Lorentz gas with small scatterers

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Abstract

1 Introduction

We aim to obtain local limit theorem and mixing as the size of the scatterers ρ goes to zero. A first statement on the limit theorem is included in Section 7.

2 Lorentz gas on $\mathbb{Z}^2$ with small scatterers

The whole billiard will be the plane $\mathbb{R}^2$ which is divided into countably many compact cells. A single cell for this model is a unit square, which can be made into the 2-dimensional torus. The phase space of the billiard map in this cell is $\mathcal{M} = \partial O \times [-\frac{\pi}{2}, \frac{\pi}{2}]$, where O is a round disk at the origin with radius ρ. The phase space representing all cells together is therefore $\hat{\mathcal{M}} = \mathcal{M} \times \mathbb{Z}^2$, and the displacement function by $\kappa : \mathcal{M} \to \mathbb{Z}^2$ indicates the difference in cell numbers going from one collision to the next. We use coordinates $\theta \in \mathbb{S}^1$ in clockwise orientation (so the corresponding point on ∂O is $(\rho \sin\theta, \rho\cos\theta)$) and $\phi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ for the outgoing angle that the billiard trajectory makes after a collision at a point with coordinate θ with the outward normal vector $\vec{N}_\theta$ at this point (so $\phi = \frac{\pi}{2}$ corresponds to an outgoing trajectory tangent to O in the positive θ-direction).

Let $T_\rho : \mathcal{M} \to \mathcal{M}$ be the corresponding billiard map. It preserves an invariant probability measure

$$d\mu = \frac{1}{4\pi} \cos\phi \, d\theta \, d\phi. \tag{1}$$

In these coordinates (θ, φ), the measure μ has the same form for all values of the radius $\rho > 0$. Integrals involving the displacement function κ, however, do depend on ρ. For instance, for indicator functions of the form $1_{\{\kappa = N\xi + \xi'\}}$, the dependence on ρ exists in the fact that the set of $(\theta, \varphi) \in \mathcal{M}$ for which the next scatterer hit is at lattice point $N\xi + \xi'$ becomes smaller when the scatterers become smaller.

The billiard map on the entire lattice $\hat{\mathcal{M}} = \mathcal{M} \times \mathbb{Z}^2$ (or $\mathcal{M} \times \mathbb{Z}$ in case of the Lorentz tube) can be modelled by a $\mathbb{Z}^d$ extension ($\mathbb{Z}$-extension for the Lorentz tube)

$$\hat{T}_\rho(\theta, \phi, \ell) = (T_\rho(\theta, \phi), \ell + \kappa(\theta, \phi)), \qquad (\theta, \phi) \in \mathcal{M}, \ \ell \in \mathbb{Z}^2.$$

In Sections 6 and 7, where we want to emphasize the ρ-dependence, we write $\kappa(\rho)$. Apart from this section we suppress the ρ in the notation.

If $\|\kappa(x)\|$ is large (for $x = (\theta, \phi) \in \mathcal{M}$), then the flight between (x, ℓ) and $(T(x), \ell + \kappa(x))$ goes through such a corridor and the angle at which the second scatterer is hit is close to $\pm\frac{\pi}{2}$. This sparks another long flight in the same corridor, i.e., $\|\kappa(Tx)\|$ is large too.

2.1 Corridors and their widths

The function $\kappa : \mathcal{M} \to \mathbb{Z}^2$ is unbounded, because the billiard in $\mathbb{Z}^2$ has infinite horizon. More precisely, as soon as the scatterer radius $\rho < \frac{1}{2}$, there are infinite corridors parallel to the coordinate axes. If $\rho < \frac{1}{4}\sqrt{2}$, then corridors at angles of $\pm 45°$ open up, and the smaller ρ becomes, the more corridors open up at rational angles.

Let O_ℓ denote the circular scatterer of radius ρ placed at lattice point $\ell \in \mathbb{Z}^2$. The computation of $\mu(x \in \partial O_0 \times [-\frac{\pi}{2}, \frac{\pi}{2}] : \kappa(x) = (p,q))$ is based on the division of the phase space in corridors. These are infinite strips in a rational directions given by $\xi \in \mathbb{Z}^2 \setminus \{0\}$ that are disjoint from all scatterers (but maximal with respect to this property), and they are periodically repeated under integer translations. Given $0 \neq \xi \in \mathbb{Z}^2$ and $\rho > 0$ sufficiently small, there are two such corridors, simultaneous tangent to O_0 and O_ξ, one corridor on either side of the arc connecting 0 and ξ. We denote the points of tangency of these two corridors in ∂O_0 by x_0 and $\tilde{x}_0$. The widths of the corridors are denoted by $d(\xi)$ and $\tilde{d}(\xi)$, see Figure 1.

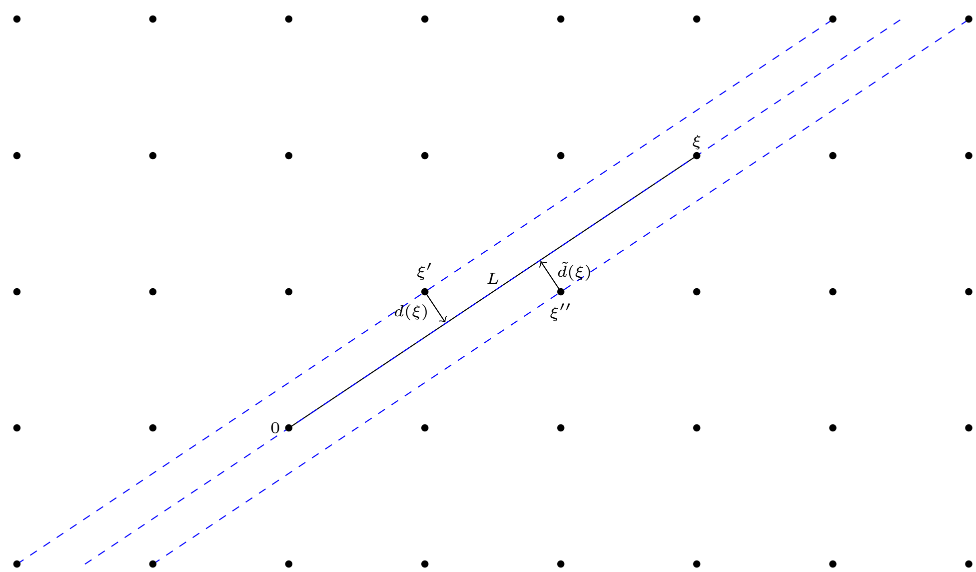

Figure 1: Corridors tangent to 0 and $\xi = (3,2)$

Lemma 2.1 *If $\rho = 0$ and $\xi = (p,q) \in \mathbb{Z}^2$ is expressed in lowest terms, then the width of the corridors for ξ satisfy*

$$d_0(\xi) = \tilde{d}_0(\xi) = \frac{1}{|\xi|}.$$

For $\rho > 0$, the actual width of the corridor is then $d_\rho(\xi) = \tilde{d}_\rho(\xi) = \max\{0, |\xi|^{-1} - 2\rho\}$.

Remark 2.2 *Let us call these two corridors in the direction ξ the ξ-corridors. They open up only when $\rho < d_0(\xi)/2 = \tilde{d}_0(\xi)/2$. For $\rho = 0$, the common boundary (called ξ-boundary) of the two ξ-corridors is the line through 0 and ξ. The other boundaries are lines parallel to the ξ-boundary, going through lattice points that are called ξ' and ξ'' in the below proof. For $\xi = (p,q)$ (with $\gcd(p,q) = 1$), these points $\xi' = (p',q')$, $\xi'' = (p'',q'')$ are uniquely determined by ξ in the sense that p'/q' and p''/q'' are convergent preceding p/q in the continued fraction expansion of p/q. In particular $|\xi'|, |\xi''| \leq |\xi|$. In the sequel, we usually only need one of these two ξ-corridors, and we take the one with ξ' in its other boundary.*

Proof. If $(p, q) = (0, \pm 1)$ or $(\pm 1, 0)$, then clearly $d(\xi) = \tilde{d}(\xi) = 1$, so we can assume without loss of generality that $p \geq q > 0$. Let L be the arc connecting $(0, 0)$ to (p, q). The corridors associated to ξ intersect $[0, p] \times [0, q]$ in diagonal strips on either side of L.

Let $\frac{q}{p} = [0; a_1, \dots, a_n = a]$ be the standard continued fraction expansion with $a \geq 1$, and the previous two convergents are denoted by q'/p' and q''/p'', say $q''/p'' < q/p < q'/p'$ (the other inequality go analogously). Therefore $q'p - qp' = 1$ and $q''p' - q'p'' = -1$. Also

$$\frac{(a-1)q' + q''}{(a-1)p' + p''} < \frac{q}{p} < \frac{q'}{p'}$$

are the best rational approximations of q/p, belonging to lattice points ξ' above L and ξ'' below L. The vertical distance between ξ' and the arc L is $|q' - p'\frac{q}{p}| = \frac{1}{p}|q'p - p'q| = \frac{1}{p}$. The vertical distance between L and ξ'' is

$$\begin{aligned} ((a-1)p' + p'')\frac{q}{p} - ((a-1)q' + q'') &= \frac{1}{p}((a-1)(qp' - q'p) + qp'' - q''p) \\ &= \frac{1}{p}(1 - a + (aq' + q'')p'' - (ap' + p'')q'') \\ &= \frac{1}{p}(1 - a + a(q'p'' - q''p')) = \frac{1}{p}. \end{aligned}$$

The corridor's diameter is perpendicular to ξ, so $d_0(\xi)$ is computed from this vertical distance as the inner product of the vector $(0, 1/p)^T$ and the vector $\xi = (p, q)^T$ rotated over $90°$:

$$\frac{1}{\sqrt{p^2 + q^2}} \left\langle \begin{pmatrix} 0 \\ 1/p \end{pmatrix} , \begin{pmatrix} -q \\ p \end{pmatrix} \right\rangle = \frac{1}{\sqrt{p^2 + q^2}} = \frac{1}{|\xi|}.$$

The computation for $\tilde{d}_0(\xi) = \frac{1}{|\xi|}$ is the same. ☐

2.2 Singularities of the billiard map

In the coordinates $(\theta, \phi, \ell) \in \mathbb{S}^1 \times [-\frac{\pi}{2}, \frac{\pi}{2}] \times \mathbb{Z}^2$ (or $\times\mathbb{Z}$ if it is a Lorentz tube), the size of the scatterers ρ doesn't appear, but it comes back in the formula of the billiard map T and in its hyperbolicity. Also the curvature of the scatterers is $\mathcal{K} \equiv 1/\rho$. We recall some notation from the Chernov & Makarian book [8] (going back to the work of Sinaĭ), bearing in mind that we have to redo several of their estimates to track the precise dependence on ρ. Let us write the phase space as $\hat{\mathcal{M}} = \mathcal{M} \times \mathbb{Z}^2 = \bigcup_{\ell \in \mathbb{Z}^2} \mathcal{M}_\ell$, where each $\mathcal{M}_\ell$ is a copy of the cylinder $\mathbb{S}^1 \times [-\frac{\pi}{2}, \frac{\pi}{2}]$, see Figure 2

Let $\mathcal{S}_0 = \{\phi = \pm\frac{\pi}{2}\}$ be the discontinuity of the billiard map corresponding to grazing collisions. The forward and backward discontinuities are

$$\mathcal{S}_n = \cup_{i=0}^n T^{-i}(\mathcal{S}_0) \quad \text{ and } \quad \mathcal{S}_{-n} = \cup_{i=0}^n T^i(\mathcal{S}_0)$$

so that $T^n : \mathcal{M} \setminus \mathcal{S}_n \to \mathcal{M} \setminus \mathcal{S}_{-n}$ is a diffeomorphism. We line the curve $\mathcal{S}_0$ with homogeneity strips $\mathbb{H}_k$ bounded by curves $| \pm \frac{\pi}{2} - \phi| = k^{-r_0}$ and $| \pm \frac{\pi}{2} - \phi| = (k+1)^{-r_0}$, $k \geq k_0$, for a fixed number $r_0 > 1$. The standard value is $r_0 = 2$, but as distortion results and some other estimates improve when r_0 is larger, we choose the optimal value of r_0 later.

The set $\mathcal{S}_{-1}$ consists of multiple curves inside $\mathcal{M}_0$, one for each scatterer from which a particle can reach O_0 in the next collision. In Figure 2 we consider the corridor in the direction of $\xi \in \mathbb{Z}^2$, and drew the parts of $\mathcal{S}_{-1}$ coming from scatterers O_ξ, $O_{-\xi}$ and $O_{-\kappa}$ for some scatterer on the other side of this corridor.

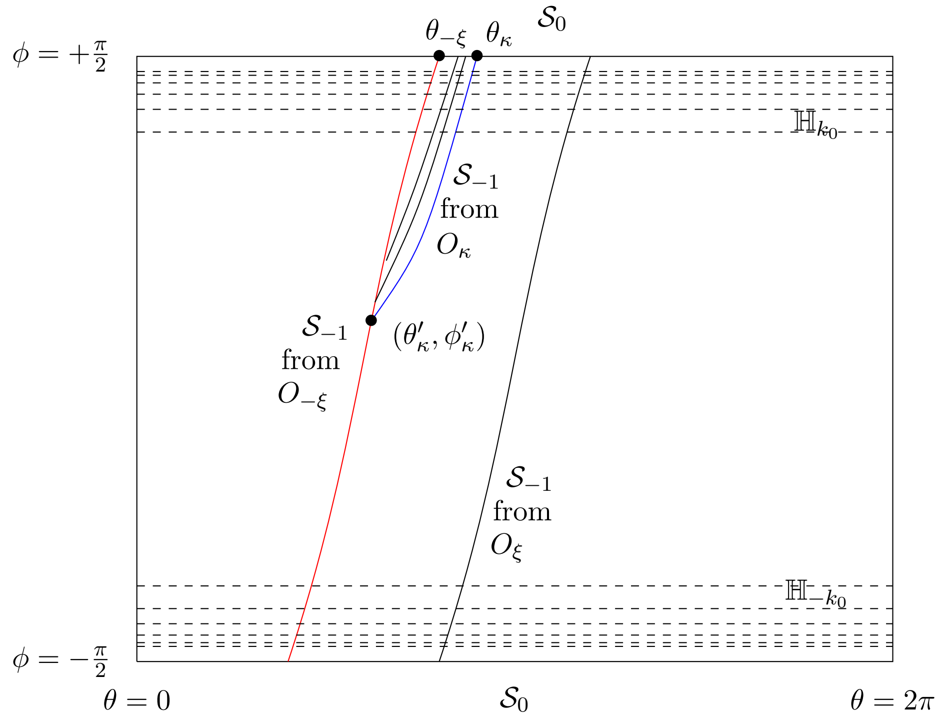

Figure 2: The parameter subset $\mathcal{M}_0$ with singularity lines and $\kappa = \xi' - M\xi$.

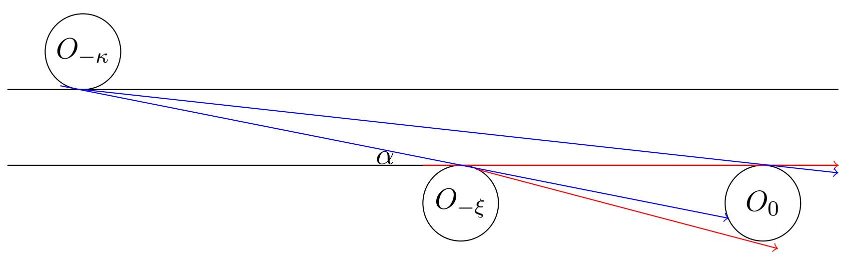

Figure 3: A corridor collision map from $O_{-\xi}$ and $O_{-\kappa}$ to O_0.

Lemma 2.3 *For the ξ-corridor, let $(\theta_{-\xi}, \frac{\pi}{2}) \in \mathcal{M}_0$ be the point of intersection of $\mathcal{S}_0$ and the part of $\mathcal{S}_{-1}$ associated to the scatterer $O_{-\xi}$, and $(\theta_\kappa, \frac{\pi}{2}) \in \mathcal{M}_0$, $\kappa = \xi' - M\xi$, be the point of intersection of $\mathcal{S}_0$ and the part of $\mathcal{S}_{-1}$ associated to the scatterer $O_\kappa = O_{\xi'-M\xi)}$ at the other side (i.e., the ξ'-boundary) of the ξ-corridor, see Figure 3. Let $(\theta'_\kappa, \phi'_\kappa)$ be the intersection of the parts of $\mathcal{S}_{-1}$ associated to the scatterers $O_{-\xi}$ and the scatterer $O_\kappa = O_{\xi'-M\xi}$. Then*

$$|\theta_{-\xi} - \theta_\kappa| = \frac{d_\rho(\xi)}{|\xi|M}\left(1 + \mathcal{O}\left(\frac{\rho}{|\xi|M}\right)\right)$$

and

$$\frac{\pi}{2} - \phi'_\kappa = \sqrt{\frac{2d_\rho(\xi)}{\rho M}}\left(1 - \mathcal{O}\left(\frac{\rho}{|\xi|} - \frac{1}{M} + \frac{\sqrt{d_\rho(\xi)\rho}}{|\xi|\sqrt{M}}\right)\right).$$

Proof. The angle $\theta_{-\xi}$ refers to the point where the common tangent line of O_0 and $O_{-\xi}$ touches O_0. For the value θ_κ, $\kappa = \xi' - M\xi$, we simply take the common tangent line to O_0 and $O_{\xi'-M\xi}$ which has slope $\frac{d_\rho(\xi)}{M|\xi|}\left(1 + \mathcal{O}(\frac{\rho}{|\xi|M})\right)$. This is then also $|\theta - \xi - \theta_\kappa|$.

