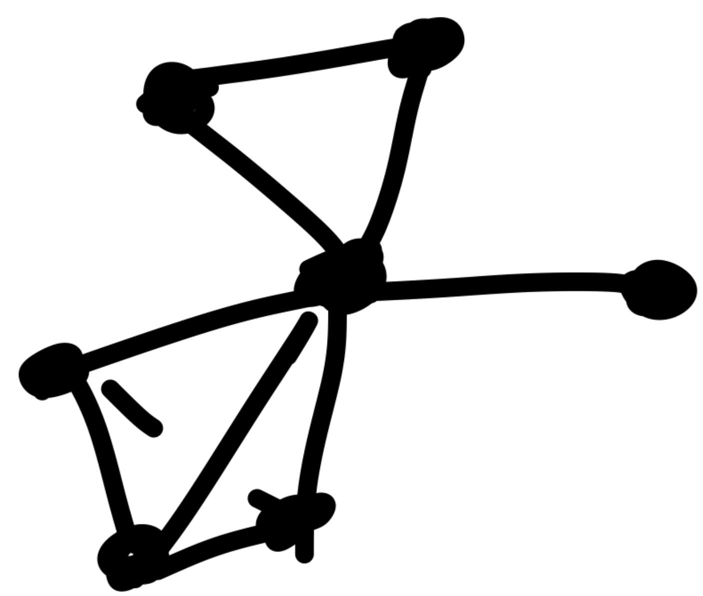
Grushko decomposition is ‘visual’: can see it just by looking at the graphs —connected components.



$$\mathbb{Z}^2 * \mathbb{Z}^3 * F_5$$

By Papasoglu-Whyte, we can restrict to connected graphs with more than one vertex = 1-ended groups.

Clay, Groves-Hull: A JSJ decomposition is also visible. Separating vertices give separating quasilines: 2–connected components give rigid vertices.



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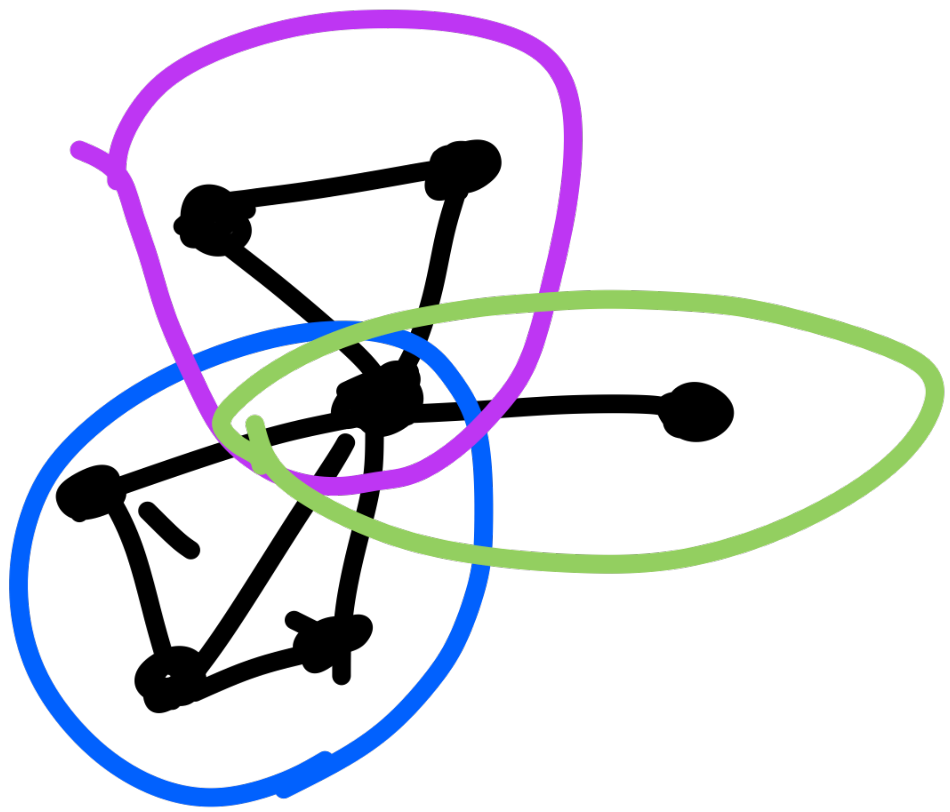

