On co-σ-porosity of the parameters with dense critical orbits for skew tent maps and matching on generalized β-transformations.

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Abstract

We prove that the critical point and the point 1 have dense orbits for Lebesgue-a.e., parameter pairs in the two-parameter skew tent family and generalised β-transformations. As an application, we show that for the generalised β-transformation with the tribonacci number as slope, there is matching (i.e., $T^n(0) = T^n(1)$ for some $n \geq 1$) for Lebesgue-a.e. translation parameter.

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1 Introduction

Tent maps and β-transformations are among the simplest interval maps that exhibit topologically chaotic behaviour, whilst having an absolutely continuous invariant measure μ (acip), provided their slopes are greater than one in absolute value. The orbit of the critical point (for the tent map) and the orbit of 1 (for β-transformations) are the most important because they delimit every other orbit. Several paper has been devoted to whether this orbit is dense, or even typical w.r.t. μ (i.e., the Birkhoff Ergodic Theorem applies to this orbit), for a prevalent set of parameters. Schmeling [12] showed that for the standard β-transformation $x \mapsto \beta x \pmod 1$, the orbit of 1 is typical w.r.t. the acip μ_β Lebesgue almost every slope $\beta > 1$. His proof relies on dimension-theoretic arguments. Bruin [4], using inducing techniques, proved analogous results for the symmetric tent family $T_s(x) = \min\{sx, s(1-x)\}$, $s \in (1, 2]$, after previous results on denseness of the critical orbit by Brucks & Misiurewicz [2] and Brucks & Buckzolich [1], who improved "almost every parameter" to "a co-σ-porous parameter set".

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Definition 1.1 *A set $A \subset \mathbb{R}^n$ has porosity constant η if for every $a \in A$ and $r > 0$ there is $r' \in (0, r)$ and a ball $B(x; 2\eta r')$ which is contained in $B(a; r') \setminus A$. We call A porous if it has a positive porosity constant, and σ-porous if it is the countable union of porous sets A_n (and hence the porosity constants $\eta_n > 0$ are allowed to tend to 0 as $n \to \infty$), see [8, 13]. The complement of a σ-porous set is called* co-σ-porous.

Porous sets are nowhere dense and have no Lebesgue density points, so σ-porous sets in $\mathbb{R}^n$ are meager and have zero n-dimensional Lebesgue measure. However, they can have full Hausdorff dimension.

In this paper we study the problem for two-parameter families, namely the skew tent family $T_{\alpha,\beta}$ (named by Misiurewicz & Visinescu [9] and defined in (1) below) and the generalised β-transformations $G_{\alpha,\beta}(x) = \beta x + \alpha \pmod 1$. These generalised β-transformations were probably first studied by Parry [11], and Faller & Pfister [6] (using methods similar to Schmeling's) proved that the orbit 1 (and in fact every $x \in [0, 1]$), is typical for the acip for Lebesgue-a.e. parameter pair.

The strategy consists of fixing one for the parameters, and showing denseness of the critical orbit (or orbit of 1) for almost every value (or in cases a co-σ-porous set) of the other parameter. This falls slightly short of two-dimensional co-σ-porosity, which is left as an open problem. Also, the result is weaker than [6, Theorem 2], for generalised β-transformations, but whereas they fix α and vary β, we prove it the other way around, fixing β and varying α. This allows the following an application, namely that if the slope $\beta > 1$ is a multinacci numbers, i.e., $1 + \beta + \beta^2 + \cdots + \beta^{N-1} = \beta^N$ for some $N \geq 2$, then for Lebesgue-a.e. translation parameter $\alpha \in [0, 1]$, there is $n \geq 1$ such that $G^n_{\alpha,\beta}(0) = G^n_{\alpha,\beta}(1)$. This property is called *matching* and has been studied in [5], where it is shown that matching occurs for all quadratic Pisot slopes and a set of translation parameters whose complement has Hausdorff dimension < 1. The tribonacci slope (i.e., $1 + \beta + \beta^2 = \beta^3$) is also treated there, but the multinacci case is still open.

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2 Skew tent maps

The skew tent maps $T_{\alpha,\beta} : [0, 1] \to [0, 1]$ are given by

$$T_{\alpha,\beta}(x) = \begin{cases} \frac{\beta}{\alpha}\, x, & x \in [0, \alpha], \\ \frac{\beta}{1-\alpha}\,(1 - x), & x \in [\alpha, 1], \end{cases} \tag{1}$$

for $0 \leq \max\{\alpha, 1 - \alpha\} < \beta \leq 1$.

Fix $\alpha \in (0, 1)$ and let $\xi_n(\beta) = T^n_{\alpha,\beta}(\alpha)$, where we note that α is the critical point of the skew tent map. We call $\xi_n : I \to [0, 1]$ a *branch of* ξ_n whenever I is a maximal interval on which ξ_n is monotone. Let

$$Q_n(\beta) := \frac{\xi_n'(\beta)}{\frac{\partial}{\partial x} T^n_{\alpha,\beta}(\alpha^-)},$$

where $\frac{\partial}{\partial x}$ denotes the space derivative and α^- the left limit $\lim_{x \nearrow \alpha}$. We can compute that $Q_1(\beta) = \frac{\alpha}{\beta}$ and $Q_2(\beta) \geq \min\{\frac{1}{\alpha}, \frac{1}{1-\alpha}\}$. We have the recursive formula

$$
\begin{aligned}
Q_n(\beta) &= \frac{\frac{\partial}{\partial\beta}T_{\alpha,\beta}(\xi_{n-1}(\beta)) + \frac{\partial}{\partial x}T_{\alpha,\beta}(\xi_{n-1}(\beta))\xi'_{n-1}(\beta)}{\frac{\partial}{\partial x}T_{\alpha,\beta}(\xi_{n-1}(\beta))\frac{\partial}{\partial x}T^{n-1}_{\alpha,\beta}(\alpha^-)} \\
&= \frac{\frac{\partial}{\partial\beta}T_{\alpha,\beta}(\xi_{n-1}(\beta))}{\frac{\partial}{\partial x}T^{n}_{\alpha,\beta}(\alpha^-)} + Q_{n-1}(\beta).
\end{aligned}
$$

Since $\left|\frac{\partial}{\partial x}T^n_{\alpha,\beta}(\alpha^-)\right| \geq \lambda^n$ for $\lambda := \min\{\frac{\beta}{\alpha}, \frac{\beta}{1-\alpha}\} > 1$, and $\left|\frac{\partial}{\partial\beta}T_{\alpha,\beta}(\xi_{n-1}(\beta))\right| \leq \max\{\frac{1}{\alpha}, \frac{1}{1-\alpha}\}$, the sequence $(Q_n(\beta))_{n\geq 1}$ is a Cauchy sequence that converges exponentially fast to its limit Q, and we can check that $Q > 0$.