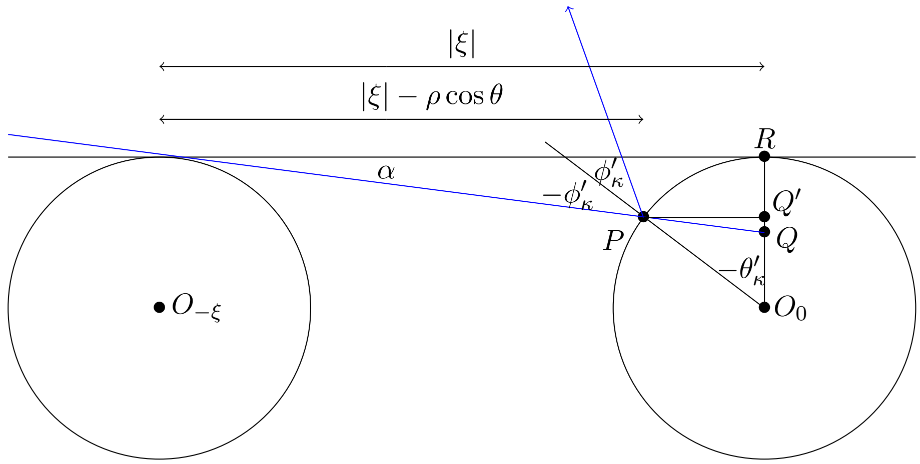

Figure 4: Illustration of the proof of Lemma 2.3

Now for the other endpoint of this piece of $\mathcal{S}_{-1}$, consider the common tangent line to $O_{-\xi}$ and $O_{\xi'-M\xi}$ which has slope $\tan\alpha := \frac{d_\rho(\xi)}{(M-1)|\xi|}(1+\mathcal{O}(\frac{\rho}{|\xi|(M-1)}))$, hitting the scatterer O_0 in point P and when extended inside O_0 hits the vertical line through the origin O in point Q. Let also R be the tangent point of O_0 to the corridor, and Q' is the point on O_0R at the same horizontal height as P, see Figure 4. Then $|RQ| = |\xi|\sin\alpha$ whereas $|O_0Q'| = \rho - (|\xi| - \rho\sin\theta'_\kappa)\sin\alpha = \rho\cos\theta$. The latter gives

$$\theta'_\kappa = \sqrt{\frac{2|\xi|}{\rho}\sin\alpha\left(1-\mathcal{O}(\frac{\rho}{|\xi|}\sin\theta)\right)} = \sqrt{\frac{2d_\rho(\xi)}{\rho M}\left(1-\mathcal{O}(\frac{\rho}{|\xi|}-\frac{1}{M})\right)}.$$

The triangle $\triangle PO_0Q$ has angles ϕ'_κ, $\alpha+\frac{\pi}{2}$ and θ'_κ, which add up to π. Hence

$$\tilde{\phi} := \frac{\pi}{2} - \phi'_\kappa = \alpha + \theta'_\kappa = \sqrt{\frac{2d_\rho(\xi)}{\rho M}}\left(1-\mathcal{O}\left(\frac{\rho}{|\xi|}-\frac{1}{M}+\frac{\sqrt{d_\rho(\xi)\rho}}{|\xi|\sqrt{M}}\right)\right) \tag{2}$$

as claimed. $\square$

2.3 Hyperbolicity of the Lorentz gas with small scatterers

The derivative $DT : \mathcal{TM} \to \mathcal{TM}$ preserves the unstable cone field

$$\mathcal{C}^u_x = \left\{(d\theta, d\phi) \in \mathcal{T}_x\mathcal{M} : 1 \le \frac{1}{2\pi}\frac{d\phi}{d\theta} \le 1 + \frac{\rho}{\tau_{\min}}\right\}. \tag{3}$$

This is [8, page 74] in the coordinates $\theta = r/2\pi\rho$, and we can sharpen this cone by replacing $\tau_{\min}$ by $\tau(x)$, the flight time at x before the next collision. The derivative of the inverse of the billiard map preserves the stable cone field

$$\mathcal{C}^s_x = \{(d\theta, d\phi) \in \mathcal{T}_x\mathcal{M} : -1 - \frac{\rho}{\tau_{\min}} \le \frac{1}{2\pi}\frac{d\phi}{d\theta} \le -1\}. \tag{4}$$

Clearly these cone-fields are transversal uniformly over $\mathcal{M}$, and $\mathcal{S}_n$ is a unstable (or stable) curve if $n > 0$ (or $n < 0$).

The expansion/contraction factor Λ, using the adapted norm of the tangent space $\|dx\|_p = 2\pi\rho\cos\phi|d\theta|$, and collision parameter $\mathcal{R}(x) = \frac{2}{\rho\cos\phi}$ satisfies

$$\Lambda \geq 1 + \tau(x)\mathcal{R}(x) \geq 1 + \tau_{\min}\mathcal{R}_{\min} = 1 + \frac{2\tau_{\min}}{\rho}.$$

This proves uniform hyperbolicity of the billiard map.

The adapted or p-norm for unstable vectors is defined as $\|dx\|_p = \cos\phi\, dr$, and the relation between the Euclidean norm is therefore

$$\|dx\| = \frac{\sqrt{1+(\frac{d\phi}{dr})^2}}{\cos\phi}\|dx\|_p = \frac{\sqrt{4\pi^2\rho^2+(\frac{d\phi}{d\theta})^2}}{2\pi\rho\cos\phi}\|dx\|_p$$

The expansion of DT of unstable vectors is uniform in the p-norm, see [8, Formula (3.40)]:

$$\frac{\|DT(dx)\|_p}{\|dx\|_p} = 1 + \frac{\tau(x)}{\cos\phi}(\mathcal{K} + \frac{d\phi}{dr}) = \frac{\tau(x)}{\rho\cos\phi}\left(1 + \frac{1}{2\pi}\frac{d\phi}{d\theta} + \frac{\rho\cos\phi}{\tau(x)}\right).$$

Expressed in Euclidean norm, this gives, for $DT(dx) = (d\theta_1, d\phi_1)$,

$$\frac{\|DT(dx)\|}{\|dx\|} = \sqrt{\frac{4\pi^2\rho^2+(\frac{d\phi_1}{d\theta_1})^2}{4\pi^2\rho^2+(\frac{d\phi}{d\theta})^2}}\,\frac{\tau(x)}{\rho\cos\phi_1}\,\left(1 + \frac{1}{2\pi}\frac{d\phi}{d\theta} + \frac{\rho\cos\phi}{\tau(x)}\right). \tag{5}$$

For later use, if $T(x)$ is in the homogeneity strip $\mathbb{H}_k$, then $\cos\phi_1 \approx k^{-r_0}$.

3 Growth lemmas

First we fix the constants for the rest of the paper. We use the same notation as in [11] except for some subscripts $_0$, and in fact some of the constants reduce to their value in [11] if $r_0 = 2$.

$$\begin{cases} r_0 \geq 2 & \text{is the exponent of the homogeneity strips:} \\ & \quad \mathbb{H}_{\pm k} = \{|\pm\frac{\pi}{2} - \varphi| \in [(k+1)^{-r_0}, k^{-r_0})\}, \\ 0 < \nu < \frac{1}{2} - \frac{1}{2r_0} & \text{the exponent of } \kappa \text{ in the continuity estimate for the} \\ & \text{transfer operator,} \\ \varsigma_0 = 1 - \frac{2r_0\nu}{r_0-1} & \text{upper bound on } \varsigma \text{in Jensen-ised growth lemma,} \\ \alpha_0 < \min\left(\frac{1}{2(r_0+1)}, \varsigma_0\right) & \text{needed for [11, Lemma 3.7] for general } r_0, \\ s_0 = \frac{1-\alpha_0(r_0+1)}{2r_0} > 0 & \text{used in Lemma 6.1,} \\ 0 < q_0 < p_0 < \frac{1}{r_0+1} & \text{cf. Lemma B.2,} \\ 0 < \beta_0 < \min\{\frac{\alpha_0}{2}, p_0 - q_0\}. & \end{cases} \tag{6}$$

We use a class $\mathcal{W}^s$ of *admissible stable leaves* defined as C^2 leaves W in the phase space such that all its tangent lines are in the stable cone bundle, their second derivative is uniformly bounded, W is contained in a single homogeneity strip, $\kappa(x)$ is constant on W and there is a ρ-dependent upper bound on $|W|$, namely

$$\sup_{W\in\mathcal{W}^s} |W| = \delta_0 := c\rho^\nu \tag{7}$$

where $c \ll 1$, to be fixed below, is independent of ρ.

Let $W \in \mathcal{M}^s$ be an admissible stable leaf. The preimage $T^{-1}(W)$ is cut by the discontinuity lines $\mathcal{S}_1$ and boundaries of homogeneity strips into at most countably many pieces V_i. Note that we may have to cut the pieces V_i further into curves W_i of length $\leq \delta_0$. In addition, we assume that

$$\delta_1 \in (0, \delta_0/2) \text{ is such that } \theta_* e^{C_d \delta_1^{1/(r_0+1)}} =: \theta_1 < 1 \tag{8}$$

for distortion constant C_d from Lemma B.2.

3.1 Growth lemma in terms of V_i

The particle can reach the scatterer O_0 at the origin from corridors in all directions, indexed by $(\xi, \xi') \in \Psi$, see Figure 3. If the previous scatterer is $\pm\xi$ itself, we call this a trajectory from the ξ-boundary; if the previous scatterer is at lattice point $\xi' - M\xi$, the trajectory comes in from the ξ'-boundary, see Remark 2.2. To each such scatterer and homogeneity strip $\mathbb{H}_k$ belongs one V_i, and the contraction $|TV_i|/|V_i|$ is governed by (5), where we use that the distortion $T : V_i \to TV_i$ is uniformly bounded, see Section B.

Proposition 3.1 *Assume $0 \leq \nu < \frac{1}{2} - \frac{1}{2r_0}$. Then there is a constant $C > 0$, uniform in ρ, ν and r_0 such that*

$$\sum_i |\kappa(V_i)|^\nu \, \frac{|TV_i|}{|V_i|} \leq C \left(\rho + \rho^{-\nu}\, \delta_0\right)$$

for every admissible homogeneous stable leaf $W^s \in \mathcal{W}^s$ on which $\kappa(W^s)$ is constant.

Remark 3.2 *(i) Since $|W^s| \leq \delta_0 \leq c\rho^\nu$, there is $\theta_* < 1$ such that*

$$\sum_{V_i} |\kappa(V_i)|^\nu \, \frac{|TV_i|}{|V_i|} \leq 3C(\rho + c) \leq \theta_*,$$

for ρ sufficiently small, and c chosen appropriately.
(ii) As later we will need $\nu \geq \frac{1}{3}$, we can take $r_0 = 4$ and $\nu = \frac{1}{3}$.

Proof. The homogeneous admissible preimage curves $T^{-1}W^s = \cup_i V_i$ are obtained by partitioning according to

- incoming corridors ξ
- for the corridor (that is, ξ) fixed, the scatterer (of the plane) on which V_i is located. Accordingly, $\kappa(V_i) = M\xi - \xi'$ for some M, and the summation is over M.
- for the scatterer fixed, the homogeneity strip in which V_i is located, that is, $V_i \subset \mathbb{H}_k$ for some k.

If W is on the scatterer O_0 and V_i is on the scatterer $O_{\xi'-M\xi}$, then both of these scatterers are tangent to the same corridor. The trajectory makes and angle $\sim \frac{d_\rho(\xi)}{M|\xi|}$ with the corridor and there is a lower bound on the collision angle given by (2). This puts restrictions on how M is related to k; as reflected by allowed intersections of homogeneity strips and M-cells on Figure 2.

In particular

$$k \geq C(\rho d_\rho(\xi)^{-1} M)^{\frac{1}{2r_0}} \tag{9}$$

which determines the range of k for M fixed.

We sum over the homogeneity strips for ξ and M fixed on the ξ' boundary.

$$\begin{aligned}
\sum_{V_i \in \mathcal{M}_{\xi'-M\xi}} |\kappa(V_i)|^\nu \, \frac{|TV_i|}{|V_i|} &\ll \frac{\rho|\xi|^\nu M^\nu}{|\xi| M} \sum_{k \geq \max\{C(\frac{\rho M}{d_\rho(\xi)},1\})^{\frac{1}{2r_0}}} \frac{1}{k^{r_0}} \\
&\ll \rho^{\frac{1}{2r_0}+\frac{1}{2}} |\xi|^{\nu-1} d_\rho(\xi)^{\frac{1}{2}-\frac{1}{2r_0}} M^{\nu-\frac{3}{2}+\frac{1}{2r_0}} \\
&\ll \rho^{\frac{1}{2r_0}+\frac{1}{2}} |\xi|^{\nu-\frac{3}{2}+\frac{1}{2r_0}} M^{\nu-\frac{3}{2}+\frac{1}{2r_0}},
\end{aligned}$$

where we used that the exponent $\frac{1}{2} - \frac{1}{2r_0}$ of $d_\rho(\xi)$ is non-negative. By our assumption that $\nu < \frac{1}{2} - \frac{1}{2r_0}$, this expression is summable over M, and therefore the sum over the ξ'-boundary of the entire ξ-corridor is

$$\sum_{\text{corridor } \xi} |\kappa(V_i)|^\nu \, \frac{|TV_i|}{|V_i|} \quad \ll \quad \rho^{\frac{1}{2}+\frac{1}{2r_0}} |\xi|^{\nu-\frac{3}{2}+\frac{1}{2r_0}}.$$

The sum over homogeneity strips for ξ fixed on the ξ-boundary is no different:

$$\sum_{V_i \in \mathcal{M}_{-\xi}} |\kappa(V_i)|^\nu \, \frac{|TV_i|}{|V_i|} \ll \frac{\rho|\xi|^\nu}{|\xi|} \sum_{k \geq 1} \frac{1}{k^{r_0}} \ll \rho|\xi|^{\nu-1}.$$

Next we sum over all opened-up corridors, indexed by all the "visible" lattice points inside a sector of angle $|W^s|/\sqrt{1+4\pi^2}$, because only trajectories from scatterers within such a narrow sector can hit O_0 at coordinates in W. It can happen that a single corridor, or even a single scatterer in a corridor blocks the entire sector, and we reserve one term for $|\xi| \geq 1$ (which is the worst case because the contraction of T is the weakest). Apart from this corridor, we replace $|W^s|$ by its upper bound δ_0. This gives

$$\begin{aligned}
\sum_{V_i} |\kappa(V_i)|^\nu \, \frac{|TV_i|}{|V_i|} &\ll \rho + \sum_{(\xi,\xi') \in \Psi_W} \rho|\xi|^{\nu-1} + \rho^{\frac{1}{2}+\frac{1}{2r_0}} |\xi|^{\nu-\frac{3}{2}+\frac{1}{2r_0}} \\
&\ll \rho + \rho^{-\nu}\delta_0 + \rho^{1-\nu}\log(1/\rho) + \rho\log^2(1/2\rho)\delta_0^{-1} \\
&\quad + \rho^{-\nu}\delta_0 + \rho^{1-\nu}\log(1/\rho) + \rho^{\frac{1}{2}+\frac{1}{2r_0}} \log^2(1/2\rho)\delta_0^{-1} \\
&\ll \rho + \rho^{-\nu}\delta_0 + \rho^{1-\nu}\log(1/\rho) + \rho^{\frac{1}{2}+\frac{1}{2r_0}} \log^2(1/2\rho)\delta_0^{-1},
\end{aligned}$$

by Lemma A.6 applied to $a = 1-\nu$ and $a = \frac{3}{2} - \nu - \frac{1}{2r_0}$. This completes the proof. $\square$

3.2 Growth lemma in terms of W_i

The pieces of preimage leaf $V_i \subset T^{-1}(W)$ emerge by natural cutting at the discontinuity set $\mathcal{S}^1$ and the homogeneity strips, but even so, their lengths can be larger than δ_0, the bound of admissible stable leaves. We therefore need to cut them into shorter pieces, denoted as W_i. In the worst case, each V_i needs to be cut into δ_0^{-1} pieces, which gives the estimate

$$\sum_i |\kappa(W_i)|^\nu \, \frac{|TW_i|}{|W_i|} \leq C\left(\rho\delta_0^{-1} + \rho^{-\nu}\right) \ll \rho^{-\nu} \tag{10}$$

Although this estimate suffices for some purposes, it is not always good enough for larger iterates T^n. The next lemma (which follows [11, Lemma 3.2] or [13, Lemma 3.3]) achieves an estimate, uniform in n, for $\nu = 0$.

Lemma 3.3 *There is a constant $C_s > 0$, independent of ρ, such that*

$$\sum_{W_i^n \in \mathcal{G}_n(W)} \frac{|T^n W_i^n|}{|W_i^n|} \le C_s, \tag{11}$$

and

$$\sum_{W_i^n \in \mathcal{G}_n(W)} \frac{|W_i^n|^\varsigma}{|W|^\varsigma} \frac{|T^n W_i^n|}{|W_i^n|} \le C_s^{1-\varsigma}. \tag{12}$$

for all $\varsigma \in [0,1)$.