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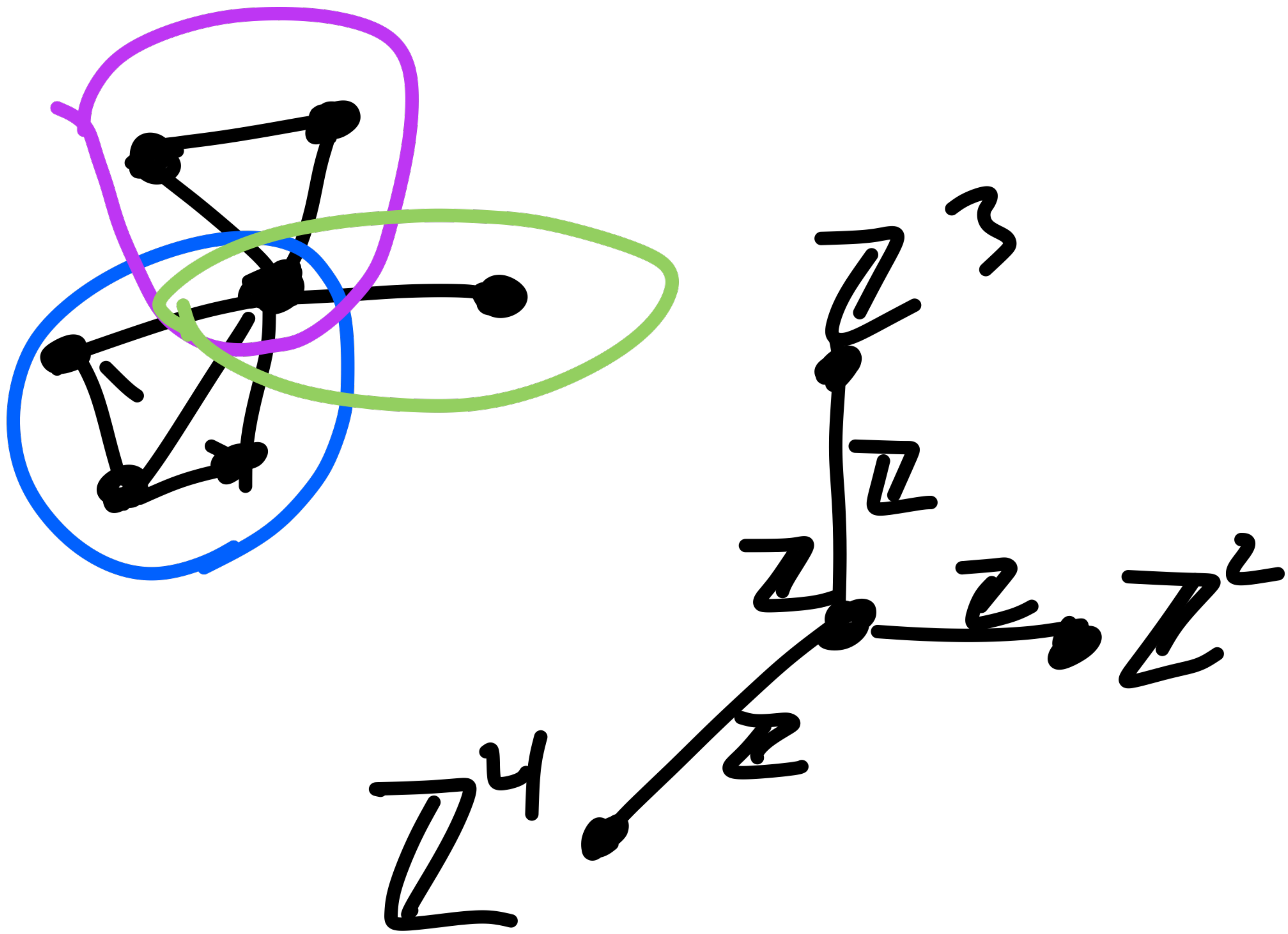
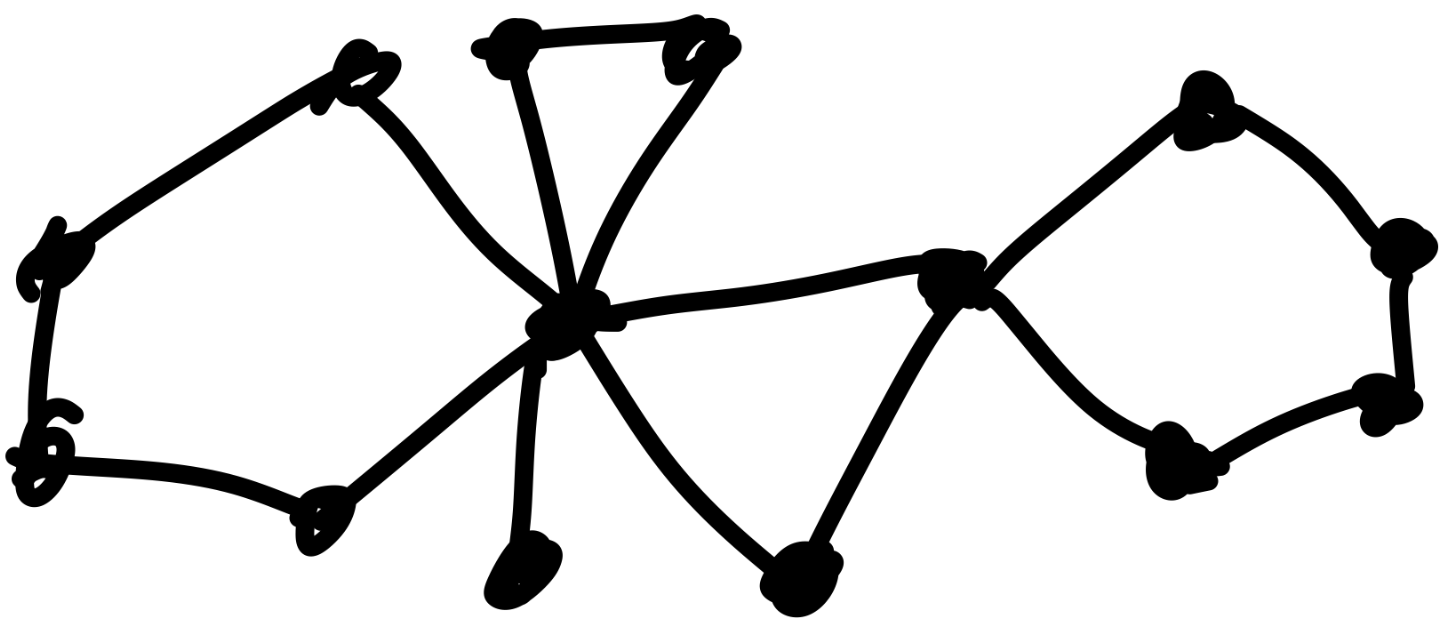