Lemma 2.1 *There are $u, C > 0$, depending only on $\alpha \in (0,1)$, such that for every n and branch $\xi_n : I \to [0,1]$ and every $\beta_1, \beta_2 \in I$,*

$$
\left|\frac{\xi'_n(\beta_2)}{\xi'_n(\beta_1)} - 1\right| < Ce^{-un}.
$$

Proof. By the exponential convergence of Q_n together with the exponential growth of $\frac{\partial}{\partial x}T^n_{\alpha,\beta}(\alpha^-)$, we know that $|I|$ is exponentially small in n. We can assume that $\beta_1 < \beta_2$, so $\Delta\beta := \beta_2 - \beta_1 \leq |I|$ is positive and also exponentially small. In general,

$$
\frac{\partial}{\partial x}T^n_{\alpha,\beta}(\alpha^-) = \beta^n \left(\prod_{j=0}^{n-1} a(\xi_j(\beta))\right)^{-1}, \qquad a(x) = \begin{cases} \alpha, & \text{for } x < \alpha, \\ 1-\alpha, & \text{for } x > \alpha. \end{cases}
$$

is exponentially large because $\beta > \max\{\alpha, 1-\alpha\}$. Therefore there are exponentially small errors $\varepsilon_{2,n}$ and $\varepsilon_{1,n}$ such that

$$
\begin{aligned}
\frac{\xi'_n(\beta_2)}{\xi'_n(\beta_1)} &= \frac{Q\frac{\partial}{\partial x}T^n_{\alpha,\beta_2}(\alpha^-) + \varepsilon_{2,n}}{Q\frac{\partial}{\partial x}T^n_{\alpha,\beta_1}(\alpha^-) + \varepsilon_{1,n}} = \left(\frac{\beta_2}{\beta_1}\right)^n \frac{Q + \varepsilon_{2,n}\beta_2^{-n}}{Q + \varepsilon_{1,n}\beta_1^{-n}} \\
&= \left(1 + \frac{\Delta\beta}{\beta_1}\right)^n \frac{Q + \varepsilon_{2,n}\beta_2^{-1}}{Q + \varepsilon_{1,n}\beta_1^{-n}} = 1 + O(n\Delta\beta).
\end{aligned}
$$

This proves the lemma. □

We continue to fix parameter α. Let $W_{n-1} = W_{n-1}(\beta)$ be te maximal neighbourhood of $\beta = T_{\alpha,\beta}(c)$ on which $T^{n-1}_{\alpha,\beta}$ is monotone, and $M_n = M_n(\beta) = T^{n-1}_{\alpha,\beta}(W_{n-1})$.

Lemma 2.2 *For $n \geq 1$, there are integers $1 \leq r_n, \tilde{r}_n < n$ such that $T^{n-1}_{\alpha,\beta}(\partial W_{n-1}) = \{T^{n-r_n}_{\alpha,\beta}(c), T^{n-\tilde{r}_n}(c)\}$*

Proof. Let $\beta \in W_{n-1} =: [b_{n-1}, \tilde{b}_n]$. By maximality of W_n, there is $r_n < n$ such that $T^{r_n-1}(b_{n-1}) = c$ and so $T^n_{\alpha,\beta}(b_{n-1}) = T^{n-r_n}_{\alpha,\beta}(c)$. Likewise for $\tilde{b}_{n-1}$ and $\tilde{r}_n$. In fact, in terms of cutting times $\{S_k\}_{k\geq 0}$ and co-cutting times $\{\tilde{S}_l\}_{l\geq 0}$, $b_n = n - \max\{S_k : S_k < n\}$ and $\tilde{b}_n = n - \max\{\tilde{S}_l : S_l < n\}$, see [3]. □

This lemma is rather trivial, but it introduces the notation for the next lemma. Given $n \geq 4$, let $Z_n(\beta)$ be the maximal interval containing the critical value β such that $T^{n-1}_{\alpha,\beta}$ is monotone on $Z_n(\beta)$. (If β is the common boundary point of two such interval, choose the left one.) Since $|\frac{\partial}{\partial x}T^n_{\alpha,\beta}|$ is exponentially large and due to Lemma 2.1, $Z_n(\beta)$ is exponentially small.

Lemma 2.3 *Let β_n and $\tilde{\beta}_n$ be the boundary points of $Z_n(\beta)$. Then (possibly after swapping β_n and $\tilde{\beta}_n$) we have $\xi_n(\beta_n) = \xi_{n-r_n}(\beta_n)$ and $\xi_n(\tilde{\beta}_n) = \xi_{n-\tilde{r}_n}(\tilde{\beta}_n)$. Moreover the quotient*

$$q : \beta' \mapsto \frac{\xi_n(\beta') - \xi_{n-r_n}(\beta')}{\xi_{n-\tilde{r}_n}(\beta') - \xi_{n-r_n}(\beta')}$$

is a monotone map from $Z_n(\beta)$ onto $[0, 1]$.

Proof. By maximality of $Z_n(\beta)$, at the boundary points β_n and $\tilde{\beta}_n$ of $Z_n(\beta)$, there must be integers r_n and $\tilde{r}_n$ such that $\xi_{r_n}(\beta_n) = c = \xi_{\tilde{r}_n}(\tilde{\beta}_n)$. Now as β' moves through the interior of $Z_n(\beta)$, we have $c \in T^{r_n}_{\alpha,\beta}(\partial W_{n-1}(\beta'))$ and $c \in T^{\tilde{r}_n}_{\alpha,\beta}(\partial W_{n-1}(\beta'))$. This shows that r_n and $\tilde{r}_n$ depend only on $Z_n(\beta)$ (that is, on β only), and by swapping the boundary points on $Z_n(\beta)$ if necessary, the integers r_n and $\tilde{r}_n$ are the same as those in Lemma 2.2. Finally, q is clearly continuous, and thus onto $[0, 1]$. Since the slope of ξ_n is larger than the slopes of ξ_{n-r_n} and $\xi_{n-\tilde{r}_n}$, the quotient q is indeed a monotone function. □

Given a point $x \in [0, 1] \setminus \{\alpha\}$, let the involution $\hat{x}$ be the point different from x such that $T_{\alpha,\beta}(\hat{x}) = T_{\alpha,\beta}(x)$. Let $p = \frac{\beta}{1-\alpha+\beta} \in [\alpha, \beta]$ be the orientation reversing fixed point of $T_{\alpha,\beta}$. Note that $T^2_{\alpha,\beta}(\alpha) < \hat{p} < p < T_{\alpha,\beta}(\alpha)$.