Proof. Fix $W \in \mathcal{W}^s$. As in [13], we construct the components $\mathcal{G}_k(W)$ of $T^{-k}W$ inductively on $k = 0, \dots, n$. In particular, to obtain $\mathcal{G}_{k+1}(W)$ first we apply Proposition 3.1 to each curve in call $\mathcal{G}_k(W)$, and then we partition curves that are longer then δ_0 into pieces of length between δ_0 and $\delta_0/2$. We will write $W_i^k \in \mathcal{G}_k(W)$ and call it the kth generation. Define $\mathcal{L}_k$ as the collection of indices such that $W_i^k \in \mathcal{G}_k(W)$ that is long, i.e., $|W_i^k| \ge \delta_1$ for $i \in \mathcal{L}_k$, and $\mathcal{I}_n(W_j^k)$ as the collection indices of W_i^n such that their most recent long ancestor is $W_j^k \in \mathcal{G}_k(W)$. If for some $W_{i_1}^n$ no such long ancestor exists, set $k(i_1) = 0$ and $W_{i_1}^n$ belongs to $\mathcal{I}_n(W)$; if $W_{i_2}^n$ is itself long, set $k(i_2) = n$. Fix some $j \in \mathcal{L}_k$. As for $W_n^i \in \mathcal{I}_n(W_j^k)$ the preimages under T^{n-k} of $T^{n-k}W_n^i$ need not be cut artificially (they are already short), and due to the distortion bound from Lemma B.2,

$$\sum_{i \in \mathcal{I}_n(W_j^k)} \frac{|T^{n-k} W_i^n|}{|W_i^n|} \le \theta_1^{n-k}, \qquad \text{for } \theta_1 = \theta_* e^{C_d |\delta_1|^{\frac{1}{r_0+1}}}. \tag{13}$$

Recall that by our assumption δ_1 is so small that $\theta_1 < 1$. In the estimate below, we group $W_i^n \in \mathcal{G}_n(W)$ according to their most recent long ancestors.

$$\begin{aligned}
\sum_i \frac{|T^n W_i^n|}{|W_i^n|} &= \sum_{k=1}^{n} \sum_{W_j^k \in \mathcal{L}_k(W)} \sum_{i \in \mathcal{I}_n(W_j^k)} \frac{|T^n W_i^n|}{|W_i^n|} + \sum_{i \in \mathcal{I}_n(W)} \frac{|T^n W_i^n|}{|W_i^n|} \\
&\le \sum_{k=1}^{n} \sum_{W_j^k \in \mathcal{L}_k(W)} \left(\sum_{i \in \mathcal{I}_n(W_j^k)} \frac{|T^{n-k} W_i^n|}{|W_i^n|} \right) e^{\delta_1^{1/r_0+1} C_d} \frac{|T^k W_j^k|}{|W_j^k|} + \theta_1^n \\
&\le \sum_{k=1}^{n} \sum_{W_j^k \in \mathcal{L}_k(W)} \theta_1^{n-k} \delta_1^{-1} |T^k W_j^k| + \theta_1^n \\
&\le C\delta_1^{-1} |W| \sum_{k=1}^{n} \theta_1^{n-k} + \theta_1^n \le C_s,
\end{aligned} \tag{14}$$

where we have used that for fixed k and $W_j^k \in \mathcal{L}_k(W)$, (i) $|W_j^k| \ge \delta_1$, (ii) the $T^k W_j^k$ are pairwise disjoint subcurves of W, and (iii) $|W| \le \delta_1$.

By Jensen's inequality and (14),

$$\sum_i \frac{|W_i^n|^\varsigma}{|W|^\varsigma} \frac{|T^n W_i^n|}{|W_i^n|} = \sum_i \left(\frac{|W|}{|W_i^n|} \right)^{1-vs} \frac{|T^n W_i^n|}{|W|} \le \left(\sum_i \frac{|T^n W_i^n|}{|W_i^n|} \right)^{1-\varsigma} \ll C_s^{1-\varsigma},$$

which proves the second statement. $\square$

It is worth including the following bound, which follows from (13) by Jensen inequality:

$$\sum_{i\in\mathcal{I}_n(W)} \frac{|W_i^n|^\varsigma}{|W|^\varsigma}\frac{|T^nW_i^n|}{|W_i^n|} \le \theta_1^{(1-\varsigma)n}, \qquad \forall\varsigma\in[0,1). \tag{15}$$

Remark 3.4 *For further reference, here we state a version of* (12) *for* $\nu > 0, n = 1$. *Let* $\varsigma_0 = 1 - \frac{2r_0\nu}{r_0-1}$.

$$\sum_i |\kappa(W_i)|^\nu \frac{|TW_i|}{|W_i|}\frac{|W_i|^\varsigma}{|W|^\varsigma} \ll \rho^{-\nu}, \qquad \forall\varsigma\in[0,\varsigma_0). \tag{16}$$

This follows by Jensen's inequality from (10)*, applied with* $\frac{\nu}{1-\varsigma}$ *in place of* ν. *Note that the condition* $\varsigma < \varsigma_0$ *ensures* $\frac{\nu}{1-\varsigma} < \frac{1}{2} - \frac{1}{2r_0}$. *For the choices* $r_0 = 4$, $\nu = \frac{1}{3}$ *we have* $\varsigma_0 = \frac{1}{9}$.

4 Banach spaces and spectral gap

For the exponents p_0 and q_0 defined in (6) we define the Banach spaces (of distributions) $C^{p_0}, \mathcal{B}, \mathcal{B}_w, (C^{q_0})'$ in analogy to [13]. [1] We recall that $(C^{q_0})'$ is the topological dual of C^{q_0}.

Given $W \in \mathcal{W}^s$, let m_W be the Lebesgue measure on W, and define

$$|\psi|_{W,\alpha,p_0} := |W|^\alpha \cos W\, |\psi|_{C^{p_0}}, \qquad |\psi|_{C^{p_0}} := |\psi|_{C^0} + H_W^{p_0}(\psi),$$

for $\alpha \ge 0$, $\cos W = |W|^{-1}\int_W \cos\phi\, dm_W$ (note that $\cos W \ll k^{-r_0}$ if $W \subset \mathbb{H}_{\pm k}$), and $H_W^{p_0}(\psi)$ the Hölder constant of ψ along W. Also let $d_W(W_1, W_2)$ stand for the distance between leaves as in [11, Section 3.1] or [13, Section 3.1]; in particular, if W_1 and W_2 belong to the same homogeneity layer, $d_W(W_1, W_2)$ is the C^1 distance of their graphs in the (θ, ϕ) coordinates, and otherwise infinite.

Given $W \in \mathcal{W}^s$ and $h \in C^1(W)$, define the *weak norm*[2]

$$\|h\|_{\mathcal{B}_w} := \sup_{W\in\mathcal{W}^s} \sup_{\substack{|\psi|\in C^{p_0}(W)\\ |\psi|_{W,0,p_0}\le 1}} \int_W h\psi\, dm_W. \tag{17}$$

With $q_0 < p_0$ fixed we define the distance between functions $d(\psi_1, \psi_2)$ in the same way as in [11, Section 3.1]. We define the *strong stable norm* by

$$\|h\|_s := \sup_{W\in\mathcal{W}^s} \sup_{\substack{\psi\in C^{q_0}(W)\\ |\psi|_{W,\alpha_0,q_0}\le 1}} \int_W h\psi\, dm_W. \tag{18}$$

Choosing $\varepsilon_0 \in (0, \delta_0)$ and $\beta_0 \in (0, \min\{\alpha_0, p_0 - q_0\})$, we define the *strong unstable norm* by

$$\|h\|_u := \sup_{\varepsilon\le\varepsilon_0} \sup_{\substack{W_1,W_2\in\mathcal{W}^s\\ d(W_1,W_2)\le\varepsilon}} \sup_{\substack{\phi_i\in C^{p_0}(W),\\ |\psi_i|_{C^1(W)}\le 1\\ d_{q_0}(\psi_2,\tilde\psi_2)\le\varepsilon}} \frac{1}{\varepsilon^{\beta_0}}\left|\int_{W_1} h\psi_1\, dm_W - \int_{W_2} h\psi_2\, dm_W\right|. \tag{19}$$

[1] Note that our setup fits the conditions (H1)-(H5) in [13, Section 2.1], with $f(x) = f(\theta, \phi) = \cos\phi$ and $\kappa = 1$ in (H1), $r_h = r_0 + 1$ in (H2), $\xi = \frac{1}{2}$ and $t_0 = 1$ in (H3), $p_0 = \frac{1}{r_0+1}$ in (H4) and $\gamma_0 = 0$ in (H5).

[2] In the definition of the weak norm [13] uses test functions with $|\psi|_{W,\gamma,p} \le 1$ for some $\gamma > 0$, and requires $p < \gamma$. However, this is needed only to ensure that the inclusion $\mathcal{B}_w \hookrightarrow (C^p)'$ is injective, cf. [13, Lemma 3.8]. Since we do not use this property, we can take $\gamma = 0$ in the definition of the weak norm, and avoid additional restrictions on p_0.

The *strong norm* is defined by $\|h\|_{\mathcal{B}} = \|h\|_s + c_u\|h\|_u$, where we will fix $c_u \ll 1$ (but independent of ρ) at the beginning of Subsection 5.2.

Since $C^{p_0} \subset \mathcal{B} \subset \mathcal{B}_w \subset (C^{q_0})'$ (see Subsection 4.1), we have $\|h\|_{\mathcal{B}_w} + \|h\|_{\mathcal{B}} \le C\|h\|_{C^1}$. As in [13], we define $\mathcal{B}$ to be the completion of C^1 in the strong norm and $\mathcal{B}_w$ to be the completion in the weak norm.

4.1 Transfer operator on $\mathcal{B}$

Throughout we let $R_\rho : L^1(m) \to L^1(m)$ be the transfer operator of the billiard map T_ρ. We recall that [11, Lemmas 3.7-3.10] ensure that: i) $R_\rho(C^1) \subset \mathcal{B}$ and as a consequence R is well defined on $\mathcal{B}$; $\mathcal{B}_w$; ii) the unit ball of $\mathcal{B}$ is compactly embedded in $\mathcal{B}_w$. and iii) $C^{p_0} \subset \mathcal{B} \subset \mathcal{B}_w \subset (C^{q_0})'$.

It follows that R_ρ is well defined on $\mathcal{B}$ and $\mathcal{B}_w$, and we also let R_ρ denote the extension of this transfer operator to $\mathcal{B}_w$.

4.2 Lasota-Yorke inequalities

Using Proposition 3.1 with $\nu = 0$ and Lemma 3.3 we obtain the analogue of the Lasota-Yorke inequality [13, Proposition 2.3]. As our setup fits [13], our only concern is the dependence on ρ. It is important to point out that our all estimates in Section 3 and Appendix B are independent of ρ, except for $\delta_1 < \delta_0 \ll \rho^\nu$.

Lemma 4.1 (Weak norm) *There exists a uniform constant $C > 0$ so that for all $h \in \mathcal{B}$ and for all $n \ge 0$,*

$$\|R_\rho^n h\|_{\mathcal{B}_w} \le C \cdot C_s \, \|h\|_{\mathcal{B}_w},$$

where C_s is given by (11).

Proof. Note that for $W \in \mathcal{W}^s$, $h \in C^1(\mathcal{M}_0)$, $\psi \in C^{p_0}(W)$ with $|\psi|_{W,\alpha_0,p_0} \le 1$,

$$\int_W R_\rho^n h\psi \, dm_W = \sum_{W_i^n \in \mathcal{G}_n(W)} \int_{W_i^n} h \frac{J_{W_i^n} T_\rho^n}{|DT^n|} \psi \circ T_\rho^n \, dm_W.$$

Using the present definition of the weak norm,

$$\int_W R_\rho^n h\psi \, dm_W \le \sum_{W_i^n \in \mathcal{G}_n(W)} \int_{W_i^n} \|h\|_{\mathcal{B}_w} \frac{|J_{W_i} T_\rho|_{C^{p_0}(W_i)}}{|DT_\rho|} |\psi \circ T_\rho|_{C^{p_0}(W_i)} \cos(W_i^n) \, dm_W.$$

From here on the argument goes almost word for word as the argument in [13, Section 4.1], except for the use of equation (11) (the analogue of [13, Lemma 3.3(a)] with $\varsigma = 0$). □

Lemma 4.2 (Strong stable norm) *Take δ_1 as in* (8) *and θ_1 as in* (13). *There exists a uniform constant $C > 0$ so that for all $h \in \mathcal{B}$ and all $n \ge 0$,*

$$\|R_\rho^n h\|_s \le C \left(\theta_1^{(1-\alpha_0)n} + C_s^{1-\alpha_0} \Lambda^{-q_0 n} \right) \|h\|_s + C\delta_1^{-\alpha_0} \|h\|_{\mathcal{B}_w}.$$

Remark 4.3 *The compact term $C\delta_1^{-\alpha_0}\|h\|_{\mathcal{B}_w}$ in Lemma 4.2 is the only point in the Lasota-Yorke inequalities where a ρ dependence arises, via $\delta_1 \ll \rho^\nu$.*

Proof. The argument goes almost word for word as the [13, Argument in Section 4.2], except for the differences:

i) We use of equation (12) with $\varsigma = \alpha_0$ instead of [13, Lemma 3.3 (b)] (also with $\varsigma = \alpha_0$) in [13, Equation (4.5)]. In particular, using the present definition of the stable norm, with the same notation as in [13, Section 4.2], we have the following analogue of [13, Equation (4.5)]:

$$\sum_{W_i^n \in \mathcal{G}_n(W)} \int_{W_i^n} h \frac{J_{W_i^n} T_\rho^n}{|DT_\rho^n|} \left(\psi \circ T_\rho^n - \bar{\psi}_i\right) dm_W$$
$$\ll \Lambda^{-q_0 n} \|h\|_s \sum_{W_i^n \in \mathcal{G}_n(W)} \frac{|W_i^n|^{\alpha_0}}{|W|^{\alpha_0}} \frac{|T_\rho^n W_i^n|}{|W_i^n|} \ll \Lambda^{-q_0 n} \|h\|_s,$$

where we have used the distortion bounds of Appendix B and Formula (12) (with $\varsigma = \alpha_0$).

ii) To obtain the analogue of [13, Equation (4.6)], as in [13, section 4.2], we split the sum

$$\sum_{k=0}^{n} \sum_{j \in L_k} \sum_{i \in \mathcal{I}_n(W_j^k)} |W|^{-\alpha_0} (\cos W)^{-1} \int_{W_i^n} h \frac{J_{W_i^n} T^n}{|DT_\rho^n|} dm_W$$

into a term for $k = 0$ and further terms for $k = 1, \ldots, n$. For $k = 0$, we use the strong stable norm and (15) (the analogue of [13, Lemma 3.3(a)]) with $\varsigma = \alpha_0$, giving a contribution $\ll \|h\|_s \theta_1^{n(1-\alpha_0)}$. For the terms $k = 1, \ldots n$ we use the weak norm, (12) (the analogue of [13, Lemma 3.3(b)]) with $\varsigma = \alpha_0$, and the fact that for $j \in \mathcal{L}_k(W)$ we have $|W_j^k| \geq \delta_1$, resulting in a contribution of $\ll \|h\|_{\mathcal{B}_w} \delta_1^{-\alpha_0}$. □

As in [13], dealing with the strong unstable norm is the most delicate part of the Lasota-Yorke inequality. The only difference from [13, Argument in Section 4.3] is that we apply (11) (instead of [13, Lemma 3.3 (b)]) multiple times. Note that our bound in (11) is independent of ρ which ensures that no ρ-dependence arises here.

Lemma 4.4 (Strong unstable norm) *There exists a uniform constant $C > 0$ so that for all $h \in \mathcal{B}$ and for all $n \geq 0$,*

$$\|R_\rho^n h\|_u \leq C \cdot C_s \cdot \Lambda^{-\beta_0 n} \|h\|_u + C \cdot C_s \cdot n \|h\|_s.$$

Proof. Given $W_1, W_2 \in \mathcal{W}^s$ with $d(W_1, W_2) \leq \varepsilon$, we may identify matched and unmatched pieces in $T_\rho^{-n} W_\ell$, $\ell = 1, 2$. The estimates of [13] on the length of the *unmatched pieces* apply, thus we may estimate their contribution by the strong stable norm using (11) (instead of [13, Lemma 3.3 (b)]). As the length estimates give $\varepsilon^{\alpha_0/2}$, $\beta_0 < \alpha_0/2$ is essential here (cf. [13, Formulas (4.10) and (4.11)], noting that $\gamma = 0$ in our case).

To bound the contribution of the *matched pieces* we use, on the one hand, the strong unstable norm (as in [13, Formula (4.14)]) and, on the other hand, the strong stable norm (as in [13, Formula (4.17)]). Here again we rely on equation (11) which plays the role of [13, Lemma 3.3 (b)]. $\beta_0 < p_0 - q_0$ ensures that after division by ε^{β_0} the proof of Lemma 4.4 can be completed. □

5 Perturbed transfer operators

A standard way of obtaining limit theorems for dynamical systems is via the perturbed transfer operator method. In Section 7 we will use the spectral properties of the family of perturbed transfer operators $\hat{R}_\rho(t), t \in \mathbb{R}$ with $\hat{R}_\rho(t)h = R(e^{it\kappa(\rho)}h)$, $h \in L^1(m)$.