Margolis: The tree of cylinders is also visible.
Cylinder stabilizers are $A(st(v))$ for cut vertex v.



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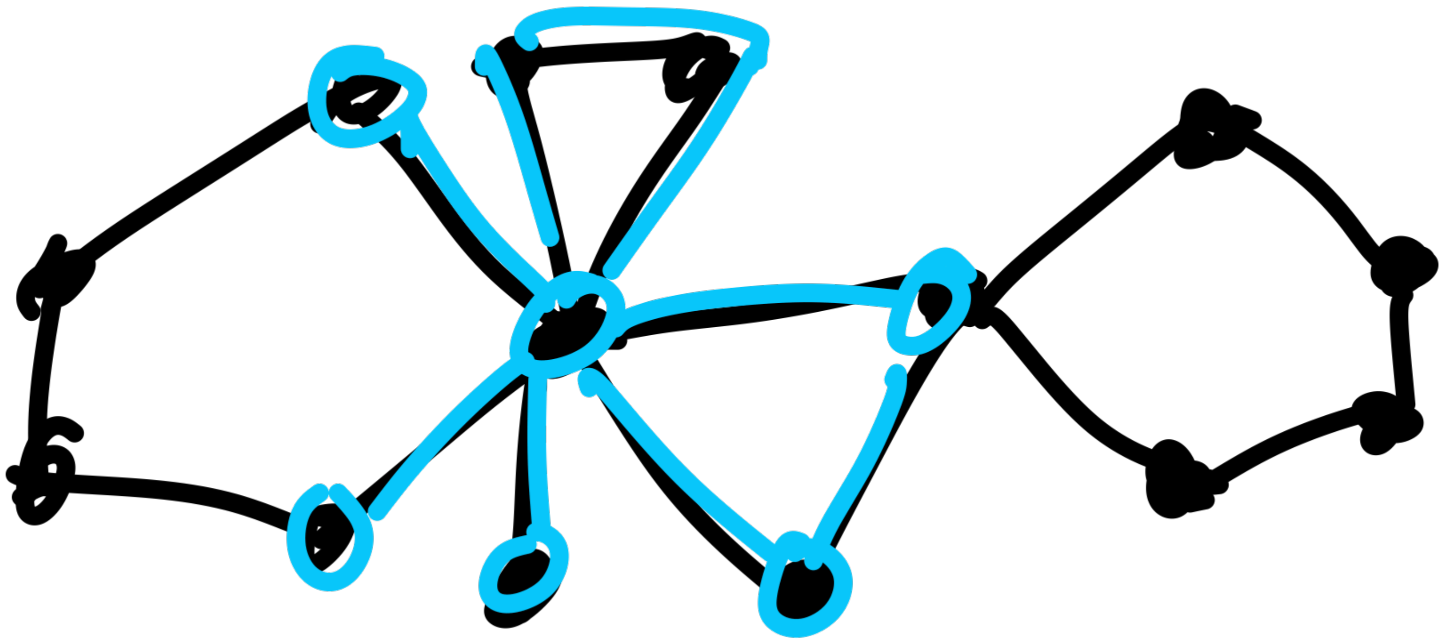

$\mathbb{Z} \times (\mathbb{Z}^2 * \mathbb{Z}^2 * F_3)$

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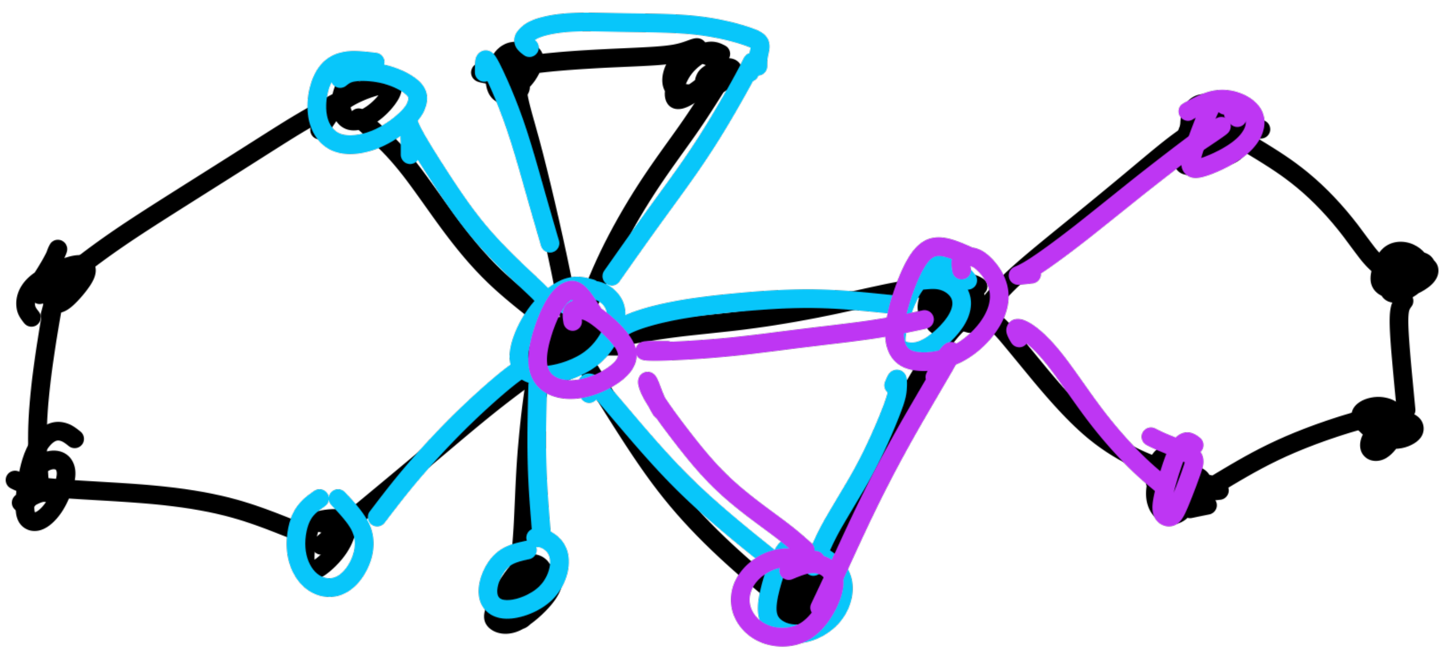