Proposition 2.1 *Let $T_{\alpha,\beta}$ be a skew tent map with $0 < \min\{\alpha, 1-\alpha\} < \beta \leq 1$. Then there exists $\eta \in (0, 1)$ and arbitrary small neighbourhoods J of the critical point $c = \alpha$ for which there are intervals $H \subset J$ with $|H| \geq \eta |J|$ and $n \in \mathbb{N}$ such that $T^n_{\alpha,\beta}$ maps H monotonically onto $[\hat{p}, p]$.*

Proof. Let $J_0 = [\hat{p}, p]$ and $H_0 \subset [c, p]$ be such that $T^2_{\alpha,\beta}(H_0) = [\hat{p}, p]$. First we assume that the critical point c is recurrent; the proof of the proposition is simple otherwise.

We construct neighbourhoods $J_k \ni c$ with subintervals H_k and $\hat{H}_k$ adjacent to the endpoints of J_k inductively. We always set the ratio

$$r_k = \frac{|H_k|}{|L_k|} = \frac{|\hat{H}_k|}{|\hat{L}_k|} = \frac{|\hat{H}_k \cup \hat{H}_k|}{|J_k|},$$

where L_k is the component of $J_k \setminus \{c\}$ containing H_k and $\hat{L}_k$ is the other component.

Suppose J_k and $H_k, \hat{H}_k \subset J_k$ are known and assume by induction that

$$\operatorname{orb}(\partial J_k) \cap \mathring{J}_k = \emptyset, \tag{2}$$

where $^\circ$ denotes the interior. Let $m_k = \min\{n \geq 1 : T^n_{\alpha,\beta}(c) \in J_k \setminus (H_k \cup \hat{H}_k)\}$, and let J'_{k+1} be the maximal neighbourhood of c such that $T^{m_k}_{\alpha,\beta}(J'_{k+1}) \subset J_k$. By the inductive assumption, this means that $T^{m_k}_{\alpha,\beta}(\partial J'_{k+1}) \subset \partial J_k$.

We claim that there are $C, u > 0$ independently of k such that

$$\frac{|J'_{k+1}|}{|J_k|} \leq C e^{-uk}. \tag{3}$$

This is because, for any $x \in J'_{k+1} \setminus \{c\}$ and J_x the component of $J'_{k+1} \setminus \{c\}$ containing x,

$$\left| \frac{\partial}{\partial x} T^{m_k}_{\alpha,\beta}(x) \right| = \frac{|T^{m_k}_{\alpha,\beta}(J'_{k+1})|}{|J_x|} \leq \frac{|J_k|}{|J'_{k+1}|}.$$

But the derivative $\frac{\partial}{\partial x}T^{m_k}_{\alpha,\beta}$ is exponentially large and $n_k \geq k$, so there are $C, u > 0$ depending only on α, β such that $\frac{|J'_{k+1}|}{|J_k|} \leq Ce^{-uk}$ which is (3). As a consequence, $\eta := r_0 \prod_{i=0}^{\infty} \left(1 - \frac{|J'_{i+1}|}{|J_i|}\right) > 0$.

Now there are two cases:

- $T^{m_k}_{\alpha,\beta}(J'_{k+1}) \not\supset J'_{k+1}$. In this case, we set $n_k = m_k$, $J_{k+1} = J'_{k+1}$ and by (2) we have

$$\text{orb}(\partial J_{k+1}) \cap \overline{J_k} \subset \partial J_k. \tag{4}$$

  Thus we can take H_{k+1} and $\hat{H}_{k+1} \subset J_{k+1}$ such that $T^{n_k}_{\alpha,\beta}(H_{k+1}) = T^{n_k}_{\alpha,\beta}(\hat{H}_{k+1})$ equals H_k or $\hat{H}_k$, say it is H_k and J_k^+ is the component of $J_k \setminus \{c\}$ containing H_k. Because the branches of $T^{n_k}_{\alpha,\beta}|_{J_{k+1}}$ are affine

$$r_{k+1} = \frac{|H_k|}{|T^{n_k}_{\alpha,\beta}(J_{k+1})|} \geq \frac{|H_k|}{|L_k \cup J_{k+1}|} \geq r_k \frac{|L_k|}{|L_k \cup J_{k+1}|} \geq r_k \left(1 - \frac{|J'_{k+1}|}{|J_k|}\right). \tag{5}$$

- $T^{m_k}_{\alpha,\beta}(J'_{k+1}) \supset J'_{k+1}$. In this case choose $n_k = \min\{n \geq 1 : T^n_{\alpha,\beta}(c) \in J'_{k+1}\}$ and let J_{k+1} be the maximal neighbourhood of c such that $T^{n_k}_{\alpha,\beta}(J_{k+1}) \subset J_k$. By (2) and (4), again $\text{orb}(\partial J_{k+1}) \cap \mathring{J}_{k+1} = \emptyset$ and $T^{n_k}_{\alpha,\beta}(\partial J_{k+1}) \subset \partial J_k$. Thus we can take H_{k+1} and $\hat{H}_{k+1} \subset J_{k+1}$ such that $T^{n_k}_{\alpha,\beta}(H_{k+1}) = T^{n_k}_{\alpha,\beta}(\hat{H}_{k+1})$ equals H_k or $\hat{H}_k$. Also here (5) can be verified in the same way.

This concludes the inductive construction, and we have

$$T^{N_k}_{\alpha,\beta}(H_k) = T^{N_k}_{\alpha,\beta}(\hat{H}_k) = [\hat{p}, p] \qquad \text{for } N_k := n_k + n_{k-1} + \cdots + n_0 + 2.$$

By (5) also $r_k \geq r_0 \prod_{i=0}^{k-1} \left(1 - \frac{|J'_{i+1}|}{|J_i|}\right) \geq \eta$, so the proposition follows. □

Lemma 2.4 *The set $A = \{(\alpha, \beta) : orb(c)$ is not dense for $T_{\alpha,\beta}\}$ is a Borel set.*

Proof. Let $\{U_j\}_{j\in\mathbb{N}}$ be countable basis of the topology on $[0,1]$. Then $\xi_n^{-1}([0,1] \setminus U_j)$ is closed and $A = \cap_j \cup_n \xi_n^{-1}([0,1] \setminus U_j)$ is Borel. □

Theorem 2.1 *The set of parameters (α, β) for which the critical point of $T_{\alpha,\beta}$ has a non-dense orbit has zero Lebesgue measure.*

Proof. First fix $\alpha \in (0,1)$. Then we show that the set of parameters β for which the critical point of $T_{\alpha,\beta}$ has a non-dense orbit is σ-porous.

Let $\{U_j\}_{j\in\mathbb{N}}$ be countable basis of the topology on $[0,1]$. Fix α and let $A_j = \{\beta : T^n_{\alpha,\beta}(c) \notin U_j$ for all $n \geq 1\}$. We first look at the sets U_j that contain the critical point, so $\beta \in A_j$ means that c is not recurrent for $T_{\alpha,\beta}$. Fix $\beta \in A_j$ and define $Z_n(\beta) \ni \beta$ to be the maximal neighbourhood of β on which ξ_n is monotone. Now take n arbitrary such that $\xi_n(Z_n(\beta)) \ni c$. Since $T^m_{\alpha,\beta}(c) \notin U_j$ for all $m \leq n$, $\xi_n(Z_n) \supset U_j$ and by Lemma 2.1, $|\xi_n^{-1}(U_j)|/|Z_n| \geq \frac{1}{2}|U_j|$. Since n can be taken arbitrarily large A_j is porous. Hence the set of β such that c is not recurrent $\cup_{c\in U_j} A_j$ is σ-porous.