5.1 Continuity properties

By definition, $\hat{R}_\rho(0) = R_\rho$. Take $0 \le \nu < \frac{1}{2} - \frac{1}{2r_0}$ as in Proposition 3.1. In this subsection we show that the following continuity estimate holds:

$$\|(\hat{R}_\rho(t) - \hat{R}_\rho(0))h\|_{\mathcal{B}} \le C\rho^{-\nu}|t|^\nu \|h\|_{\mathcal{B}} \tag{20}$$

for some uniform constant C.

The argument goes parallel to Subsection 4.2, just as this time we need the estimates (i) for $\nu > 0$ and (ii) only for $n = 1$, we rely on (10) and (16) instead of Lemma 3.3.

Lemma 5.1 *Assume* (7)*. Then there exists a uniform constant $C > 0$ so that for all $h \in \mathcal{B}$,*

$$\|R_\rho(e^{it\kappa(\rho)} - 1)h)\|_{\mathcal{B}_w} \le C\rho^{-\nu}|t|^\nu \|h\|_{\mathcal{B}_w}.$$

Proof. The argument goes similarly to the argument in [13, Section 4.1] restricted to the case $n = 1$. More precisely, for $W \in \mathcal{W}^s$, $h \in C^1(\mathcal{M}_0)$, $\psi \in C^{p_0}(W)$ with $|\psi|_{W,\alpha_0,p_0} \le 1$,

$$\int_W R_\rho(e^{it\kappa(\rho)} - 1)h\psi \, dm_W = \sum_{i \in \mathcal{G}_1(W)} \int_{W_i} (e^{it\kappa(\rho)} - 1)h \frac{J_{W_i}T_\rho}{|DT|} \psi \circ T_\rho \, dm_W.$$

Using the definition of the weak norm and the inequality $|e^{ix} - 1| \le x^\nu$,

$$\int_W R_\rho(e^{it\kappa(\rho)} - 1)h\psi \, dm_W \le |t|^\nu \sum_{i \in \mathcal{G}_1(W)} \int_{W_i} \|h\|_{\mathcal{B}_w} |\kappa(W_i)|^\nu$$
$$\times \frac{|J_{W_i}T_\rho|_{C^{p_0}(W_i)}}{|DT_\rho|} |\psi \circ T_\rho|_{C^{p_0}(W_i)} \cos(W_i) \, dm_W.$$

From here on the proof goes the same as the argument in [13, Section 4.1] except for the use of equation (10) instead of [13, Lemma 3.3 (b)]. □

Lemma 5.2 *There exists a uniform constant $C > 0$ so that for all $h \in \mathcal{B}$ and for all $n \ge 0$,*

$$\|R_\rho(e^{it\kappa(\rho)} - 1)h)\|_s \le C|t|^\nu \rho^{-\nu} \|h\|_s.$$

Proof. This time we are only concerned with $n = 1$, and do not need a contraction of the strong stable norm. Hence, an argument analogous to the proof of Lemma 5.1 suffices, with the weak replaced by the strong stable norm. Accordingly, we use equation (16) with $\varsigma = \alpha_0$ instead of [13, Lemma 3.3 (b)]. □

Lemma 5.3 *There exists a uniform constant $C > 0$ so that for all $h \in \mathcal{B}$,*

$$\|R_\rho(e^{it\kappa(\rho)} - 1)h)\|_u \le C|t|^\nu \left(\rho^{-\nu} \cdot \|h\|_u + \rho^{-\nu} \cdot \|h\|_s\right).$$

Proof. As with the proof of Lemma 4.4, the argument goes similar to [13, Argument in Section 4.3], restricted to the case $n = 1$. The matched and unmatched pieces can be again identified, this time for $T^{-1}W_\ell$, $\ell = 1, 2$. Then, as in the proof of Lemma 5.1, the factors $|t|^\nu$ and $|\kappa(\rho)|^\nu$ arise. Clearly κ is constant on each of the (matched or unmatched) pieces, and takes the same value on any two pieces that are matched. Accordingly, the various contributions can be estimated in the same way as in proof of Lemma 4.4, with the only difference that, by the presence of the factor $|\kappa(\rho)|^\nu$, throughout the argument (10) is used instead of (11) . □

Equation (20) follows from the definition of the norm in $\mathcal{B}$ together with Lemmas 5.1, 5.2 and 5.3.

5.2 Peripheral spectrum and spectral gap

Choose $1 > \sigma > \max\{\Lambda^{-\beta_0}, \theta_1^{(1-\alpha_0)}, \Lambda^{-q_0}\}$. By Lemmas 4.1, 4.2 and 4.4 and arguing as in [13, Equation (2.14)], we obtain the traditional Lasota-Yorke inequality for some $N \geq 1$, provided c_u in the definition of $\| \, \|_{\mathcal{B}}$ is chosen small enough in terms of N. That is,

$$\|R_\rho^N h\|_{\mathcal{B}} \leq \sigma^N \|h\|_{\mathcal{B}} + C\delta_1^{-\alpha_0}\|h\|_{\mathcal{B}_w}. \tag{21}$$

Combined with the properties collected in Subsection 4.1 (that is, the relative compactness of the unit ball of $\mathcal{B}$ in $\mathcal{B}_w$), equation (21) shows that the essential spectral radius of R_ρ is bounded by σ and that the spectral radius is 1.

Let Π_ρ be the eigenprojection (that is, the projection on the eigenspace of R_ρ) corresponding to the eigenvalue 1. In particular, $\Pi_\rho 1 = \mu$ is the invariant measure for T_ρ. Since for every ρ, T_ρ is mixing, the peripheral spectrum of R_ρ consists of just the simple eigenvalue at 1. Thus, for every $\rho > 0$, the eigenprojection Π_ρ corresponding to the eigenvalue 1 of R_ρ can be also characterized by

$$\Pi_\rho h = \lim_{m\to\infty} R_\rho^m h, \tag{22}$$

for all $h \in \mathcal{B}$.

Let Q_ρ is complementary spectral projection. From here onwards, we exploit that for every $\rho > 0$, there exists $\gamma(\rho) \in (0,1)$ and $C_\rho > 0$ so that

$$\|Q_\rho^m\|_{\mathcal{B}} \leq C_\rho(1-\gamma(\rho))^m \tag{23}$$

for every $m \geq 1$. Altogether, $R_\rho^m = \Pi_\rho + Q_\rho^m$, where Q_ρ satisfies (23).

6 Asymptotics of the dominant eigenvalue

A standard route for proving limit (in particular, local) theorems is by means of the Fourier transform. For establishing limit theorems (such as Theorem 7.1 below) we study the asymptotics of $\mathbb{E}_\mu(e^{it\kappa_m(\rho)}1) = \mathbb{E}_\mu(\hat{R}_\rho(t)^m 1)$, as $t \to 0$ and $m \to \infty$. We recall that $\hat{R}_\rho(t)h = R_\rho(e^{it\kappa(\rho)}h)$, $h \in \mathcal{B}$.

We already know that for every ρ, 1 is a simple eigenvalue of $\hat{R}_\rho(0) = R_\rho$ when viewed as an operator from $\mathcal{B}$ to $\mathcal{B}$. Due to (20), $\hat{R}_\rho(t)$ is C^ν (in t) from $\mathcal{B}$ to $\mathcal{B}$. It follows that for t in a neighbourhood of 0, $\hat{R}_\rho(t)$ has a dominant eigenvalue $\lambda_\rho(t)$ (with $\lambda_\rho(0) = 1$).

Let $\gamma(\rho)$ be as in equation (23). The continuity properties together with (23) ensure that there exists $\delta \in (0, \gamma(\rho))$ so that for all $t \in B_\delta(0)$,

$$\hat{R}_\rho(t)^m = \lambda_\rho(t)^m \Pi_\rho(t) + Q_\rho(t)^m, \quad \|Q_\rho(t)^m\|_{\mathcal{B}} \leq C_\rho(1-\gamma(\rho))^m, \tag{24}$$

for some $C_\rho > 0$ and $\Pi_\rho(t)^2 = \Pi_\rho(t)$, $\Pi_\rho(t)Q_\rho(t) = 0$. Further, for all $t \in B_\delta(0)$,

$$\Pi_\rho(t) = \int_{|u-1|=\delta} (u - \hat{R}_\rho(t))^{-1}\, du, \tag{25}$$

for all t small enough. A standard consequence of (20) and (23) is that for every $\delta \in (0, \gamma(\rho)$ and for all u so that $|u-1| = \delta$,

$$\begin{aligned}\|(u-\hat{R}_\rho(t))^{-1} - (u-\hat{R}_\rho(0))^{-1}\|_{\mathcal{B}} &\leq C\rho^{-\nu}|t|^\nu \|(u-\hat{R}_\rho(t))^{-1}\|_{\mathcal{B}}\|(u-\hat{R}_\rho(0))^{-1}\|_{\mathcal{B}} \\ &\leq C\rho^{-\nu}\gamma(\rho)^{-2}|t|^\nu. \end{aligned} \tag{26}$$

Hence, $\|\Pi_\rho(t) - \Pi_\rho(0)\|_{\mathcal{B}} \leq C\rho^{-\nu}|t|^\nu \rho^{-\nu}\gamma(\rho)^{-2}|t|^\nu$.

The rest of this section is allocated to the study the asymptotics of $\lambda_\rho(t)$ as $t \to 0$.

The following property was used in [16, 4, 5] (see [5, assumption (H2)]) for the study of eigenvalues of perturbed transfer operators in the Banach spaces introduced in [10]. Here we use it to obtain an adequate analogue for the present setup.

Lemma 6.1 *Take $s_0 = \frac{1-\alpha_0(r_0+1)}{2r_0}$ as in (6). Let $h \in \mathcal{B}$ and $v \in C^{p_0}$. For every corridor with boundaries determined by O_ξ and $O_{\xi'}$, there exists constant $C > 0$ independent of ρ and ξ so that*

$$\left| \int hv1_{\{\kappa(\rho)=\xi'+N\xi\}}\, dm \right| \le C\|h\|_s |v|_{C^{q_0}} d_\rho(\xi)^{\frac{3}{2}-s_0} |\xi|^{-1} \rho^{-\frac{1}{2}+s_0} N^{-\frac{5}{2}+s_0}.$$

Proof. Let $\{W_\ell\}_{\ell\in L}$ be the foliation of the set $\{\kappa(\rho) = \xi' + \xi N\}$ into stable leaves. We can parametrise these leaves by their endpoints $(\ell, \frac{\pi}{2})$ in $\mathcal{S}_0$, then L is an interval of length $c \ll \frac{d_\rho(\xi)}{N^2|\xi|}$ according to Lemma 2.3. The lengths of these stable leaves $|W_\ell| \le c'$ for another constant $c' \ll \sqrt{\frac{2d_\rho(\xi)}{\rho N}}$, again by Lemma 2.3. The measure dm_{W_ℓ} is Lebesgue on the C^1 stable leaf W_ℓ, and it can be parametrised as $(w_\ell(\phi), \phi)$ where w is C^1 with $-\frac{1}{2\pi}\frac{\rho+\tau_{\min}}{\tau_{\min}} < w'(\phi) < -\frac{1}{2\pi}$ because of the direction of the stable cones, see (4).

Let ν be a measure on L that produces the decomposition of Lebesgue measure m on $\{\kappa(\rho) = \xi' + \xi N\}$ along stable leaves. We have $\nu \ll m_L$ (and $d\nu/dm_L$ is bounded above). Since we need to partition stable leaves W_ℓ by the homogeneity strips $\mathbb{H}_k$ near $\mathcal{S}_0$ into pieces $W_{\ell,k} := W_\ell \cap \mathbb{H}_k$, we get an extra sum over $k \ge k(c') := \lfloor (c')^{-1/r_0} \rfloor$. Then

$$\begin{aligned}
\left| \int hv1_{\{\kappa(\rho)=N\xi+\xi'\}}\, dm \right| &= \left| \int_L \sum_{k\ge k(c')} \int_{W_{\ell,k}} h\, v\, dm_{W_\ell}\, d\nu(\ell) \right| \\
&\ll \left| \int_L |v|_{C^{q_0}} \sum_{k\ge k(c')} \int_{W_{\ell,k}} h\, \frac{v}{|v|_{C^{q_0}}}\, dm_{W_\ell}\, d\ell \right| \\
&\le |v|_{C^{q_0}} \|h\|_s \int_L \left(\sum_{k\ge k(c')} |W_{\ell,k}|^{\alpha_0-1} \int_{W_{\ell,k}} \cos\phi \sqrt{1+|w'(\phi)|^2}\, d\phi \right) d\ell \\
&\ll |v|_{C^{q_0}} \|h\|_s \int_L \sum_{k\ge k(c')} k^{-r_0-(r_0+1)\alpha_0}\, d\ell \\
&\le |v|_{C^{q_0}}\, \|h\|_s\, c\, k(c')^{1-\alpha_0(r_0+1)-r_0} \\
&\ll |v|_{C^{q_0}} \|h\|_s |\xi|^{-1} d_\rho(\xi)^{\frac{3}{2}-s_0} \rho^{-\frac{1}{2}+s_0} N^{-\frac{5}{2}+s_0},
\end{aligned}$$

for $s_0 = \frac{1-\alpha_0(r_0+1)}{2r_0}$, as claimed. □

Using (26), Lemma A.2 and Lemma 6.1 we obtain the asymptotics of the eigenvalue in Proposition 6.3 below.

Lemma 6.2 *For $t \in \mathbb{R}^2$, let $\bar{A}(t,\rho) = \sum_{|\xi|\le 1/(2\rho)} \frac{d_\rho(\xi)^2\langle t,\xi\rangle^2}{|\xi|}$. Then*

$$\lim_{\rho\to 0} \frac{\rho}{2}\bar{A}(t,\rho) = \frac{|t|^2}{2\pi} = \langle \Sigma t, t\rangle \quad \textit{for} \quad \Sigma = \begin{pmatrix} \frac{1}{2\pi} & 0 \\ 0 & \frac{1}{2\pi} \end{pmatrix}.$$

Proof. The coordinate axes $p = 0$ and $q = 0$, and the two diagonals $p = q$ and $p = -q$ divide the plane into eight sectors. Here we count counter-clockwise with the first sector Ψ_1 directly above the positive p-axis. Let $\gamma = \gamma(t,\xi)$ be the angle between the vectors t and ξ. Let $\alpha = \arctan q/p$ and θ be the polar angles of ξ and $t \in \mathbb{R}^2$ respectively, so $\gamma = \theta - \alpha$. For the first sector Ψ_1, taking into account that for every ξ there are two ξ', we have

$$\begin{aligned}\sum_{(\xi,\xi')\in\Psi_1} \frac{d_\rho(\xi)^2\langle t,\xi\rangle^2}{|\xi|} &= 2|t|^2 \sum_{(\xi,\xi')\in\Psi_1} \frac{d_\rho(\xi)^2(|\xi|\cos\gamma)^2}{|\xi|} \\ &= 2|t|^2 \sum_{(\xi,\xi')\in\Psi_1} \frac{d_\rho(\xi)^2(\cos\theta\cos\alpha|\xi| + \sin\theta\sin\alpha|\xi|)^2}{|\xi|} \\ &= 2|t|^2 \sum_{\xi\in S_1} \frac{d_\rho(\xi)^2(p\cos\theta + q\sin\theta)^2}{|\xi|}.\end{aligned}$$

The eighth sector Ψ_8 directly below the positive p-axis gives the same result with $-q$ instead of q, and sectors Ψ_4 and Ψ_5 above and below the negative p-axis give the same results as sectors Ψ_8 and Ψ_1. Therefore

$$\sum_{(\xi,\xi')\in\Psi_1\cup\Psi_4\cup\Psi_5\cup\Psi_8} \frac{d_\rho(\xi)^2}{|\xi|} = 4|t|^2 \sum_{(\xi,\xi')\in\Psi_1} \frac{d_\rho(\xi)^2}{|\xi|}(p^2\cos^2\theta + q^2\sin^2\theta).$$

The same result holds the remaining sectors with $\cos\theta$ replaced by $\sin\theta$ and vice versa. Putting the results on all eight sectors together, we get by Lemma A.4

$$\begin{aligned}\sum_{(\xi,\xi')\in\Psi} \frac{d_\rho(\xi)^2\langle t,\xi\rangle^2}{|\xi|} &= \frac{|t|^2}{2} \sum_{(\xi,\xi')\in\Psi} \frac{(|\xi|^{-1} - 2\rho)^2}{|\xi|}(p^2+q^2) \\ &= \frac{|t|^2}{2} \sum_{(\xi,\xi')\in\Psi} |\xi|^{-1} - 4\rho + 4\rho^2|\xi| \\ &= \frac{|t|^2}{2}\frac{2\pi}{\zeta(2)}\frac{1}{2\rho}(1 - \frac{2}{2} + \frac{1}{3})(1+o(1)) = \frac{2|t|^2}{\rho\pi}(1+o(1).\end{aligned}$$

Hence Σ is a diagonal matrix with diagonal entries $\lim_{\rho\to 0}\frac{\rho}{2}\bar{A}(t,\rho) = \frac{|t|^2}{2\pi}$. $\square$

For the result on the asymptotics of the eigenvalue in Proposition 6.3, we will also assume some correlation decay type results. Namely, we assume that there exists $\hat{\gamma}(\rho) \in (0,1)$ and some non-uniform constants $\hat{C}_\rho$ so that for every $j \geq 1$,

$$\left|\int_{\mathcal{M}_0} (e^{it\kappa(\rho)} - 1)\,(e^{it\kappa(\rho)} - 1)\circ T_\rho^j\,d\mu - \Big(\int_{\mathcal{M}_0} (e^{it\kappa(\rho)} - 1)\,d\mu\Big)^2\right| \leq \hat{C}_\rho|t|^2(1-\hat{\gamma}(\rho))^j. \qquad (27)$$

More generally, we assume that that there exist $\hat{\gamma}(\rho) \in (0,1)$ and some non-uniform constant $\hat{C}_\rho$ so that for every $j \geq 1$ and every $m \geq 0$

$$\begin{aligned}\Big|\int_{\mathcal{M}_0} &(e^{it\kappa(\rho)} - 1)\cdot R_\rho(0)^m(e^{it\kappa(\rho)} - 1)\,(e^{it\kappa(\rho)} - 1)\circ T_\rho^j\,d\mu \qquad (28)\\ &- \int_{\mathcal{M}_0} (e^{it\kappa(\rho)} - 1)R_\rho(0)^m(e^{it\kappa(\rho)} - 1)\,d\mu \int_{\mathcal{M}_0} (e^{it\kappa(\rho)} - 1)\,d\mu \\ &- C\Big(\int_{\mathcal{M}_0} (e^{it\kappa(\rho)} - 1)\,d\mu\Big)\int_{\mathcal{M}_0} (e^{it\kappa(\rho)} - 1)\,(e^{it\kappa(\rho)} - 1)\circ T_\rho^j\,d\mu \\ &- C\Big(\int_{\mathcal{M}_0} (e^{it\kappa(\rho)} - 1)\,d\mu\Big)^3\Big| \leq \hat{C}_\rho|t|^2(1-\hat{\gamma}(\rho))^{m+j},\end{aligned}$$

where $C = 0$ if $m = 0$ and $C = 1$ if $m \geq 1$. As justified in Proposition C.1 in Appendix C via the argument used in [9, Proof of Proposition 9.1], assumptions (27) and (28) are natural.