$\mathbb{Z}\times(\mathbb{Z}^2 * \mathbb{Z}^2 * F_3)$

$\mathbb{Z}\times(\mathbb{Z}^2 * F_2)$

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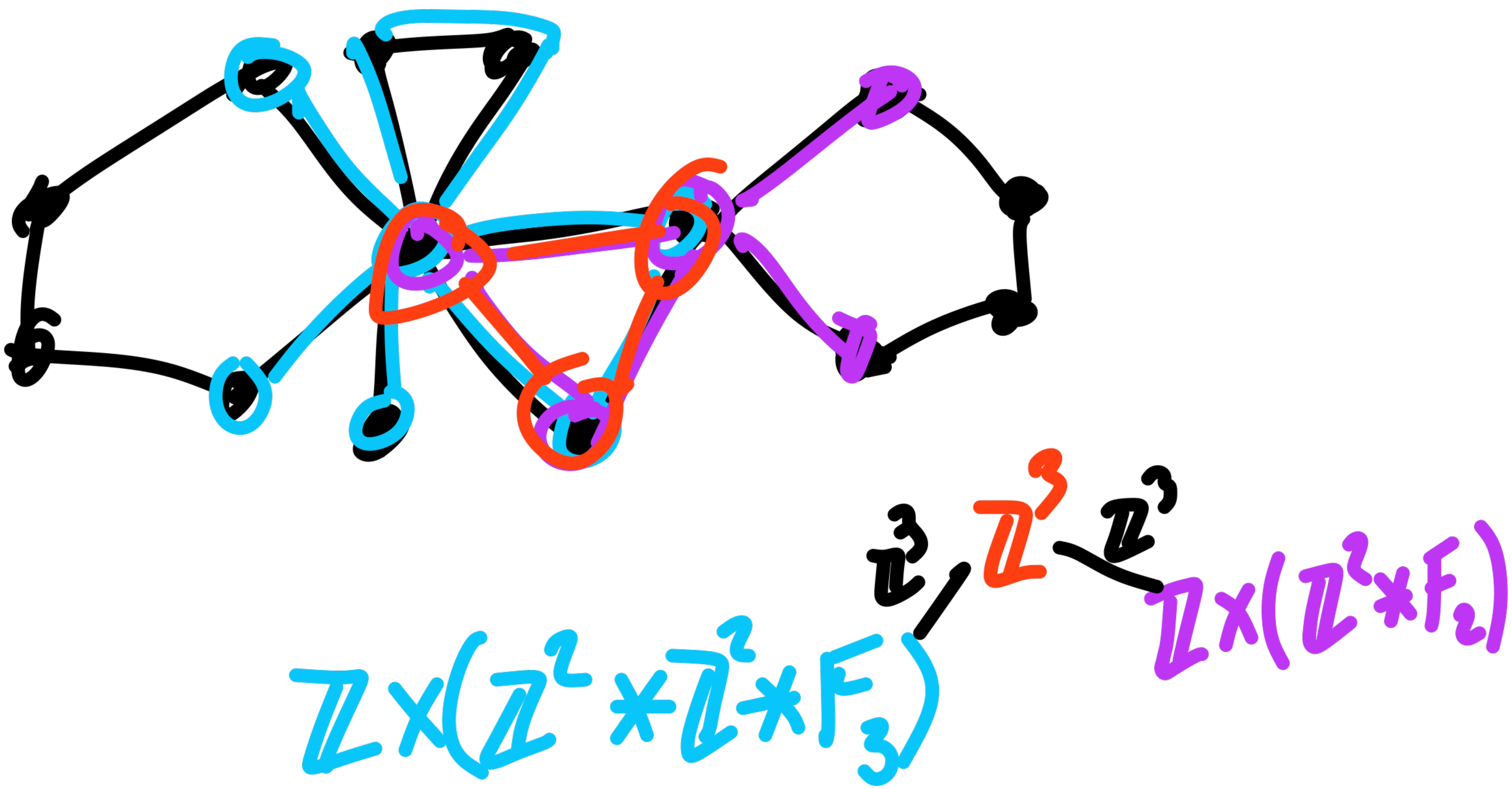