So for the rest of the proof we can assume that c is recurrent and we consider the U_js that don't contain c. We call n a closest approach time if $|\xi_{n+1}(\beta) - \beta| < |\xi_{m+1}(\beta) - \beta|$ for all $m < n$. Let n' such a time and pick k maximal such that $J_k \ni \xi_n(\beta)$, where J_k

are the intervals in Proposition 2.1. Once this k is fixed we can take the smallest closest approach time $n \le n'$ such that $\xi_n(\beta) \in J_k$. Then $\xi_n(Z_n(\beta)) \supset J_k \supset H_k$. Therefore $|\xi_n^{-1}(H_k)|/|\xi_n^{-1}(J_k)| \ge \eta$ and therefore $|\xi_n^{-1}(H_k \cap T_{\alpha,\beta}^{-N_k}(U_j))|/|\xi_n^{-1}(J_k)| \ge \eta|U_j|$. Since n can be taken arbitrarily large A_j is porous. Therefore the set of β for which c is recurrent but its orbit avoids some U_j is σ-porous too.

Recall that σ-porous sets have zero Lebesgue measure. Because we are speaking of Borel sets (see Lemma 2.4), the result for all α follows from Fubini's Theorem. □

3 Generalised β-transformations

The generalised β-transformation $G_{\alpha,\beta} : [0,1] \to [0,1]$ is given by

$$G_{\alpha,\beta}(x) = \beta x + \alpha \pmod 1,$$

for $\alpha \in [0,1]$ and $\beta > 1$. Due to the symmetry $G_{\alpha,\beta}(1-x) = 1 - G_{1-(\alpha+\beta \pmod 1)),\beta}(x)$ it suffices to study only parameters $\alpha \le (1 + \lfloor \alpha+\beta \rfloor - \beta)/2$.

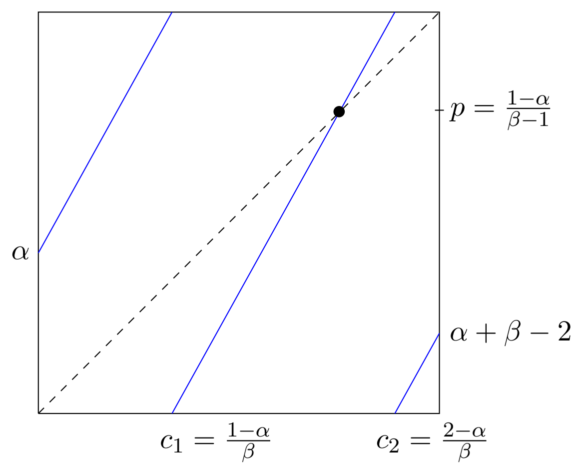

Figure 1: The generalised β-transformation with some important points indicated.

When we consider this as map on the circle $[0,1]/\sim$, then it has a single discontinuity point $c = 0 = 1$. We call the left and right limit of the images $G^k_{\alpha,\beta}(c^-)$ and $G^k_{\alpha,\beta}(c^+)$.

Lemma 3.1 *For each $\beta > 1$ and $\alpha \in [0,1]$, $G_{\alpha,\beta}$ has a unique smallest invariant union $V_{\alpha,\beta}$ of non-trivial intervals. Furthermore, for every $\delta > 0$ there is L such that $\bigcup_{j=0}^{L-1} G^j_{\alpha,\beta}(M) = V_{\alpha,\beta}$ for every interval M of length $|M| \ge \delta$.*

Proof. If J is a non-trivial interval, that $|G_{\alpha,\beta}(J)| = \beta|J|$ unless 0 is an interior point of J. Hence J keeps growing under iteration of $G_{\alpha,\beta}$ until it contains 0 in its interior, and in particular, there cannot be two disjoint $G_{\alpha,\beta}$-invariant unions of intervals. We denote the smallest such by $V_{\alpha,\beta}$.

Now for the second statement, choose $r \in \mathbb{N}$ such that $\beta^r \ge 4$. If $\alpha = 0$, then $G_{\alpha,\beta}$ is the normal β-transformation and there is nothing to prove. So take $\alpha \in (0,1)$ and choose $\varepsilon > 0$ such that $G^j_{\alpha,\beta}(B(0;\varepsilon)) \not\ni 0$ for $0 < j < r$. This implies that if the interval $M \subset B(0;\varepsilon)$ and $j \ge 1$ are such that $G^j_{\alpha,\beta}(M) \ni 0$, then $|G^j_{\alpha,\beta}(M)| \ge 2|M|$. For general intervals $M \subset V_{\alpha,\beta}$ of length $|M| \le \varepsilon$, we obtain $|G^j_{\alpha,\beta}(M)| \ge 2^{j/r}|M|$ as long as $G^j_{\alpha,\beta}(M)|$ does not contain a component of $B(0;\varepsilon) \setminus \{0\}$.

If x is a left endpoint of (a component of) $V_{\alpha,\beta}$, then at least one point of $G^{-1}_{\alpha,\beta}(x)$ is equal to c or a left endpoint of $V_{\alpha,\beta}$. If the latter is true for all left endpoints, then these left endpoints contain a periodic point, say of period m, which is expanding. Hence $G^m_{\alpha,\beta}(p+\eta) > p+\eta > p$ for all sufficiently small $\eta > 0$, and therefore $V_{\alpha,\beta} \setminus \bigcup_{j=0}^{m-1}[G^j_{\alpha,\beta}(p), G^j_{\alpha,\beta}(p+\eta))$ is forward invariant, contradicting the minimality of $V_{\alpha,\beta}$. The same argument applies to the right endpoints. Therefore $\partial V_{\alpha,\beta}$ is contained in the forward orbit of the left and right limit of the discontinuity point: there is $L_0 \in \mathbb{N}$ such that $\partial V_{\alpha,\beta} \subset \cup_{j=0}^{L_0} G^j_{\alpha,\beta}(\{\alpha, \alpha+\beta \pmod 1\})$. Since $G_{\alpha,\beta}$ is expanding, we can find $L_1 \in \mathbb{N}$ such that

$$V_{\alpha,\beta} \subset \bigcup_{j=0}^{L_1} G^j_{\alpha,\beta}([0,\varepsilon]) \quad \text{and} \quad V_{\alpha,\beta} \subset \bigcup_{j=0}^{L_1-1} G^j_{\alpha,\beta}([-\varepsilon,0]).$$