Proposition 6.3 *Assume* (23) *and* (27)*, and let $\bar{A}(t,\rho)$ be the matrix defined in Lemma 6.2. Let $\delta \in (0, \gamma(\rho)$ so that* (24) *holds. Then there exists $\delta_0 < \delta^8$ so that for all $t \in B_{\delta_0}(0)$,*

$$1 - \lambda_\rho(t) = \bar{A}(t,\rho)\frac{\log(1/|t|)}{8\pi\rho} + E(t,\rho),$$

where $|E(t,\rho)| \leq \hat{C}_\rho\, \hat{\gamma}(\rho)^{-1}\, |t|^2 + C|t|^2\rho^{-2}$ for $\hat{C}_\rho$ and $\hat{\gamma}(\rho)$ as in (27) *and some uniform constant C.*

Remark 6.4 *f Let $\Delta\rho)$ be the flight function taking values in $\mathbb{R}^2$ as opposed to the displacement function $\kappa(\rho)$ taking values in $\mathbb{Z}^2$. A similar statement hold for the the dominant eigenvalue of the perturbed operator $R_\rho(e^{it\Delta(\rho)}$. The proof is similar to the one below using that $|\Delta(\rho) - |\kappa(\rho)|| \leq 1$.*

Proof of Proposition 6.3. In the notation of Banach spaces of distributions (see, for instance, [16]) for $h \in C^{q_0}$ we write $\langle h, 1\rangle = \langle 1, h\rangle = \int h1\, dm$ and $\langle m, h\rangle = \int h\, dm$.

Let $v_\rho(t) = \frac{\Pi_\rho(t)1}{\langle \Pi_\rho(t)1, 1\rangle}$ and recall that $v_\rho(0) = \mathbf{1}$, where $\mathbf{1}$ is both an element of $\mathcal{B}$ and of $(C^{q_0})'$. Recall that for every ρ, $\lambda_\rho(t)v_\rho(t) = \hat{R}_\rho(t)v_\rho(t)$ for t small enough, and that $\lambda_\rho(0) = 1$. Since $\langle v_\rho(t), \mathbf{1}\rangle = 1$,

$$\begin{aligned}1 - \lambda_\rho(t) &= 1 - \langle \hat{R}_\rho(t)v_\rho(t), \mathbf{1}\rangle = \mu(1 - e^{it\kappa(\rho)}) + \langle (\hat{R}_\rho(t) - \hat{R}_\rho(0))(v_\rho(t) - \mathbf{1}), \mathbf{1}\rangle \\ &=: \mu(1 - e^{it\kappa(\rho)}) + V(t,\rho).\end{aligned}$$

With the meaning of inner product clarified, for ease of notation from here on we will write $V(t,\rho) = \int_{\mathcal{M}}(e^{it\kappa(\rho)} - 1)(v_\rho(t) - \mathbf{1})\, dm$. To continue, we recall the terminology in Remark 2.2. For $\xi = (p,q)$ with $\gcd(p,q) = 1$), we let $\xi' = (p', q')$ be the point uniquely determined by ξ in the sense that p'/q' is convergent preceding p/q in the continued fraction expansion of p/q; in particular $|\xi'| \leq |\xi|$. Recall that Ψ is the set of all such pairs (ξ, ξ') with $|\xi| \leq 1/(2\rho)$. With this specified, we write

$$\mu(1 - e^{it\kappa(\rho)}) = \sum_{(\xi,\xi')\in\Psi}\sum_{N=1}^{\infty}(e^{it(\xi'+N\xi)} - 1)\mu(\{\kappa(\rho) = \xi' + N\xi\}).$$

Using the fact that $\int \kappa(\rho)\, d\mu = 0$, we compute that

$$\begin{aligned}\mu(1 - e^{it\kappa(\rho)}) &= \sum_{(\xi,\xi')\in\Psi}\sum_{N=1}^{\infty}\left(e^{it(\xi'+N\xi)} - 1 - it(\xi' + N\xi)\right)\mu(\{\kappa(\rho) = \xi' + N\xi\}) \\ &= \sum_{(\xi,\xi')\in\Psi}\sum_{N=1}^{1/|t|}\left(e^{it(\xi'+N\xi)} - 1 - it(\xi' + N\xi)\right)\mu(\{\kappa(\rho) = \xi' + N\xi\}) \\ &\quad + O\left(|t|\sum_{(\xi,\xi')\in\Psi}|\xi|\sum_{N>1/|t|}N\mu(\{\kappa(\rho) = \xi' + N\xi\})\right) \\ &= \sum_{(\xi,\xi')\in\Psi}\sum_{N=1}^{1/|t|}\frac{1}{2}\langle t, \xi' + N\xi\rangle^2\mu(\{\kappa(\rho) = \xi' + N\xi\}) + O(|t|^2) := I(t,\rho) + O(|t|^2),\end{aligned}$$

where the involved constants in the last big O are independent of ρ. Further, using Lemma A.2,

$$\begin{aligned}
I(t,\rho) &= \frac{1}{8\pi\rho}\sum_{|\xi|\le 1/(2\rho)}\frac{d_\rho(\xi)^2}{|\xi|}\langle t,\xi\rangle^2\sum_{N=\max\{1,d_\rho(\xi)/(2\rho)\}}^{1/|t|}\frac{1}{N}\\
&\quad + O\left(|t|^2\sum_{(\xi,\xi')\in\Psi}\frac{1}{4\pi|\xi|\rho}\sum_{N<\max\{1,d_\rho(\xi)/(2\rho)\}}4\rho^2\,N\,|\xi|\right)\\
&= \frac{1}{8\pi\rho}\sum_{|\xi|\le 1/(2\rho)}\frac{d_\rho(\xi)^2}{|\xi|}\langle t,\xi\rangle^2\sum_{N=\max\{1,d_\rho(\xi)/(2\rho)\}}^{1/|t|}\frac{1}{N}+O\left(|t|^2\rho^{-1}\right)\\
&= \frac{\log(1/|t|)}{8\pi\rho}\sum_{|\xi|\le 1/(2\rho)}\frac{d_\rho(\xi)^2}{|\xi|}\langle t,\xi\rangle^2+O\left(|t|^2\rho^{-1}\log(1/\rho)\right).
\end{aligned}$$

Hence, with the quantity $\bar{A}(t,\rho)$ as in Lemma 6.2,

$$\mu(1-e^{it\kappa(\rho)}) = \bar{A}(t,\rho)\frac{\log(1/|t|)}{8\pi\rho}+O\left(|t|^2\rho^{-1}\log(1/\rho)\right).$$

Hence, $1-\lambda_\rho(t) = \bar{A}(t,\rho)\frac{\log(1/|t|)}{4\pi\rho}+E(t,\rho)$, where $E(t,\rho) = O\left(|t|^2\rho^{-1}\log(1/\rho)\right)+V(t,\rho)$. It remains to estimate $V(t,\rho)$. Note that

$$v_\rho(t)-\mathbf{1} = \frac{\mu((\Pi_\rho(t)-\Pi_\rho(0))\mathbf{1})}{\mu(\Pi_\rho(t)\mathbf{1})}\Pi_\rho(0)\mathbf{1}+\frac{(\Pi_\rho(t)-\Pi_\rho(0))1}{\mu_\rho(\Pi_\rho(t)1)}.$$

Hence,

$$\begin{aligned}
V(t,\rho) &= \frac{\mu((\Pi_\rho(t)-\Pi_\rho(0))\mathbf{1})}{\mu(\Pi_\rho(t)\mathbf{1})}\int_{\mathcal{M}_0}(e^{it\kappa(\rho)}-1)\,d\mu+\frac{\int_{\mathcal{M}_0}(e^{it\kappa(\rho)}-1)(\Pi_\rho(t)-\Pi_\rho(0))\mathbf{1}\,dm}{\mu(\Pi_\rho(t)\mathbf{1})}\\
&= I_1(t,\rho)+I_2(t,\rho).
\end{aligned}$$

Estimating $I_1(t,\rho)$. Since $\int_{\mathcal{M}_0}\kappa(\rho)\,d\mu=0$, we have

$$I_1(t,\rho) = \frac{\mu((\Pi_\rho(t)-\Pi_\rho(0))\mathbf{1})}{\mu(\Pi_\rho(t)\mathbf{1})}\int_{\mathcal{M}_0}(e^{it\kappa(\rho)}-1-it\kappa(\rho))\,d\mu.$$

Now, by (26) and Lemma 6.1,

$$\begin{aligned}
\int_{\mathcal{M}}|(\Pi_\rho(t)-\Pi_\rho(0))\mathbf{1}|\,d\mu &= \sum_{(\xi,\xi')\in\Psi}\sum_{N=1}^{\infty}\int_{\mathcal{M}}1_{\{\kappa(\rho)=\xi'+N\xi\}}|(\Pi_\rho(t)-\Pi_\rho(0))\mathbf{1}|\\
&\le \sum_{(\xi,\xi')\in\Psi}|\xi|^{-\frac{5}{2}+s_0}\rho^{-\frac{1}{2}+s_0}\|\Pi_\rho(t)-\Pi_\rho(0)\|_s\sum_{N=1}^{\infty}N^{-\frac{5}{2}}\\
&\le C\rho^{-\nu}\gamma(\rho)^{-2}|t|^{\nu}
\end{aligned}$$

for some uniform C. Using also that $|e^{ix}-1-ix|\le x^y$, for any $y\in(0,2]$,

$$|I_1(t,\rho)|\le C\rho^{-\nu}\gamma(\rho)^{-2}|t|^{\nu}|t|^{2-\nu/2}\int_{\mathcal{M}_0}|\kappa(\rho)|^{2-\nu}\,d\mu\le C\rho^{-\nu}\gamma(\rho)^{-2}|t|^{\nu/2+2}.$$

Note that for $|t|\in B_{\delta_0}(0)$ with $\delta_0\le\gamma(\rho)^8$, as in the statement, $|t|^{\nu/2}<\gamma(\rho)^2$ for all $\nu<1/2$. Thus, $|I_1(t,\rho)|\le C\rho^{-\nu}|t|^2$.

Estimating $I_2(t,\rho)$**.** Recall that (23) holds and that δ is chosen so that (25) holds, i.e., $\delta < \gamma(\rho)$. Using the definition of $\Pi_\rho(t)$ and noting that for every ρ, $(u - \hat{R}_\rho(0))^{-1}\mathbf{1} = (1-u)^{-1}$,

$$
\begin{aligned}
(\Pi_\rho(t) - \Pi_\rho(0)1 &= \int_{|u-1|=\delta} (u - \hat{R}_\rho(t))^{-1}(\hat{R}_\rho(t) - \hat{R}_\rho(0))(u - \hat{R}_\rho(0))^{-1}\mathbf{1}\, du \\
&= \int_{|u-1|=\delta} (1-u)^{-1}(u - \hat{R}_\rho(t)^{-1}(\hat{R}_\rho(t) - \hat{R}_\rho(0))\mathbf{1}\, du.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&I_2(t,\rho) = \int_{\mathcal{M}_0} (e^{it\kappa(\rho)} - 1) \int_{|u-1|=\delta} (1-u)^{-1}(u - \hat{R}_\rho(t))^{-1}(\hat{R}_\rho(t) - \hat{R}_\rho(0))\mathbf{1}\, du\, dm \\
&= \int_{\mathcal{M}_0} (e^{it\kappa(\rho)} - 1) \int_{|u-1|=\delta} (1-u)^{-1}(u - \hat{R}_\rho(0))^{-1}(\hat{R}_\rho(t) - \hat{R}_\rho(0))\mathbf{1}\, du\, dm \\
&+ \int_{\mathcal{M}_0} (e^{it\kappa(\rho)} - 1) \int_{|u-1|=\delta} (1-u)^{-1} \Big((u - \hat{R}_\rho(t))^{-1} - (u - \hat{R}_\rho(0))^{-1}\Big) (\hat{R}_\rho(t) - \hat{R}_\rho(0))\mathbf{1}\, du\, dm \\
&:= J_1(t,\rho) + J_2(t,\rho).
\end{aligned} \tag{29}
$$

Now,

$$
\begin{aligned}
J_2(t,\rho) = \int_{\mathcal{M}_0} (e^{it\kappa(\rho)} - 1) \int_{|u-1|=\delta} &(1-u)^{-1}(u - \hat{R}_\rho(t))^{-1}(\hat{R}_\rho(t) - \hat{R}_\rho(0)) \\
&\times (u - \hat{R}_\rho(0))^{-1}(\hat{R}_\rho(t) - \hat{R}_\rho(0))\mathbf{1}\, du\, dm = K_1(t,\rho) + K_2(t,\rho),
\end{aligned}
$$

where

$$
\begin{aligned}
K_1(t,\rho) = \int_{\mathcal{M}_0} (e^{it\kappa(\rho)} - 1) \int_{|u-1|=\delta} &(1-u)^{-1}(u - \hat{R}_\rho(0))^{-1}(\hat{R}_\rho(t) - \hat{R}_\rho(0)) \\
&\times (u - \hat{R}_\rho(0))^{-1}(\hat{R}_\rho(t) - \hat{R}_\rho(0))\mathbf{1}\, du\, dm
\end{aligned} \tag{30}
$$

and

$$
\begin{aligned}
K_2(t,\rho) = \int_{\mathcal{M}_0} (e^{it\kappa(\rho)} - 1) \int_{|u-1|=\delta} &(1-u)^{-1} \Big((u - \hat{R}_\rho(t))^{-1} - (u - \hat{R}_\rho(0))^{-1}\Big) \\
&\times (\hat{R}_\rho(t) - \hat{R}_\rho(0))(u - \hat{R}_\rho(0))^{-1}(\hat{R}_\rho(t) - \hat{R}_\rho(0))\mathbf{1}\, du\, dm.
\end{aligned}
$$

We first treat $K_2(t,\rho)$. Note that for u in the chosen contour, $\|(u - R_\rho(t))^{-1}\|_{\mathcal{B}} \le \gamma(\rho)^{-1}$. Using (26), for all such u,

$$
\begin{aligned}
&\left\| \Big((u - \hat{R}_\rho(t))^{-1} - (u - \hat{R}_\rho(0))^{-1}\Big) (\hat{R}_\rho(t) - \hat{R}_\rho(0))(u - \hat{R}_\rho(0))^{-1}(\hat{R}_\rho(t) - \hat{R}_\rho(0)) \right\|_{\mathcal{B}} \\
&\le C\rho^{-2\nu}|t|^{3\nu}\gamma(\rho)^{-3}.
\end{aligned}
$$

This together with Lemma 6.1 gives that

$$
\begin{aligned}
|K_2(t,\rho)| \le &\sum_{(\xi,\xi')\in\Psi} \sum_{N=1}^{\infty} \int_{\mathcal{M}_0} \int_{|u-1|=\delta} |1-u|^{-1} 1_{\{\kappa(\rho)=\xi'+N\xi\}} |e^{it\kappa(\rho)} - 1| \\
&\times \left| (u - \hat{R}_\rho(t))^{-1} - (u - \hat{R}_\rho(0))^{-1}(\hat{R}_\rho(t) - \hat{R}_\rho(0)(u - \hat{R}_\rho(0))^{-1}(\hat{R}_\rho(t) - \hat{R}_\rho(0))\mathbf{1} \right| du\, dm \\
&\le |t|^{3\nu}\rho^{-3\nu}\gamma(\rho)^{-3} \sum_{(\xi,\xi')\in\Psi} |\xi|^{-\frac{5}{2}+s_0}\rho^{-\frac{1}{2}+s_0} \sum_{N=1}^{\infty} |t| N^{-\frac{3}{2}} \le C\rho^{-3\nu}\gamma(\rho)^{-3}|t|^{3\nu+1}.
\end{aligned}
$$

Since $\nu \in (1/3, 1/2)$, we have $|K_2(t,\rho)| \le C\rho^{-1}\gamma(\rho)^{-3}|t|^{5/2}$. Thus, for all $|t| \in B_{\delta_0}$ with $\delta_0 < \gamma(\rho)^8$, as in the statement, we have $|t|^{1/2} < \gamma(\rho)^4$. It follows that $|K_2(t,\rho)| \le C\rho^{-1}|t|^2$.