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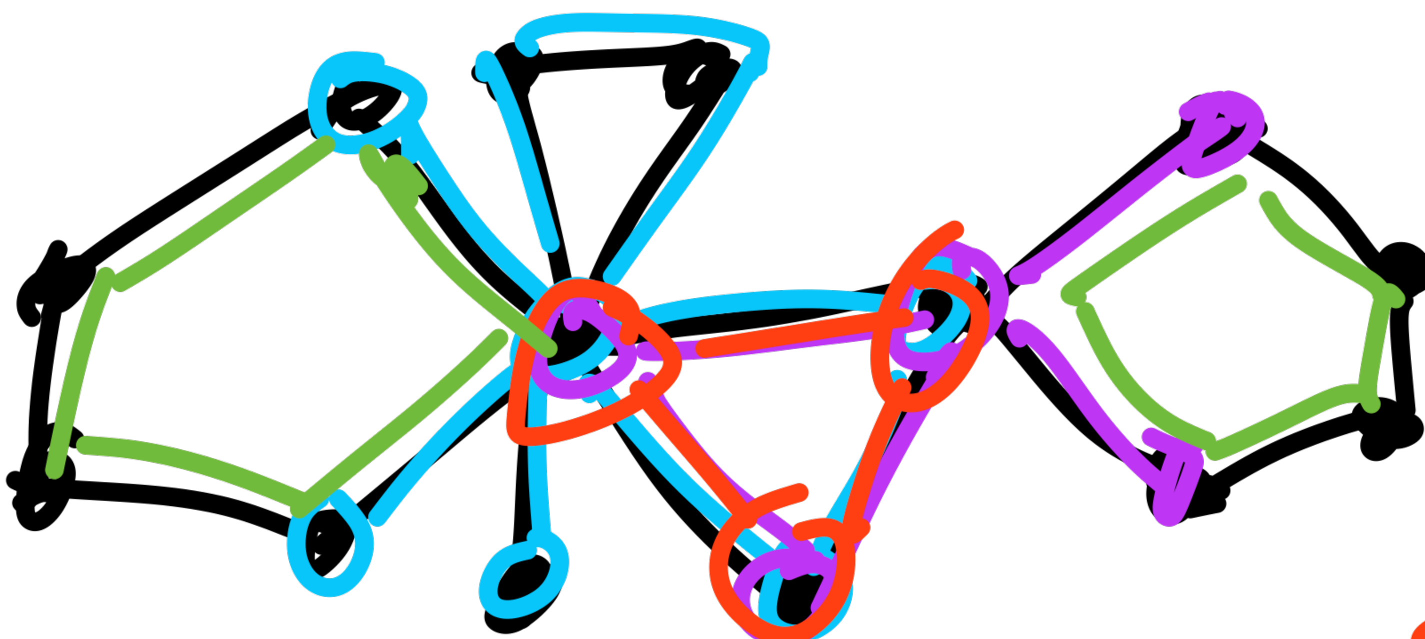