Then the claimed property holds for $L(\delta) = L_1 - \log_2 \delta^r$. □

In [6] it is shown that for every $\alpha \in [0,1]$ and $x \in [0,1]$, the set of $\beta > 1$ such that x is a typical point w.r.t. the measure of maximal entropy (i.e., the absolutely continuous invariant probability measure (acip)) of $G_{\alpha,\beta}$ has full Lebesgue measure. From this it follows that for Lebesgue-a.e. pair (α,β), the point 0 is typical, and in particular has a dense orbit in $V_{\alpha,\beta}$ from Lemma 3.1. This is in many ways stronger than what we will prove, but for our purposes later on, it is important to first fix β (and $x = 0$ but any other x would work equally well) and then vary α. In this way, we can use particular values of β, such as Pisot numbers. Namely, if $\beta > 1$ that are Pisot numbers, the techniques to prove this result can also be used, to prove that $G_{\alpha,\beta}$ has matching for a full measure set of α, cf. [5]. For us, only the typical denseness of the orbit of 0 is of interest, not the stronger property of being typical w.r.t. its own acip, nor shall we prove that the set of α with a non-dense orbit is σ-porous.

For the generalised β-transformations, $\frac{\partial}{\partial x} G^n_{\alpha,\beta}(x) = \beta^n$ and for fixed $\beta > 1$, $\xi_n(\alpha) := G^n_{\alpha,\beta}(0)$ has derivative $\xi_n'(\alpha) = \frac{\beta^n - 1}{\beta - 1}$. Therefore

$$Q_n(\alpha) := \frac{\xi_n'(\alpha)}{\frac{\partial}{\partial x} G^n_{\alpha,\beta}(0)} = \frac{(\beta^n - 1)}{\beta^n(\beta - 1)} \to \frac{1}{\beta - 1} \qquad \text{as} \quad n \to \infty,$$

and we can derive the same (uniform) distortion properties for ξ_n as for the tent-map case. In particular Lemma 2.1 holds.

For fixed $\beta > 1$, let $W_{n-1} = W_{n-1}(\alpha)$ be the maximal neighbourhood of $\alpha = G_{\alpha,\beta}(c^+)$ on which $T^{n-1}_{\alpha,\beta}$ is monotone.

Lemma 3.2 *For $n \ge 1$, there are integers $1 \le r_n^+, r_n^- < n$ such that $G^{n-1}_{\alpha,\beta}(\partial W_{n-1}) = \{G^{n-r_n^+}_{\alpha,\beta}(c^+), G^{n-r_n^-}_{\alpha,\beta}(c^-)\}$.*

The proof is analogous to that of Lemma 2.2 and thus omitted. The parallel result holds for the maximal neighbourhood of $\alpha + \eta \pmod 1 = G_{\alpha,\beta}(c^-)$, but we will not need it.

Given $n \ge 4$, let $Z_n(\alpha)$ be the maximal interval containing α such that ξ_{n-1} is monotone on $Z_n(\alpha)$. Since $|\frac{\partial}{\partial x} G^n_{\alpha,\beta}|$ is exponentially large and due to Lemma 2.1, $Z_n(\alpha)$ is exponentially small. The next lemma is the analogue of Lemma 2.3, proven in the same way.

Lemma 3.3 *Let α_n and $\tilde{\alpha}_n$ the boundary points of $Z_n(\alpha)$. Then (after swapping α_n and $\tilde{\alpha}_n$ if necessary) we have $\xi_n(\alpha_n) = \xi_{n-r_n^+}(\alpha_n)$ and $\xi_n(\tilde{\alpha}_n) = \xi_{n-r_n^-}(\tilde{\alpha}_n)$. Moreover the quotient*

$$q : \alpha' \mapsto \frac{\xi_n(\alpha') - \xi_{n-r_n^+}(\alpha')}{\xi_{n-r_n^-}(\alpha') - \xi_{n-r_n^+}(\alpha')}$$

is a monotone map from $Z_n(\alpha)$ onto $[0,1]$.

Proposition 3.1 *For any fixed $\beta > 1$ there is $\delta \in (0, 1/\beta)$ such that for Lebesgue-a.e. $\alpha \in [0,1]$, there is a sequence (n_i) such that $|\xi_{n_i}(Z_{n_i}(\alpha))| \geq \delta$.*

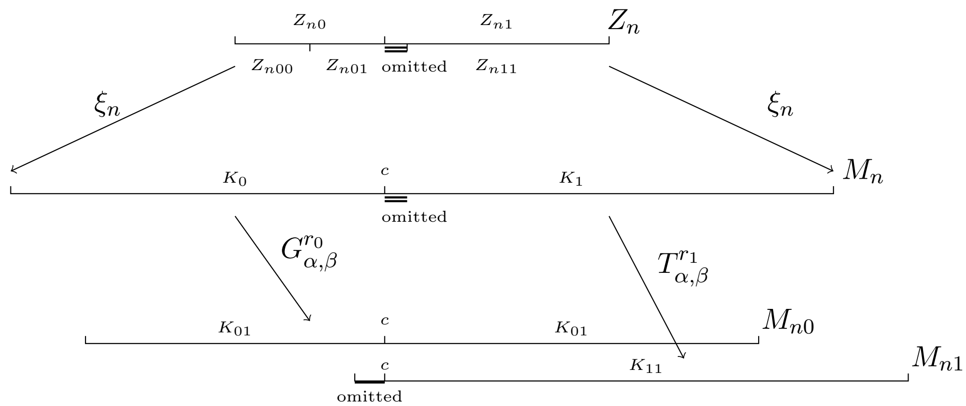

Figure 2: Intervals used in the proof of Theorem 3.1.

Proof. It suffices to show that no $\alpha \in [0,1]$ can be a density point of the set of parameters such that $\limsup_n |\xi_n(Z_n(\alpha))| = 0$. Take α_0 and $n \in \mathbb{N}$ arbitrary. We will show that for a definite (i.e., independent of n and α_0) fraction of the set $Z_n(\alpha_0)$, there is n' such that $|\xi_{n'}(Z_{n'}(\alpha)| > \delta$.

First set $M_n = \xi_n(Z_n(\alpha_0))$. Since $\xi_n' = \frac{\beta^n - 1}{\beta - 1}$ we have $|M_n| = \frac{\beta^n-1}{\beta-1}|Z_n(\alpha_0)| \geq C\beta^n |Z_n(\alpha_0)|$ for some $C > 0$. Without loss of generality we can assume that $c \in M_n$, and denote the two components of $M_n \setminus \{c\}$ by K_0 and K_1, and let $Z_{n0}(\alpha_0), Z_{n1}(\alpha_0) \subset Z_n(\alpha_0)$ be the subintervals such that $\xi_n(Z_{ni}(\alpha_0) = K_i$. If $|K_i| \leq C\beta^{n/2}|Z_n(\alpha_0)|$, then we omit $Z_{ni}(\alpha_0)$ from Z_n. Since $|K_0| + |K_1| \geq C\beta^n |Z_n(\alpha_0)|$, at most one of them can be omitted, and the omitted fraction is $\leq \beta^{-n/2}$.