Estimating $J_1(t,\rho)$ in (29) **and $K_1(t,\rho)$ in** (30)**.** These terms are in, some sense, independent of the Banach space $\mathcal{B}$ (see the explanation below) and can be analysed either directly via the correlation function (27) or some form closely related to it. The rest of the proof is allocated to this type of analysis.

We start with $J_1(t,\rho)$ defined in (29), which is easier using (27). Recall that $\hat{R}_\rho(0) = R_\rho$ and $\int_{|u-1|=\delta}(1-u)^{-2}\,du = 0$ due to Cauchy's theorem. This gives

$$\begin{aligned} J_1(t,\rho) &= \int_{\mathcal{M}_0}(e^{it\kappa(\rho)}-1)\int_{|u-1|=\delta}(1-u)^{-1}\sum_{j=0}^{\infty}u^{-j-1}R_\rho^j\,R_\rho(e^{it\kappa(\rho)}-1)\mathbf{1}\,du\,dm \\ &\quad - \Big(\int_{\mathcal{M}_0}(e^{it\kappa(\rho)}-1)\,d\mu\Big)^2\int_{|u-1|=\delta}(1-u)^{-1}\sum_{j=0}^{\infty}u^{-j-1}\,du \\ &= \int_{|u-1|=\delta}(1-u)^{-1}\sum_{j=0}^{\infty}u^{-j-1}\int_{\mathcal{M}_0}(e^{it\kappa(\rho)}-1)R_\rho^j\,R_\rho(e^{it\kappa(\rho)}-1)\mathbf{1}\,dm\,du \\ &\quad - \Big(\int_{\mathcal{M}_0}(e^{it\kappa(\rho)}-1)\,d\mu\Big)^2\int_{|u-1|=\delta}(1-u)^{-1}\sum_{j=0}^{\infty}u^{-j-1}\,du. \end{aligned}$$

Swapping the order of the integrals is allowed because due to (27). The quantity

$$\Big(\int_{\mathcal{M}_0}(e^{it\kappa(\rho)}-1)R_\rho^{j+1}(e^{it\kappa(\rho)}-1)\,d\mu - \int_{\mathcal{M}_0}(e^{it\kappa(\rho)}-1)\,d\mu\Big)^2$$

decays exponentially fast. Hence, we can write

$$\begin{aligned} J_1(t,\rho) &= \int_{|u-1|=\delta}(1-u)^{-1}\sum_{j=0}^{\infty}u^{-j-1} \\ &\quad \times\Big(\int_{\mathcal{M}_0}(e^{it\kappa(\rho)}-1)\,(e^{it\kappa(\rho)}-1)\circ T_\rho^{j+1}\,d\mu - \Big(\int_{\mathcal{M}_0}(e^{it\kappa(\rho)}-1)\,d\mu\Big)^2\Big)\,du \end{aligned}$$

Using Lemma A.5 to control the dependence on ρ, $\Big(\int_{\mathcal{M}_0}(e^{it\kappa(\rho)}-1)\,d\mu\Big)^2 \le C|t|^2\rho^{-2}$. Next, recall that (23) holds and set $\delta = \min\{\gamma(\rho), \hat{\gamma}(\rho)\}/2$. Note that for $|u-1| = \delta$, we have $|u|^{-(j+1)} \ll (1-\hat{\gamma}(\rho)/2)^{-(j+1)}$. This together with (27) gives

$$\begin{aligned} |J_1(t,\rho)| &\le C_\rho\,|t|^2\int_{|u-1|=\delta}|1-u|^{-1}\sum_{j=0}^{\infty}|u|^{-j-1}\,(1-\hat{\gamma}(\rho))^{j+1} \\ &\ll \hat{C}_\rho\,|t|^2\sum_{j=1}^{\infty}\Big(\frac{1-\hat{\gamma}(\rho)}{1-\hat{\gamma}(\rho)/2}\Big)^{j+1} \le 2\hat{C}_\rho\,|t|^2\,\hat{\gamma}(\rho)^{-1}. \end{aligned}$$

An argument similar to the one above used in estimating $J_1(t,\rho)$ with (28) instead of (27) allows us to deal with $K_1(t,\rho)$ defined in (30). Compute that

$$\begin{aligned} K_1(t,\rho) = \int_{\mathcal{M}_0}(e^{it\kappa(\rho)}-1)\int_{|u-1|=\delta}&(1-u)^{-1}\sum_{m\ge1}u^{-m}\sum_{j\ge1}u^{-j}\hat{R}_\rho(0)^j(e^{it\kappa(\rho)}-1) \\ &\times\hat{R}_\rho(0)^m(e^{it\kappa(\rho)}-1)\,du\,dm. \end{aligned}$$

Let

$$
\begin{aligned}
E(t,\rho) &= \int_{\mathcal{M}_0} (e^{it\kappa(\rho)}-1)\,d\mu \int_{|u-1|=\delta} (1-u)^{-1} \\
&\qquad \times \sum_{j\ge 1} u^{-j} \sum_{m\ge 1} u^{-m} \int_{\mathcal{M}_0} (e^{it\kappa(\rho)}-1) R_\rho(0)^m (e^{it\kappa(\rho)}-1)\,d\mu\,du \\
&\quad - \int_{|u-1|=\delta} (1-u)^{-1} \sum_{j\ge 1} u^{-j} \sum_{m\ge 1} u^{-m} \int_{\mathcal{M}_0} (e^{it\kappa(\rho)}-1)\,d\mu \\
&\qquad \times \int_{\mathcal{M}_0} (e^{it\kappa(\rho)}-1)\,(e^{it\kappa(\rho)}-1)\circ T_\rho^j\,d\mu\,du \\
&\quad - \Big(\int_{\mathcal{M}_0} (e^{it\kappa(\rho)}-1)\,d\mu\Big)^3 \int_{|u-1|=\delta} (1-u)^{-1} \sum_{j\ge 1} u^{-j} \sum_{m\ge 1} u^{-m}\,du \\
&\quad = (E_1(t,\rho)-E_2(t,\rho)) \int_{\mathcal{M}_0} (e^{it\kappa(\rho)}-1)\,d\mu - E_3(t,\rho).
\end{aligned}
$$

Using (28),

$$
\Big|K_1(t,\rho)-E(t,\rho)\Big| \le \hat{C}_\rho |t|^2 \sum_{m\ge 1} |u|^{-m} \sum_{j\ge 1} |u|^{-j} (1-\hat{\gamma}(\rho))^{m+j} \le 4\hat{C}_\rho |t|^2 \hat{\gamma}(\rho)^{-2},
$$

where in the last inequality we proceeded as in estimating J_1 above.

Finally, we need to argue that E is bounded by $|t^2|$. First,

$$
\begin{aligned}
E_1(t,\rho) &= \int_{|u-1|=\delta} (1-u)^{-1} \sum_{j\ge 1} u^{-j} \sum_{m\ge 1} u^{-m} \int_{\mathcal{M}_0} (e^{it\kappa(\rho)}-1) R_\rho(0)^m (e^{it\kappa(\rho)}-1)\,d\mu\,du \\
&= \int_{|u-1|=\delta} (1-u)^{-2} \sum_{m\ge 1} u^{-m} \int_{\mathcal{M}_0} (e^{it\kappa(\rho)}-1)\,(e^{it\kappa(\rho)}-1)\circ T_\rho^m\,d\mu\,du \\
&= \int_{|u-1|=\delta} (1-u)^{-2} \sum_{m\ge 1} u^{-m} \\
&\quad \times \left(\int_{\mathcal{M}_0} (e^{it\kappa(\rho)}-1)\,(e^{it\kappa(\rho)}-1)\circ T_\rho^m\,d\mu\,du - \left(\int_{\mathcal{M}_0} (e^{it\kappa(\rho)}-1)\,d\mu\right)^2 d\mu\right) \\
&\quad + \left(\int_{\mathcal{M}_0} (e^{it\kappa(\rho)}-1)\,d\mu\right)^2 \int_{|u-1|=\delta} (1-u)^{-2} \sum_{m\ge 1} u^{-m}\,du = E_1^1(t,\rho)+E_1^2(t,\rho).
\end{aligned}
$$

Using (27), we have that $|E_1^1(t,\rho)| \le 2\hat{C}_\rho\,|t|^2\,\hat{\gamma}(\rho)^{-1}$.

Also, $E_2(t,\rho) = \int_{|u-1|=\delta}(1-u)^{-2}\sum_{j\ge 1} u^{-j}\int_{\mathcal{M}_0}(e^{it\kappa(\rho)}-1)\,(e^{it\kappa(\rho)}-1)\circ T_\rho^j\,d\mu$ and using again (27) and Cauchy's theorem, $|E_2(t,\rho)| \le 4\hat{C}_\rho\,|t|^2\,\hat{\gamma}(\rho)^{-2}$. Finally, $E_3(t,\rho)=0$. Altogether, $|K_1(t,\rho)| \le 8\hat{C}_\rho\,|t|^2\,\hat{\gamma}(\rho)^{-2}$. □

7 Limit theorem and mixing as $\rho \to 0$

The first result below is the non standard Gaussian limit law, known to hold when the horizon is infinite. Our main contribution is in characterizing the limit path allowed as $\rho \to 0$; this is done up to the unknown $\gamma(\rho)$, C_ρ in (23) and $\hat{C}_\rho$, $\hat{\gamma}(\rho)$ as in (27).

We let $\Longrightarrow$ stand for convergence in distribution with respect to the invariant measure μ.

Theorem 7.1 *Let $\gamma(\rho)$, C_ρ be as in (23), let $\hat{C}_\rho$, $\hat{\gamma}(\rho)$ be as in (27) and let C be as in Proposition 6.3. Let Σ be the variance matrix defined in Lemma 6.2.*

Set $b_{n,\rho} = \frac{\sqrt{n\log(n/\rho^2)}}{\sqrt{4\pi}\ \rho}$. Then for $n \gg \exp\left(\max\{C_\rho\gamma(\rho)^{-1}, \hat{C}_\rho\hat{\gamma}(\rho)^{-1}\} + C\rho^{-2}\right)$,

$$\frac{\kappa_n(\rho)}{b_{n,\rho}} \implies \mathcal{N}(0,\Sigma) \qquad \textit{as } \rho \to 0.$$

Remark 7.2 *A similar statement holds for the flight function $\Delta(\rho)$. The only change in the proof is the use of Remark 6.4 instead of Proposition 6.3.*

Proof. By equation (24), for t small enough,

$$\begin{aligned}\mathbb{E}_\mu(e^{it\kappa_n(\rho)}1) = \mathbb{E}_\mu(\hat{R}_\rho(t)^n 1) &= \lambda_\rho(t)^n \int_{\mathcal{M}_0} \Pi_\rho(t)1\, d\mu + \int_{\mathcal{M}_0} Q_\rho(t)^n 1\, d\mu \\ &= \lambda_\rho(t)^n \int_{\mathcal{M}_0} \Pi_\rho(t)1\, d\mu + O(C_\rho\,(1-\gamma(\rho))^n).\end{aligned}$$

Hence, as $n \to \infty$ and given the range of n, equivalently as $\rho \to 0$,

$$\left|\mathbb{E}_\mu\left(\exp\left(it\frac{\kappa_n(\rho)}{b_{n,\rho}}\right)\right) - \lambda_\rho\left(\frac{t}{b_{n,\rho}}\right)^n \int_{\mathcal{M}_0} \Pi_\rho\left(\frac{t}{b_{n,\rho}}\right) 1\, d\mu\right| \to 0.$$

Also, it follows from (26) that $\|\Pi_\rho(t) - \Pi_\rho(0)\|_{\mathcal{B}} \to 0$, as $t \to 0$. Thus, a standard argument based on the dominated convergence theorem shows that as $n \to \infty$, equivalently as $\rho \to 0$,

$$\left|\mathbb{E}_\mu\left(\exp\left(it\,\frac{\kappa_n(\rho)}{b_{n,\rho}}\right)\right) - \lambda_\rho\left(\frac{t}{b_{n,\rho}}\right)^n\right| \to 0.$$

It remains to understand $\lambda_\rho\left(\frac{t}{b_{n,\rho}}\right)^n$ as $\rho \to 0$. By Proposition 6.3,

$$\begin{aligned}n\left(1-\lambda_\rho\left(\frac{t}{b_{n,\rho}}\right)\right) =& \frac{n}{4\pi\rho}\,\bar{A}\left(\frac{t}{b_{n,\rho}}\,,\,\rho\right)\log(b_{n,\rho}/|t|) \\ &+ n\,O\left(\left(\hat{C}_\rho\hat{\gamma}(\rho)^{-1} + C\rho^{-2}\right)\left(\frac{|t|}{b_{n,\rho}}\right)^2\right).\end{aligned}$$

By assumption, $n \gg \exp\left(\hat{C}_\rho\hat{\gamma}(\rho)^{-1} + C\rho^{-2}\right)$. Hence, as $\rho \to 0$,

$$n\,\left(\hat{C}_\rho\hat{\gamma}(\rho)^{-1} + C\rho^{-2}\right)\left(\frac{|t|}{b_{n,\rho}}\right)^2 = \left(\hat{C}_\rho\hat{\gamma}(\rho)^{-1} + C\rho^{-2}\right)\frac{4\pi|t|^2\rho}{\log(n/\rho^2)} = o(|t|^2).$$

Now, given that $\bar{A}$ is as in Lemma 6.2,

$$\frac{n}{4\pi\rho}\,\bar{A}\left(\frac{t}{b_{n,\rho}}\,,\,\rho\right) = \frac{1}{\log(n/\rho^2)}\,\frac{1}{\rho}\,\rho^2\,\bar{A}\,(t,\rho) = \frac{\rho\bar{A}(t,\rho)}{\log(n/\rho^2)}.$$

Also, using Lemma 6.2 and recalling the range of n,

$$\begin{aligned}&\lim_{\rho\to 0}\frac{n}{4\pi\rho}\,\bar{A}\left(\frac{t}{b_{n,\rho}}\,,\,\rho\right)\log\left(\frac{b_{n,\rho}}{|t|}\right) \\ &\quad= \lim_{\rho\to 0}\frac{\rho\bar{A}\,(t,\rho)}{\log(n/\rho^2)}\log\left(\frac{b_{n,\rho}}{|t|}\right) \\ &\quad= \lim_{\rho\to 0}\frac{\rho}{2}\,\frac{\bar{A}(t,\rho)}{\log\left(\frac{\sqrt{n}}{\rho}\right)}\log\left(\frac{\sqrt{n}}{\rho}\,\frac{\sqrt{\log(n/\rho^2)}}{\sqrt{4\pi}|t|}\right) = \langle\Sigma t, t\rangle,\end{aligned}$$

where in the last equality we have used Lemma 6.2 and the uniform convergence theorem for slowly varying functions.

Putting the above together,

$$\lim_{\rho\to 0}\lambda_\rho\left(\frac{t}{b_{n,\rho}}\right)^n=\lim_{\rho\to 0}\exp\left(n\left(1-\lambda_\rho\left(\frac{t}{b_{n,\rho}}\right)\right)\right)=\exp\left(\langle\Sigma t,t\rangle\right), \tag{31}$$

as required. □

The next result gives a local limit theorem as $\rho\to 0$, again up to the unknown $\gamma(\rho)$, C_ρ, $\hat{C}_\rho$ and $\hat{\gamma}(\rho)$. This is possible due to the present method of proof based on spectral methods which produces the fine control of the eigenvalue in Proposition 6.3. We remark that the present proof of local limit theorem for the infinite horizon is new even for ρ fixed. We recall that the only proof of such a local limit is given in [22] via the abstract results in [2] for Young towers. Our proof relies on Proposition 6.3, which is new in the setup of the Banach spaces considered here and it heavily relies on Proposition 3.1.

In the notation of Theorem 7.1 we let Φ_Σ be the density of a Gaussian random variable distributed according to $\mathcal{N}(0,\Sigma)$ and recall from Section 4.1 that $C^{p_0}\subset\mathcal{B}$.

Theorem 7.3 *Assume the assumptions and notation of Theorem 7.1. Let $v\in C^{p_0}(\mathcal{M})$ and let $w\in L^a(\mathcal{M})$, for $a>1$. Then for $n\gg\exp\left(\max\{C_\rho\gamma(\rho)^{-1},\hat{C}_\rho\hat{\gamma}(\rho)^{-1}\}+C\rho^{-2}\right)$,*

$$\left|\int_{\mathcal{M}}v1_{\{\kappa_n(\rho)=N\}}w\circ T_\rho^n\,d\mu-\frac{\mathbb{E}_\mu(v)\,\mathbb{E}_\mu(w)}{(b_{n,\rho})^2}\Phi_\Sigma\left(\frac{N}{b_{n,\rho}}\right)\right|\to 0.$$

uniformly in $N\in\mathbb{Z}^2$ as $\rho\to 0$.