'rigid' in JSJ terms, but 'flexible' for QI

$$A(5) \overset{\mathbb{Z}\times F_2}{\text{———}} \mathbb{Z}\times(\mathbb{Z}^2 * \mathbb{Z}^2 * F_3) \overset{\mathbb{Z}^3}{\text{———}} \mathbb{Z}^3 \overset{\mathbb{Z}^3}{\text{———}} \mathbb{Z}\times(\mathbb{Z}^2 * F_2) \overset{\mathbb{Z}\times F_2}{\text{———}} A(5)$$

rigid — cylinder — cylinder — rigid

Notice:

- Have a different type of flexible vertex, ‘rigid’ $\mathbb{Z}^n$ instead of hanging.
- Graph of cylinders is ***not*** a JSJ decomposition. Not all edge stabilizers are 2-ended.
- Richer peripheral pattern; consider ***both*** intersection of vertex group with adjacent cylinder group and the parallel family of separating quasilines it contains.
- All of the relevant vertex stabilizers and separating quasilines are standard subgroups.

What about rigidity?

$\mathrm{Out}(A(\Gamma))$ is generated by:

- Graph isomorphism
- Generator inversion
- Partial conjugation: if $St(v)$ separates Γ and Γ' is a component then send generators $w \in \Gamma'$ to $v^{-1}wv$ and fix the other generators.
- Transvection: If $lk(w) \subset St(v)$ send $w \mapsto wv$ and fix the other generators.

The subgroup generated by the first two types is finite, while elements of the last two types have infinite order.

Bestvina-Kleiner-Sageev: Atomic RAAGs are QI if and only if isomorphic.

$\Downarrow$

Huang I: RAAGs with finite outer automorphism groups are QI if and only if isomorphic.

$\Downarrow$

Huang II: If $\mathrm{Out}(A(\Gamma))$ does not contain a nonadjacent transvection and $A(\Gamma')$ is QI to $A(\Gamma)$ then they are commensurable.

Idea of proof of Huang I:

Kim-Koberda: Define *extension complex* where $k-$dim simplex corresponds to coarse equivalence class of $(k+1)-$dim standard flat.

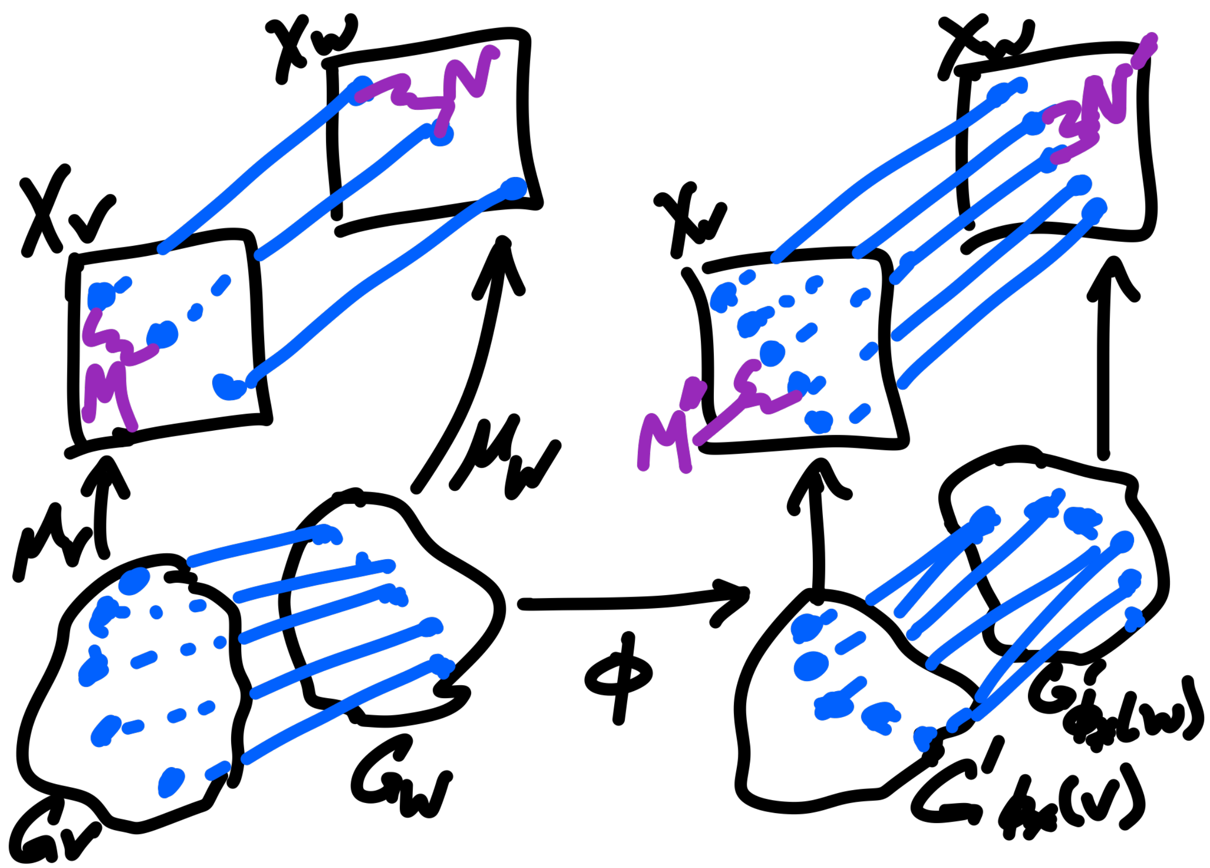
It is a quasitree on which $A(\Gamma)$ acts acylindrically.
“Curve complex for RAAGs”.