Next let $r_i \in \mathbb{N}$ be the minimal integers such that $\xi_{n+r_i}(Z_{ni}) \ni c$ and set $M_{ni} = \xi_{n+r_i}(Z_{ni})$ with components K_{ij}, $j = 0, 1$, of $M_{ni} \setminus \{c\}$, and corresponding subintervals $Z_{nij}(\alpha_0) \subset Z_i(\alpha_0)$. Similar to the above, $M_{ni} = |K_{i0}| + |K_{i1}| = \beta^{r_i}|K_i|$ and we omit $Z_{nij}(\alpha_0)$ if $|K_{i0}| < \beta^{-r_i/2}|K_i|$. Thus the relative Lebesgue measure of omitted parameters in this round is $\leq \beta^{-r_1/2} := \min\{\beta^{-r_0/2}, \beta^{-r_1/2}\}$.

Continue inductively, until the images $M_{ni_1\dots i_k}$ are finally longer than δ. The non-omitted proportion is $\prod_m (1 - \beta^{-r_m/2}) \geq \exp(-\sum_m \beta^{-r_m/2})$. Since each next $M_{ni_1\dots i_m}$ is much larger than the previous $M_{ni_1\dots i_{m-1}}$, the sequence $(r_m)_m$ is strictly decreasing and naturally all the (finitely many) factors in the product are < 1. Hence the proportion of non-omitted parameters is always at least $\exp(-\sum_m \beta^{-m/2}) = e^{-1/(\sqrt{\beta}-1)} =: \eta > 0$, independently of α_0 and n.

For each non-omitted parameter $\alpha \in Z_n(\alpha_0)$ there is some $n' \leq n + r_1 + \cdots + r_k$ such that $|\xi_{n'}(Z_{n'}(\alpha))| > \delta$, and this concludes the proof. $\square$

In general, we would like to have the stronger statement where $\delta = 1$. This is not always possible. First, of course, the largest branch may not have length 1. If $0 < \alpha < \beta + \alpha < 2$, then the largest branch has length $\max\{1-\alpha, \alpha+\beta-1\} < 1$. Recall from Lemma 3.1 that $V_{\alpha,\beta} = \omega(x)$ for some $x \in [0,1]$.

But even so, the lack of topological mixing can prevent δ from being 1, see e.g. Theorems 4.5-4.8 and also Theorem 6.6 of [10]. However, with a single exception $\beta = \sqrt{2}$, $\alpha = (2-\sqrt{2})/2$, every **two-branched** $G_{\alpha,\beta}$ is topologically mixing for $\beta \geq \sqrt{2}$.

Theorem 3.1 *Recall the union of intervals $V_{\alpha,\beta}$ from Lemma 3.1. For every $\beta > 1$ and Lebesgue-a.e. $\alpha \in [0,1]$, the $G_{\alpha,\beta}$-orbit of $c = 0$ is dense in $V_{\alpha,\beta}$.*

Proof. Fix $\beta > 1$ and take $\delta > 0$ as in Proposition 3.1. Lemma 3.1 stated that there is $L = L(\delta) \in \mathbb{N}$ so that $\bigcup_{j=0}^{L-1} G^j_{\alpha,\beta}(M) = V_{\alpha,\beta}$ for every interval with diameter $|M| \geq \delta$. Let $\{U_k\}_k$ be a countable basis of the topology of $V_{\alpha,\beta}$. Then $\text{Leb}(\bigcup_{j=0}^{L-1} G^{-j}_{\alpha,\beta}(U_k) \cap M) \geq L^{-1}\beta^{-L}|U_k|$ for each k.

By Proposition 3.1, each neighbourhood $Z_n(\alpha_0)$ contains an η-proportion of points α such that $|Z_{n'}(\alpha)| > \delta$, and therefore also for an $\eta L^{-1}\beta^{-L}|U_k|$ proportion of points $\alpha \in Z_n(\alpha_0)$, the $G_{\alpha,\beta}$-orbit of 0 will visit U_k. Since α_0 is not a density point of the complement, it follows that for Lebesgue full measure set A_k of $\alpha \in [0,1]$, the $G_{\alpha,\beta}$-orbit of 0 will visit U_k. Now take $A = \cap_k A_k$. Then A has full Lebesgue measure, and the $G_{\alpha,\beta}$-orbit of 0 is dense in $V_{\alpha,\beta}$. □

3.1 Matching

In this section we show how the previous result can help in proving prevalent matching for generalised β-transformations with Pisot slopes. We say that $G_{\alpha,\beta}$ has *matching* if there is an iterate $\kappa \geq 1$, called *matching index* such that $G^\kappa_{\alpha,\beta}(0) = G^\kappa_{\alpha,\beta}(1)$, or, when viewed on the circle with discontinuity $c = 0$, $G^\kappa_{\alpha,\beta}(c^-) = G^\kappa_{\alpha,\beta}(c^+)$. It was shown in [5] that if β is a quadratic Pisot unit, then there is matching for Lebesgue almost every $\alpha \in [0,1]$. In fact, matching occurs on an open and dense set (*prevalent matching*) and the set of parameters where matching fails has Hausdorff dimension < 1.

It is expected that matching is prevalent for every Pisot slope β. Recall that $\beta > 1$ is a degree N *Pisot unit* if it is the leading root of an irreducible polynomial

$$P(\beta) = \beta^N - \sum_{i=0}^{N-1} a_i \beta^i, \qquad a_i \in \mathbb{Z}, \tag{6}$$

and all the algebraic conjugates of β lie strictly inside the unit disk.

The Pisot numbers we are trying to tackle are the multinacci numbers, i.e., the leading roots of the polynomials

$$P(\beta) = \beta^N - (\beta^{N-1} + \beta^{N-2} + \cdots + \beta + 1) = \beta^N - \frac{\beta^N - 1}{\beta - 1}. \tag{7}$$

Thus $\beta < 2$ (in fact, for $N = 2$, β is the golden mean, and for $N = 3$, $\beta = 1.8392867552\ldots$ is the tribonacci number) and $\beta \nearrow 2$ as $N \to \infty$. It can be easily computed that

$$1 = \beta^{-1} + \beta^{-2} + \cdots + \beta^{-N} \qquad \text{and} \qquad 2 - \beta = \beta^{-N}. \tag{8}$$

In [5] it was shown that for $N = 3$, i.e., the tribonacci number, the non-matching set has Hausdorff dimension < 1. For all $N \geq 4$, prevalence of matching is still an open question.