Remark 7.4 *For $N=0$, a similar statement holds for the flight function $\Delta(\rho)$. The only change in the proof is the use of Remark 6.4 instead of Proposition 6.3. Also, the uniform estimate in N can be obtained by a straightforward adaptation of the argument used in [18, Proof of Theorem 2.7].*

It is known that for for very $\rho>0$, $\kappa(\rho)$ is aperiodic, that is that there exits no trivial solution to the equation $e^{it\kappa(\rho)}g\circ T_\rho=g$. The aperiodicity of $\kappa(\rho)$ has been used in [22] to provide LLT for fixed ρ. Given Proposition 6.3 and the aperiodicity of $\kappa(\rho)$, the proof of Theorem 7.3 is classic, see [1] and for a variation of it that provides the uniformity in N, see, for instance, [20, First part of Proof of Theorem 2.2]. The proof below recalls the main elements needed to obtain the range of n in the statement.

Proof. of Theorem 7.3. Let $\delta_0<\delta$ be so that (25), (23) and Proposition 6.3 hold for all $|t|\in(0,\delta_0)$. Since $\kappa\rho)$ is aperiodic, a known argument (see [Lemma 4.3 and Theorem 4.1][1]) shows that $\|\hat{R}_\rho(t)^n\|_{\mathcal{B}}\le C_\rho(1-\gamma(\rho)^n$, for all $|t|\ge\delta_0$. It follows that $|\mathbb{E}_\mu(\hat{R}_\rho(t)^n 1)|\le\|\hat{R}_\rho(t)^n\|_{\mathcal{B}}\le C_\rho\,(1-\gamma(\rho))^n$ for any $|t|\in(\delta,\pi)$. Thus, using that $v\in C^{p_0}\subset\mathcal{B}$,

$$\begin{aligned}
\int_{\mathcal{M}}v1_{\{\kappa_n(\rho)=N\}}w\circ T_\rho^n\,d\mu&=\frac{1}{4\pi^2}\int_{[-\pi,\pi]^2}e^{-itN}\int_{\mathcal{M}}\hat{R}_\rho(t)^n v\,w\,d\mu\,dt\\
&=\frac{1}{4\pi^2}\int_{[-\delta_0,\delta_0]^2}e^{-itN}\int_{\mathcal{M}}\hat{R}_\rho(t)^n v\,w\,d\mu\,dt+O\left(C_\rho\,(1-\gamma(\rho))^n\right)\\
&=\frac{1}{4\pi^2}\int_{[-\delta_0,\delta_0]^2}e^{-itN}\lambda_\rho\,(t)^n\int_{\mathcal{M}}\Pi_\rho(t)v\,w\,d\mu\,dt+O\left(C_\rho\,(1-\gamma(\rho))^n+\hat{C}_\rho\,(1-\hat{\gamma}(\rho))^n\right)\\
&=\frac{1}{4\pi^2}I(\rho,t)+O\left(C_\rho\,(1-\gamma(\rho))^n+\hat{C}_\rho\,(1-\hat{\gamma}(\rho))^n\right).
\end{aligned} \tag{32}$$

Using that $w \in L^a$, $a > 1$,

$$\begin{aligned} I(\rho,t) &= \int_{[-\delta,\delta]^2} e^{-itN}\lambda_\rho(t)^n \, dt \int_{\mathcal{M}} v\, d\mu \int_{\mathcal{M}} w\, d\mu \\ &\quad + \int_{[-\delta,\delta]^2} e^{-itN}\lambda_\rho(t)^n \int_{\mathcal{M}} (\Pi_\rho(t) - \Pi_\rho(0)) v\, w\, d\mu\, dt \\ &= \int_{[-\delta,\delta]^2} e^{-itN}\lambda_\rho(t)^n \, dt \int_{\mathcal{M}} v\, d\mu \int_{\mathcal{M}} w\, d\mu \\ &\quad + O\left(\|w\|_{L^a(\mu)} \int_{[-\delta,\delta]^2} |\lambda_\rho(t)^n| \int_{\mathcal{M}} |(\Pi_\rho(t) - \Pi_\rho(0))v| \, d\mu\, dt \right). \end{aligned}$$

Using (26), (20) and Lemma 6.1 and proceeding as in the estimate of I_1 in the proof of Proposition 6.3, $\int_{\mathcal{M}} |(\Pi_\rho(t) - \Pi_\rho(0))v|\, d\mu \leq C\rho^{-2}\gamma(\rho)^{-2}|t|^\nu \leq C\rho^{-2}|t|^\varepsilon$, for some uniform C and some $\varepsilon > 0$. Thus,

$$I(\rho,t) = \int_{[-\delta_0,\delta_0]^2} e^{-itN}\lambda_\rho(t)^n \, dt \int_{\mathcal{M}} v\, d\mu \int_{\mathcal{M}} w\, d\mu + O\left(\|w\|_{L^a(\mu)}\rho^{-2} \int_{[-\delta_0,\delta_0]^2} |t|^\varepsilon \, |\lambda_\rho(t)^n| \, dt \right)$$

With a change of variables,

$$\begin{aligned} I(\rho,t) &= \frac{1}{(b_{n,\rho})^2} \int_{[-\delta_0 b_{n,\rho},\delta_0 b_{n,\rho}]^2} e^{-iu\frac{N}{b_{n,\rho}}} \lambda_\rho\left(\frac{u}{b_{n,\rho}}\right)^n du \int_{\mathcal{M}} v\, d\mu \int_{\mathcal{M}} w\, d\mu \\ &\quad + O\left(\|w\|_{L^a(\mu)} \frac{\rho^{-2}}{(b_{n,\rho})^3} \int_{[-\delta_0 b_{n,\rho},\delta_0 b_{n,\rho}]^2} |u|^\varepsilon \left| \lambda_\rho\left(\frac{u}{b_{n,\rho}}\right)^n \right| du \right) \end{aligned} \tag{33}$$

Given the range of n in the statement, we use (31) to obtain

$$\lim_{\rho\to 0} \left| 4\pi^2 \int_{[-\delta b_{n,\rho},\delta b_{n,\rho}]^2} e^{-iu\frac{N}{b_{n,\rho}}} \lambda_\rho\left(\frac{u}{b_{n,\rho}}\right)^n du - \Phi_\Sigma\left(\frac{N}{b_{n,\rho}}\right) \right| = 0.$$

To deal with the big O term in (33), we use that by (31) there exists a uniform constant C so that

$$\frac{\rho^{-2}}{(b_{n,\rho})^3} \int_{[-\delta_0 b_{n,\rho},\delta_0 b_{n,\rho}]^2} |u|^\varepsilon \left| \lambda_\rho\left(\frac{u}{b_{n,\rho}}\right)^n \right| du \leq \frac{\rho^{-2}}{(b_{n,\rho})^{2+\varepsilon}} \int_{[-\delta_0 b_{n,\rho},\delta_0 b_{n,\rho}]^2} |u|^\varepsilon e^{-C|u|^2} du.$$

Given that $n \gg \exp\left(C\rho^{-2}\right)$, $\frac{\rho^{-2}}{(b_{n,\rho})^{2+\varepsilon}} \ll \frac{\log n}{(b_{n,\rho})^{2+\varepsilon}} = o\left(\frac{1}{(b_{n,\rho})^2}\right)$ as $\rho \to 0$. Putting these together and using (33),

$$\lim_{\rho\to 0} \left| 4\pi^2 I(\rho,t) - \Phi_\Sigma\left(\frac{N}{b_{n,\rho}}\right) \int_{\mathcal{M}} v\, d\mu \int_{\mathcal{M}} w\, d\mu \right| = 0.$$

This together with (32) gives that as $\rho \to 0$,

$$\begin{aligned} &\left| \int_{\mathcal{M}} v 1_{\{\kappa_n(\rho)=N\}} w \circ T_\rho^n \, d\mu - \frac{1}{(b_{n,\rho})^2} \Phi_\Sigma\left(\frac{N}{b_{n,\rho}}\right) \int_{\mathcal{M}} v\, d\mu \int_{\mathcal{M}} w\, d\mu \right| \\ &\quad = O\left((b_{n,\rho})^2 \left(C_\rho (1-\gamma(\rho))^n + \hat{C}_\rho (1-\hat{\gamma}(\rho))^n \right) \right) = o(1), \end{aligned}$$

where in the last equation $n \gg \exp\left(\max\{C_\rho\gamma(\rho)^{-1}, \hat{C}_\rho\hat{\gamma}(\rho)^{-1}\}\right)$ and that $\gamma(\rho), \hat{\gamma}(\rho) \in (0,1)$. This concludes the proof. □

It is known that the local limit theorem for κ and the billiard map T (with ρ fixed) implies mixing for the planar Lorentz map $\hat{T}$ (again ρ fixed): see [20]. In fact, sharp error rates in local limit theorems and mixing are also known: see [20] for the finite horizon case and [21] for the infinite horizon case.

We recall from Section 1 that the Lorentz map $\hat{T}_\rho$ defined on $\hat{\mathcal{M}} = \mathcal{M} \times \mathbb{Z}^2$ is given by $\hat{T}_\rho(\theta, \phi, \ell) = (T_\rho(\theta, \phi), \ell + \kappa(\theta, \phi))$ for $(\theta, \phi) \in \mathcal{M}$, $\ell \in \mathbb{Z}^2$. Let $\hat{\mu} = \mu \times Leb_{\mathbb{Z}^2}$, where $Leb_{\mathbb{Z}^2}$ is the counting measure on $\mathbb{Z}^2$.

An immediate consequence of Theorem 7.3 is

Corollary 7.5 *Assume the assumptions and notation of Theorem 7.3. Let $v \in C^{p_0}(\mathcal{M})$ and let $w \in L^a(\mathcal{M})$, for $a > 1$. Then for $n \gg \exp\Big(\max\{C_\rho \gamma(\rho)^{-1}, \hat{C}_\rho \hat{\gamma}(\rho)^{-1}\} + C\rho^{-2}\Big)$,*

$$\lim_{\rho \to 0} \left| (b_{n,\rho})^2 \int_{\hat{\mathcal{M}}} v\, w \circ \hat{T}_\rho \, d\hat{\mu} - \int_{\hat{\mathcal{M}}} v \, d\mu \int_{\hat{\mathcal{M}}} w \, d\mu \right| = 0.$$

Remark 7.6 *The class of functions in Corollary 7.5 is rather restrictive as the functions v, w are supported on the cell $\mathcal{M}$. Given the work [20] (see also [21, Section 6]), it is very plausible that the present mixing result can be generalized to a suitable class of dynamically Hölder functions supported on the whole of $\hat{\mathcal{M}}$. Since the involved argument is rather delicate and not a main concern of the present work, we omit this.*

A Estimates on Corridors

A.1 Estimating $\mathbb{P}(\kappa = \xi' + N\xi)$

Given a corridor associated to ξ, there a neighborhood U_0 of $x_0 = x_0(\xi)$ in $\partial O_0 \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ of initial conditions x such that the next collision occurs at a scatterer on the opposite side of the corridor. For this situation, Szász & Varjú [22] use the coordinates (α, z), where α is the angle the trajectory of some $x \in \partial O_0$ makes with the tangent line at x_0, and the intersection point is $y = x_0 + z\xi$, see Figure 5.

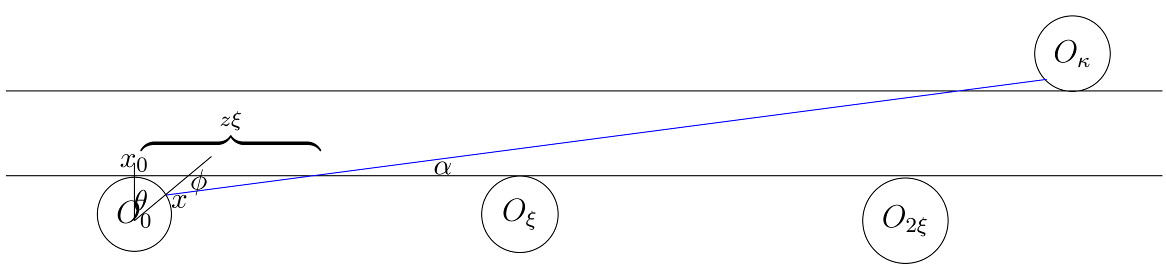

Figure 5: A corridor and coordinates (α, θ).

Lemma A.1 *In coordinates (z, α) the volume form in a neighborhood of $x_0 = x_0(\xi)$ is*

$$\frac{|\xi|}{4\pi\rho} \sin\alpha \, d\alpha \, dz = \frac{1}{4\pi} \cos\phi \, d\theta \, d\phi.$$

Proof. The part $\sin\alpha \, d\alpha \, dz$ can be understood because the Liouville measure of the billiard flow projects to a form $\cos\varphi \, d\varphi \, dr$ for any transversal section parametrised by arc-length r and with φ the angle of the trajectory to the normal vector at the collision point. When this section is the line $y = x_0 + x\xi$, we have $\alpha = \frac{\pi}{2} - \varphi$, so $\cos\varphi = \sin\alpha$. But to get the correct normalizing constant, we give a more extensive argument. From Figure 5 we have

$$\frac{\pi}{2} = \theta + \alpha + \phi, \qquad \tan\alpha = \frac{\rho(1-\cos\theta)}{z|\xi| - \rho\sin\theta}. \tag{34}$$

After making α and z subject of these equations, we see that the change of coordinates involved is

$$(\alpha, z) = F(\theta,\phi) = \left(\frac{\pi}{2} - \theta - \phi, \frac{\rho}{|\xi|}\left(\frac{1-\cos\theta}{\tan(\frac{\pi}{2}-\theta-\phi)} + \sin\theta\right)\right).$$

The Jacobian determinant is

$$|\det(dF)| = \left|\det\begin{pmatrix} -1 & -1 \\ \frac{\partial F_2}{\partial\theta} & \frac{\partial F_2}{\partial\phi}\end{pmatrix}\right| = \left|\frac{\partial F_2}{\partial\theta} - \frac{\partial F_2}{\partial\phi}\right| = \frac{\rho}{|\xi|}\left(\frac{\cos\theta}{\tan(\frac{\pi}{2}-\theta-\phi)} + \cos\theta\right).$$

Thus, using (34) and some trigonometric formulas,

$$\begin{aligned}
\frac{|\xi|}{4\pi\rho}\sin\alpha\, d\alpha\, dz &= \frac{|\xi|\sin\alpha}{4\pi\rho}\frac{\rho}{|\xi|}\left(\frac{\sin\theta}{\tan(\frac{\pi}{2}-\theta-\phi)} + \cos\theta\right) d\theta\, d\phi \\
&= \frac{1}{4\pi}(\cos\alpha\sin\theta + \sin\alpha\cos\theta)\, d\theta\, d\phi \\
&= \frac{1}{4\pi}\sin(\alpha+\theta)\, d\theta\, d\phi = \frac{1}{4\pi}\cos(\phi)\, d\theta\, d\phi,
\end{aligned}$$

as claimed. □

The following is [22, Proposition 6] in more detail and precision:

Lemma A.2 *Suppose that the scatterers have radius $\rho > 0$ and the width of the corridor given by ξ is $d_\rho(\xi)$. Then*

$$\mu(\{x \in \partial O_0 \times [-\frac{\pi}{2}, \frac{\pi}{2}] : \kappa(x) = N|\xi| + \xi'\}) = \frac{1}{4\pi N\rho}\min\{4\rho^2, d_\rho(\xi)^2 N^{-2}\}(1 + \mathcal{O}(N^{-1})),$$

where ξ' as in Remark 2.2 is the integer vector on the boundary of the corridor opposite to the ξ-boundary.

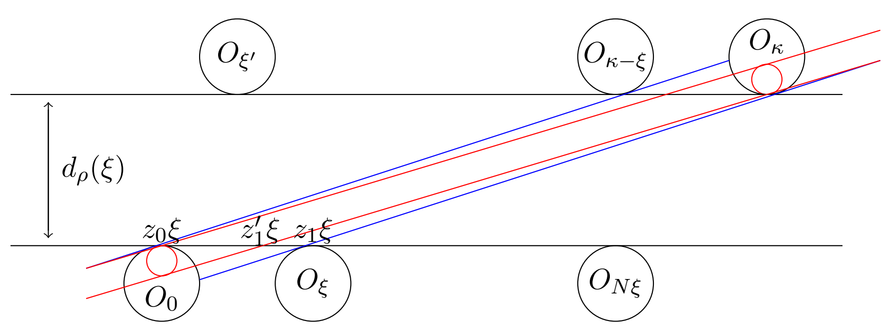

Figure 6: $[z_0, z_1]$ given by two tangent lines for $2\rho > \frac{d_\rho(\xi)}{N}$ (blue) or $2\rho < \frac{d_\rho(\xi)}{N}$ (red).

Proof. We take the region in (z,α)-coordinates where $\kappa = N\xi + \xi'$. In the z-direction this is an interval $[z_0, z_1]$, where for $z = z_0$, there is only one line connecting O_0 and O_κ, namely the common tangent line of O_0 and $O_{\kappa-\xi}$. For $z = z_1$ there is also is only one line, namely the common tangent line of O_ξ and O_κ, see Figure 6. These two lines are obtained from each other by translation over one unit ξ, so $z_1\xi - z_0\xi = |\xi|$. However, if ρ is small compared to N, these two tangent lines are the common tangent lines at the upper sides of O_0 and O_κ and at the lower sides of O_0 and O_κ. In this case

$$|z_1\xi - z_0\xi| = \frac{2\rho}{\sin\alpha} = \frac{2\rho(N|\xi| + |\xi'|)}{d_\rho(\xi) + 2\rho} + \mathcal{O}\left(\frac{\rho}{d_\rho(\xi) + 2\rho}\right). \tag{35}$$

This also shows that the transition between the two cases is when $2\rho = \frac{d_\rho(\xi)}{N}$.