Idea of proof of Huang I:

1. Top dimensional flats map by QI to within bounded Hausdorff distance of top dimensional flats.
2. No transvections $\Longrightarrow$ standard flats map by QI to within bounded Hausdorff distance of standard flats $\Longrightarrow$ QI induces isomorphism of extension complex.
3. If $\mathrm{Out}(A(\Gamma))$ is finite, can reconstruct from isomorphism of extension complexes an isomorphism of universal covers of Salvetti complexes.

Back to JSJs

Recall:



$$\frac{M}{N} = \frac{M'}{N'}$$

Motivation for definition of relative rigidity property was to define QI invariant stretch factors.

Margolis:

1. Just need the conclusion, not the hypothesis.

2. Using arguments from Huang II, the conclusion is true for ***standard geodesics,*** provided $A(\Gamma)$ is Type II centerless.

Type II: Γ connected and no pair v, w such that $lk(v) \cap lk(w)$ separates Γ.

Margolis:

RAAG is *dovetail* if every standard geodesic is either rigid or flexible.

For dovetail RAAGs there is a version of structure invariant that is complete QI invariant.

In particular, if all noncylindrical vertices of graph of cylinders are either free Abelian or finite Out, there is algorithm to compute this invariant, so QI problem is solved for this case.

Status of QI problem for RAAGs

- By Papasoglu-Whyte, reduce to case defining graph is connected with more than one vertex.

- Abelian case.

- Two finite Out RAAGs are QI if and only if same defining graph.

- If $A(\Gamma)$ is Type II then $A(\Gamma')$ is QI if and only if commensurable, and this is algorithmically checkable.

- If two RAAGs are of the form: every noncylindrical vertex in tree of cylinders is either Abelian or finite Out then QI is algorithmically checkable.

- If every RAAG is dovetail then QI problem reduces to understanding relative QI problem for rigid, infinite Out vertex RAAGs.

What remains to be done?

- Extend relative QI algorithm to work for type II. This would finish the QI problem for the case that the only violations of the type II condition are due to separating vertices.

- More complicated type II violations:

  A. $lk(v) \cap lk(w)$ contains a separating complete graph. This gives splitting over Abelian subgroup. Groves-Hull+Margolis says there is QI invariant visual JSJ over Abelian groups, so can get QI invariants from a structure invariant. Are these complete invariants? ‘Stretch factors’ should now be matrices. To what extent are they QI invariant?

  B. $lk(v) \cap lk(w)$ contains a more complicated separating graph. This case is wide open. Need a theory of JSJ decompositions over fundamental groups of special cube complexes.