Let $p = \frac{\alpha-1}{\beta-1}$ be the fixed point. Due to symmetry, we can assume that $T(0) \leq p$, i.e., $\alpha \leq \frac{1-\alpha}{\beta-1}$ or equivalently $\alpha \leq \beta^{-1}$. If $\alpha \leq 2 - \beta$, then $G_{\alpha,\beta}$ has only two branches on $[0, 1]$. In this case, for $1 \leq n < N$, we have

$$G^n_{\alpha,\beta}(0) = \alpha\frac{\beta^n - 1}{\beta - 1} \leq \alpha\frac{\beta^n - 1}{\beta - 1} + \beta^n - \frac{\beta^n - 1}{\beta - 1} = G^n_{\alpha,\beta}(1),$$

and therefore (using from (7) that $\beta^N = \frac{\beta^N-1}{\beta-1}$) there is matching at step N.

From now on, take $\alpha > 2 - \beta$ and define

$$d(n) := |G^n_{\alpha,\beta}(1) - G^n_{\alpha,\beta}(0)| = \sum_{i=1}^{N} e_i(n)\,\beta^{-i}, \qquad e_i(n) \in \{0, 1\},$$

so $d(1) = |\alpha + \beta - 2 - \alpha| = 2 - \beta = \beta^{-N}$ by (8). The iteration of $d(n)$ is given by

$$d(n+1) = \begin{cases} \sum_{i=1}^{N-1} e_{i+1}(n)\beta^{-i} & \text{if this is positive;} \\ \beta^{-N} + \sum_{i=1}^{N-1}(1 - e_{i+1}(n))\beta^{-i} & \text{otherwise.} \end{cases}$$

That is: we either shift the string $e = (e_1, \dots, e_N)$ or shift it and swap all 0s to 1s and vice versa. In particular, if $e(n) = e_1 00 \dots 0$, then $e(n + 1) = 000 \dots 0$ and $d(n + 1) = 0$, so we have matching. This is easy to see by noting that $G^n_{\alpha,\beta}(0)$ and $G^n_{\alpha,\beta}(1)$ lie $|e_1|/\beta$ apart so their images are the same. Therefore, if

$$G^{n+i}_{\alpha,\beta}(0) - G^{n+i}_{\alpha,\beta}(1) \text{ doesn't change sign for } 0 \leq i < N, \tag{9}$$

there is matching for some $i < N$. Converse, if $G^n_{\alpha,\beta}(0) - G^n_{\alpha,\beta}(1)$ has just switched sign, so $e_N(n) = 1$, then matching after N step implies (9).

It suffices to find an interval U such that if $G^m_{\alpha,\beta}(0) \in U$, then (9) holds for some $n \geq m$. Indeed, if such U exists, then Theorem 3.1 implies that for a.e. α, there is indeed m such that $G^m_{\alpha,\beta}(0) \in U$. Taking this viewpoint, we give a simpler proof of prevalence of matching than provided by [5, Theorem 5.1].

Proposition 3.2 *The generalised β-transformation $G_{\alpha,\beta}$ with β the tribonacci number has matching for Lebesgue-a.e. $\alpha \in [0, 1]$.*

Proof. As mentioned before, there is matching if T has only two branches, so we assume $\alpha > 2 - \beta$. If α is still so small that the fixed point $p = \frac{1-\alpha}{\beta-1} > 1 - \beta^{-N} = \beta - 1$, and if $G^n_{\alpha,\beta}(0)$ is very close to p, then also (9) holds for the next N steps, because there is no place in $[p, 1]$ for $G^{n+i}_{\alpha,\beta}(1)$. Combined with Theorem 3.1, this means that we have almost sure matching for $\alpha \in [0, \beta(2 - \beta)]$.

So from now on we assume that $\beta^{-1} > \alpha > \beta(2 - \beta) = \beta^{1-N}$, where the equality follows by (8). These assumptions give (recalling that $c_1 = \frac{1-\alpha}{\beta}$)

$$\frac{1}{\beta^2} < p - c_1 \quad = \quad \frac{1 - \alpha}{\beta(\beta - 1)} < \frac{1 - \beta^{1-N}}{\beta(\beta - 1)} = \frac{\beta - 1}{\beta}, \tag{10}$$

where the last equality follows since $1 - \beta^{1-N} = \beta^{1-N}(\beta - 1)(\beta^{N-2} + \beta^{N-3} + \dots + 1) = (\beta - 1)(\beta^{-1} + \dots + \beta^{-N} - \beta^{-N}) = (\beta - 1)(1 - \beta^{-N}) = (\beta - 1)^2$ by (8). Therefore

$$\frac{1}{\beta} - \frac{1}{\beta^2} > \frac{1}{\beta} - (p - c_1) = c_2 - p > \frac{2 - \beta}{\beta} = \frac{1}{\beta^{N+1}} \tag{11}$$

by (8). Also note that

$$\hat{p} := p - \frac{1}{\beta} < c_1 = \frac{1-\alpha}{\beta} < p = \frac{1-\alpha}{\beta - 1} < c_2 = \frac{2-\alpha}{\beta} < 1,$$

so $p - c_1 < \frac{1}{\beta}$.

Assume that $G^n_{\alpha,\beta}(0) = p$ (the case $G^n_{\alpha,\beta}(1) = p$ goes likewise). Since $G_{\alpha,\beta}(0) = \beta - 1 < \frac{1-\alpha}{\beta-1} = p_1$, taking a finite number of iterates if necessary, we can assume that $G^n_{\alpha,\beta}(1) < G^n_{\alpha,\beta}(0)$.

1. If $d(n) = \frac{1}{\beta}$, then there is matching at the next iterate.
2. If $d(n) > \frac{1}{\beta}$, and therefore $G^n_{\alpha,\beta}(1) < \hat{p}$, then $G^n_{\alpha,\beta}(1) < G^{n+1}_{\alpha,\beta}(1) \leq G^n_{\alpha,\beta}(0)$ and $d(n+1) = \beta d(n) - 1$.
3. If $d(n) \leq \frac{1}{\beta^2}$ and therefore $d(n) \leq p - c_1$, then $c_1 < G^n_{\alpha,\beta}(1) \leq G^n_{\alpha,\beta}(0)$, $d(n+1) = \beta d(n)$ and $G^{n+1}_{\alpha,\beta}(1) < G^n_{\alpha,\beta}(1)$.
4. The remaining case is $\frac{1}{\beta^2} < d(n) < \frac{1}{\beta}$. Here we have to make further case distinctions on β. Since β is the tribonacci number, $d(n) = \frac{1}{\beta^2} + \frac{1}{\beta^3}$ is the only possibility. If $p - c_1 > d(n) = \frac{1}{\beta^2} + \frac{1}{\beta^3}$, then this case goes as part 3., and we find $G^{n+1}_{\alpha,\beta}(1) = G^{n+1}_{\alpha,\beta}(0) - \frac{1}{\beta} - \frac{1}{\beta^2}$ and by part 1. above, we have matching in two iterates. So assume that $p - c_1 \leq \frac{1}{\beta^2} + \frac{1}{\beta^3}$. We have

   $$\frac{1}{\beta^3} < 1 - \frac{1}{\beta^2} - \frac{1}{\beta^3} \leq c_2 - p = \frac{1}{\beta} - (p_1 - c_1) = \frac{1}{\beta}(1 - \frac{1-\alpha}{\beta-1}) = \frac{1}{\beta}\frac{\alpha+\beta-2}{\beta-1}.$$

   We distinguish two cases:

   (i) $c_2 - p_1 > \frac{1}{\beta^2}$ which happens when $\alpha > \frac{3\beta-\beta^2-1}{\beta}$. Then

   $$\begin{aligned} G^{n+1}_{\alpha,\beta}(1) &= G^{n+1}_{\alpha,\beta}(0) + \frac{1}{\beta^3} < c_2 \\ G^{n+2}_{\alpha,\beta}(1) &= G^{n+2}_{\alpha,\beta}(0) + \frac{1}{\beta^2} < c_2 \\ G^{n+3}_{\alpha,\beta}(1) &= G^{n+3}_{\alpha,\beta}(0) + \frac{1}{\beta}, \end{aligned}$$

   and matching occurs at the next iterate.

   (ii) $\frac{1}{\beta^3} \leq c_2 - p \leq \frac{1}{\beta^2}$ which happens when $\frac{\beta^2-2}{\beta^2} \leq \alpha \leq \frac{3\beta-\beta^2-1}{\beta}$. In this case,

   $$\begin{aligned} G^{n+1}_{\alpha,\beta}(1) &= G^{n+1}_{\alpha,\beta}(0) + \frac{1}{\beta^3} < c_2 \\ G^{n+2}_{\alpha,\beta}(1) &= G^{n+2}_{\alpha,\beta}(0) + \frac{1}{\beta^2} > c_2 \\ G^{n+3}_{\alpha,\beta}(1) &= G^{n+3}_{\alpha,\beta}(0) - \frac{1}{\beta^2} - \frac{1}{\beta^3} < c_1. \end{aligned}$$

   Hence, if $G^n_{\alpha,\beta}(0) = p$ exactly, then $(G^n_{\alpha,\beta} + k(0), G^{n+k}_{\alpha,\beta}(1))_{k\geq 0}$ is a sequence of period 3, and there is no matching. However, for every $k \geq 1$, there is a small interval $V \subset p - \varepsilon, p)$ to the left of p such that $G^{3k+1}_{\alpha,\beta}(V) = V' := (p - \frac{1}{\beta^3}, p - \frac{1}{\beta^3})$. This means that if $G^n_{\alpha,\beta}(0) \in V$, the $G^{n+3k+1}_{\alpha,\beta}(0) \in (p-\varepsilon, p)$, $G^{n+3k+1}_{\alpha,\beta}(1) \in (p - \varepsilon, p)$, so after $3k+1$ iterates, the roles of 0 and 1 have swapped. By part 3. above, we have matching in three steps.

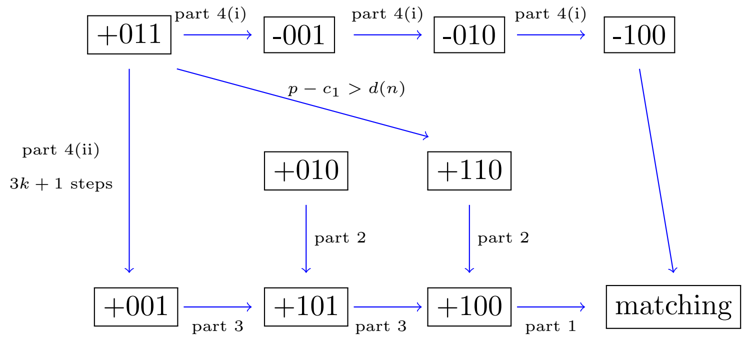

Figure 3: Flow-chart with codes $\pm e_1 e_2 e_3$, and $\pm$ indicates $\mathrm{sign}(G^n_{\alpha,\beta}(0) - G^n_{\alpha,\beta}(1))$.

In other words, $G^n_{\alpha,\beta}(1)$ cannot lie in the region $(\hat{p}, c_1)$ where $T(x) > p$, and therefore, symbolically, the map T acts as the shift on $e = e_1 \dots e_N$. Hence we have matching within N iterates.

This pattern persists if $G^n_{\alpha,\beta}(0) \in U = (p - \varepsilon, p)$ for $\varepsilon > 0$ small. By Theorem 3.1, there is matching for Lebesgue-a.e. $\alpha \in [0, 1]$. $\square$

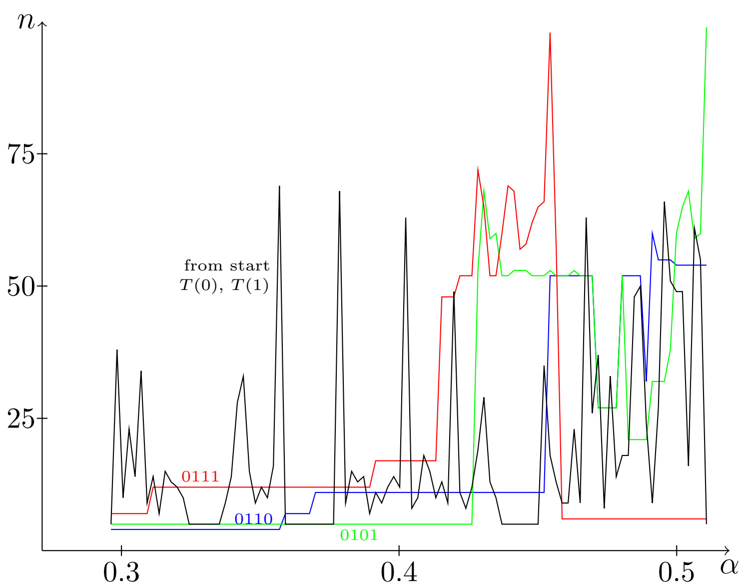

Figure 4: Number of iterates before matching for the tetrabonacci number.

For the tetrabonacci number (i.e., $N = 4$) a similar proof seems possible, but the number of case distinctions becomes very large. Instead, in Figure 4, we give some numerics on the number of iterates needed before matching occurs. The black curve has starting point $(T(0), T(1))$, and the other curves are in the gist of the proof of Proposition 3.2, namely they start when $G^n_{\alpha,\beta}(0$ is close to the fixed point: $G^n_{\alpha,\beta}(0) = p - \varepsilon$ for $\varepsilon = 0.01$ and $G^(_{\alpha,\beta}1) = G^n_{\alpha,\beta}(y) - d(N)$ for $d(n) = \sum_{i=1}^4 e_i(n)\beta^{-i}$ with $e = 0110,\ 0101$ and 0111. The range $\alpha \in [\beta^{-3}, \beta^{-1}]$ with 100 grid-points in the horizontal direction.

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