For each $z \in [z_0, z_1]$, the range of possible values of α is again bounded by the α's obtained at the tangent lines to $O_{\kappa-\xi}$ and O_κ. Therefore, see Figure 7,

$$\alpha \in [\alpha_0(z), \alpha_1(z)] := \left[\arctan\left(\frac{d_\rho(\xi)}{N|\xi| + |\xi'| - z}\right)\ ,\ \arctan\left(\frac{d_\rho(\xi)}{N|\xi| - |\xi| + |\xi'| - z}\right)\right].$$

Since $|\xi'| \leq |\xi|$ (see Remark 2.2) and $z \leq |\xi|$ as well, each α in this interval satisfies $\alpha = \frac{d_\rho(\xi)}{N|\xi|}(1 + \mathcal{O}(N^{-1}))$ and

$$\alpha_1(z) - \alpha_0(z) = \frac{d_\rho(\xi)}{N^2|\xi|}(1 + \mathcal{O}(N^{-1})). \tag{36}$$

Integrating the density given in Lemma A.1 for the case $2\rho \geq \frac{d_\rho(\xi)}{N}$ (so $|z_1 - z_0| = |\xi|$) and using $|z_1 - z_0| = |\xi|$ and the approximation $\cos\alpha_0 - \cos\alpha_1 \sim \frac{1}{2}(\alpha_1 + \alpha_0)(\alpha_1 - \alpha_0)$ gives:

$$\begin{aligned}
\int_{z_0}^{z_1}\int_{\alpha_0(z)}^{\alpha_1(z)} \frac{|\xi|}{4\pi\rho}\sin\alpha\, d\alpha\, dz &= \frac{|\xi|}{4\pi\rho}\int_{z_0}^{z_1} \cos(\alpha_0(z)) - \cos(\alpha_1(z))\, dz \\
&= \frac{|\xi|}{4\pi\rho}\frac{d_\rho(\xi)}{N|\xi|}\frac{d_\rho(\xi)}{N^2|\xi|}(1 + \mathcal{O}(N^{-1})) \\
&= \frac{1}{4\pi N\rho}\frac{d_\rho(\xi)^2}{|\xi|N^2}\left(1 + \mathcal{O}(N^{-1})\right).
\end{aligned}$$

Now for the case $2\rho < \frac{d_\rho(\xi)}{N}$, see Figure 7 with small version of O_κ, we have

$$\alpha \in [\alpha_0(z), \alpha_1(z)] := \left[\arctan\left(\frac{d_\rho(\xi)}{N|\xi| + Q - z - 2\rho\sin\alpha}\right)\ ,\ \arctan\left(\frac{d_\rho(\xi) + 2\rho\cos\alpha_1(z)}{N|\xi| + Q - z - 2\rho\sin\alpha}\right)\right],$$

so still $\alpha = \frac{d_\rho(\xi)}{N|\xi|} + \mathcal{O}(N^{-2})$ and

$$\alpha_1(z) - \alpha_0(z) = \frac{2\rho}{N|\xi|}(1 + \mathcal{O}(N^{-1}). \tag{37}$$

Integrating as before gives, using (35) and the fact that $d_\rho(\xi) + 2\rho = |\xi|^{-1}$ from Lemma 2.1:

$$\begin{aligned}
\int_{z_0}^{z_1}\int_{\alpha_0(z)}^{\alpha_1(z)} \frac{|\xi|}{4\pi\rho}\sin\alpha\, d\alpha\, dz &= \frac{|\xi|}{4\pi\rho}\int_{z_0}^{z_1} \cos(\alpha_0(z)) - \cos(\alpha_1(z))\, dz \\
&= \frac{|\xi|}{4\pi\rho}\frac{2\rho N}{d_\rho(\xi) + 2\rho}\frac{d_\rho(\xi)}{N|\xi|}\frac{2\rho}{N|\xi|}(1 + \mathcal{O}(N^{-1})) \\
&= \frac{4\rho^2}{4\pi|\xi|N\rho}\left(1 + \mathcal{O}(N^{-1})\right).
\end{aligned}$$

as required. $\square$

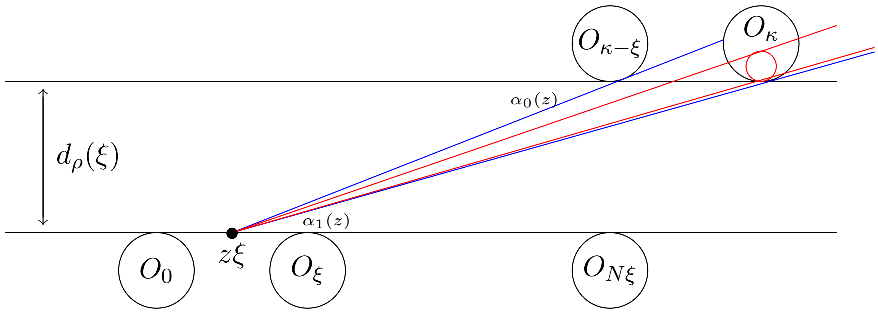

Figure 7: The parameter interval $[\alpha_0(z), \alpha_1(z)]$ given by angles between two tangent lines.

A.2 Corridors sums

Let φ be Euler's totient function, i.e., the number of integers $1 \leq q \leq p$ coprime with p. The following lemma is classical number theory, but we couldn't locate a proof of the full statement.

Lemma A.3 *For every $a > -2$, we have*

$$\sum_{n=1}^{N} n^a \varphi(n) = \frac{N^{a+2}}{a+2} \frac{1}{\zeta(2)} (1 + o(1))$$

where ζ is the Riemann ζ-function, so $\zeta(2) = \frac{\pi^2}{6}$.

Proof. Let μ be the Möbius function. A standard equality is $\varphi(n) = \sum_{d|n} \mu(d) \frac{n}{d}$. Therefore

$$\begin{aligned}
\sum_{n=1}^{N} n^a \varphi(n) &= \sum_{n=1}^{N} \sum_{d|n} n^a \mu(d) \frac{n}{d} = \sum_{n=1}^{N} \sum_{d|n} d^a \mu(d) \left(\frac{n}{d}\right)^{a+1} \\
&= \sum_{d=1}^{N} \sum_{m=1}^{\frac{N}{d}} d^a \mu(d) m^{a+1} = \sum_{d=1}^{N} d^a \mu(d) \frac{1}{a+2} \left(\frac{N}{d}\right)^{a+2} (1 + o(1)) \\
&= \frac{N^{a+2}}{a+2} \sum_{d=1}^{N} \frac{\mu(d)}{d^2} (1 + o(1)) = \frac{N^{a+2}}{a+2} \frac{1}{\zeta(2)} (1 + o(1))
\end{aligned}$$

where we used the Dirichlet series identity $\sum_{d=1}^{\infty} \frac{\mu(d)}{d^s} = \frac{1}{\zeta(s)}$ for $s = 2$.

As an aside, there are asymptotic formulas for $s > 2$

$$\sum_{p \geq 1} \frac{\varphi(p)}{p^s} = \frac{\zeta(s-1)}{\zeta(s)} \quad \text{and} \quad \sum_{p=1}^{N} \frac{\varphi(p)}{p} = \frac{N}{\zeta(2)} + \mathcal{O}((\log N)^{\frac{2}{3}} (\log \log N)^{\frac{4}{3}}), \tag{38}$$

where ζ the Riemann ζ-function, see [15, Theorem 288]. $\square$

In the course of this paper we denote the set of pairs (ξ, ξ') that are "visible" from the origin by Ψ, i.e., $\xi = (p, q)$, $\gcd(p, q) = 1$ and $|\xi| \leq (2\rho)^{-1}$. Sums of the type in the following lemma were used throughout the paper.

Lemma A.4 *We have*

$$\sum_{(\xi,\xi')\in\Psi} |\xi|^a \begin{cases} \sim \frac{1}{a+2}\frac{2\pi}{\zeta(2)}(2\rho)^{-(a+2)} & \text{if } a > -2; \\ \asymp |\log\rho| & \text{if } a = -2; \\ \leq -\frac{1}{a+2} & \text{if } a < -2. \end{cases}$$

where ζ is the Riemann ζ-function.

Proof. Using the two coordinate axes and their bisectrices, we divide the plane into eight sectors and for each sector, we sum the scatterers in $\mathcal{S}$. Circular sections of radius R have asymptotically $\frac{\pi}{4}$ as many points as triangular sectors with base R. By Lemma A.3, their sum is, for $a > -2$,

$$\sum_{(\xi,\xi')\in\Psi} |\xi|^a \quad\sim\quad \frac{8\pi}{4} \sum_{0\leq q\leq p\leq(2\rho)^{-1}} |\xi|^a = 2\pi \sum_{1\leq p\leq(2\rho)^{-1}} \phi(p)p^a \sim \frac{2\pi}{2+a}\frac{1}{\zeta(2)}(2\rho)^{-(2+a)}$$

If $a = -2$, then a similar computation gives $\asymp |\log\rho|$, and for $a < -2$, the series is summable: $2\pi\sum_{1\leq p\leq(2\rho)^{-1}}\phi(p)p^a \leq 2\pi\int_1^\infty x^a\,dx = -\frac{2\pi}{2+a}$. $\square$

Lemma A.5 *For $p \in [1,2)$, the p-norm of the displacement function satisfies*

$$\|\kappa\|_{L^p} \ll (p(2-p))^{-1/p}\ \rho^{-1}.$$

Proof. Take $p \in [1,2)$. We estimate over all ξ-corridors similarly as in Lemma A.2:

$$\begin{aligned} \int |\kappa|^p\,d\mu &\ll \sum_{|\xi|\leq(2\rho)^{-1}}\sum_{N\geq 1} |\xi|^p N^p \frac{1}{4\pi|\xi|N\rho}\min\{4\rho^2, d_\rho(\xi)^2N^{-2}\} \\ &\leq \frac{1}{4\pi\rho}\sum_{|\xi|\leq(2\rho)^{-1}} |\xi|^{p-1}\left(\sum_{N=1}^{\lfloor d_\rho(\xi)/(2\rho)\rfloor} 4\rho^2N^{p-1} + \sum_{N=\lfloor d_\rho(\xi)/(2\rho)\rfloor}^{\infty} d_\rho(\xi)^2N^{p-3}\right) \\ &\leq \frac{1}{4\pi\rho}\left(\frac{1}{p}(2\rho)^{2-p} + \frac{1}{2-p}(2\rho)^{2-p}\right)\sum_{|\xi|\leq(2\rho)^{-1}}|\xi|^{-1} \\ &\sim \frac{1}{\zeta(2)}\left(\frac{1}{p}+\frac{1}{2-p}\right)(2\rho)^{-p}, \end{aligned}$$

by Lemma A.4. Taking the p-th root gives the result. $\square$

Lemma A.6 *Suppose $\delta > 0$ is such that Mertens' function $M(n) = \sum_{d=1}^n \mu(d)$ satisfies $|M(n)| \leq n^{\frac{1}{2}+\delta}$. Let $W \in \mathcal{W}^s$ be a stable leaf, and let Ψ_W stand for all lattice points $\xi = (p,q) \in \Psi$ that can be reached from O_0 with coordinates in W. Then for every $a \in (\frac{1}{2}+\delta, 1)$,*

$$\sum_{(\xi,\xi')\in\Psi_W} |\xi|^{-a} \ll |W|\rho^{a-2} + \rho^{a-1}\log(1/\rho) + \frac{\log^2(1/2\rho)}{|W|}.$$

Validity of the Riemann hypothesis is equivalent to $|M(n)| \ll n^{\frac{1}{2}+\varepsilon}$, and the numerical evidence available int the literature shows no counter-example to $|M(n)| \leq \frac{3}{2}\sqrt{n}$ for $n \leq 10^{1000}$. Given that $n \sim (2\rho)^{-1}$ in the way we will apply this, it seems safe to use of Lemma A.6 for $\delta = \frac{1}{6}$, as we will.

Proof. There is an arc $\tilde{W} \in \mathbb{S}^1$ of length $|\tilde{W}| \ll |W|$ such that every lattice point that can be reached from O_0 with coordinates in W has its polar angle in $\tilde{W}$. Due to the symmetries in the $\mathbb{Z}^2$, it suffices to study $\tilde{W} \subset [0, \pi/2]$, so the lattice point $\xi = (p, q)$ in this sector satisfy $0 \le q \le p$ and $\tan(\tilde{W}) \subset [0, 1]$. In fact, we will start by assuming that $\tan(\tilde{W}) \in \frac{1}{10}, \frac{9}{10}]$.

Because $p^2+q^2 \ge 2pq$ for all $(p, q) = \xi$, we have $\sum_{(\xi,\xi')\in\Psi_W} |\xi|^{-a} \ll 2^{-a/2} \sum_{(\xi,\xi')\in\Psi} \frac{1}{(pq)^{a/2}} 1_{\tilde{W}}(\frac{p}{q})$. We will apply an estimate from [23, Theorem 2.2], which, in our terminology, reduces to

$$\begin{aligned}\sum_{\xi,\xi')\in\Psi} \frac{1}{(pq)^{a/2}} \psi\left(\frac{p}{q}\right) &= C_a\, \rho^{a-2} \int \psi(x)\, dx + O(\rho^{1-a} \log(1/\rho)) \\ &\quad + O\left(\sum_{\ell\neq 0} c_\psi(\ell) \sum_{\substack{d\le(2\rho)^{-1}\\ d|\ell}} d^{1-a} \sum_{k\le(2\rho d)^{-1}} \frac{\mu(k)}{k^a} \right), \end{aligned} \tag{39}$$

where C_a is a constant depending only on a, and $c(\ell)$ is the ℓ-th Fourier coefficient of $x \mapsto \psi(x)x^{-a}$.

If $\psi = 1_{\tilde{W}}$, then these Fourier coefficients are not summable, so we first smoothen $1_{\tilde{W}}$ to a function ψ supp(ψ) is concentric to $\tilde{W}$ and $|\operatorname{supp}(\psi)| = |\tilde{W}| =: 3w$. On $\tilde{W}$ itself, $\psi \equiv 1$ and on the two interval components ψ is a translated copy of the function $f : [-\frac{w}{2}, \frac{w}{2}] \to \mathbb{R}$ defined by

$$f_w(x) = \frac{1}{2} - \frac{1}{2\pi} \sin \frac{2\pi x}{w} + \frac{x}{w}.$$

Then $\int \psi\, dx = 2w$ and integrating by parts twice gives an estimate of the Fourier coefficients of $x \mapsto \psi(x)x^{-a}$.

$$|c_\psi(\ell)| \ll \left| \int \frac{(\psi(x)x^{-a})''}{(2\pi\ell)^2}\, e^{2\pi i \ell x}\, dx \right| \ll \frac{1}{w\ell^2}$$

because supp(ψ) is bounded away from $\{0, 1\}$ (so x^{-a} doesn't blow up) and $(\psi(x)x^{-a})'' = 0$ outside supp(ψ).

For $a > \frac{1}{2} + \delta$, the Dirichlet series of the Möbius function can be estimated using the Abel summation formula:

$$\sum_{k=1}^{n} \mu(k)k^{-a} = M(n)n^{-a} - M(1) + a \int_1^n M(x)x^{-1-a}\, dx \ll n^{\frac{1}{2}+\varepsilon-a} + a \int_1^n x^{\varepsilon-\frac{1}{2}-a}\, dx < \infty$$

where we used our assumption on Mertens' function $M(n) = \sum_{k=1}^{n} \mu(k)$. In fact, $\sum_k \frac{\mu(k)}{k^a} = \zeta(a)^{-1}$ for $a > \frac{1}{2}$, again provided the Riemann hypothesis holds.

Grönwall's Theorem (see [15, Theorem 323]) implies that $\sum_{d|\ell} d \ll \ell \log\log \ell$. We use this to estimate the last big O-term in (39).

$$\begin{aligned}\sum_{\ell\in\mathbb{N}} |c_\psi(\ell)| \sum_{\substack{d\le(2\rho)^{-1}\\ d|\ell}} d^{1-a} &\ll \frac{1}{w} \sum_{2\le\ell<(2\rho)^{-1}} \frac{\log\log \ell}{\ell} + \frac{1}{w} \sum_{\ell>(2\rho)^{-1}} \frac{(2\rho)^{-1} \log\log(1/2\rho)}{\ell^2} \\ &\ll \frac{\log^2(1/2\rho)}{w}.\end{aligned}$$

Hence (39) becomes

$$\sum_{(\xi,\xi')\in\Psi} \frac{1}{(pq)^{a/2}} 1_W(\frac{p}{q}) \le \sum_{\xi,\xi')\in\Psi} \frac{1}{(pq)^{a/2}}\, \psi(\frac{p}{q}) \ll |W|\rho^{a-2} + \rho^{a-1} \log(1/\rho) + \frac{\log^2(1/2\rho)}{|W|},$$

as required.

It remains to consider the cases that $\tan(\tilde{W}) \not\subset [\frac{1}{10}, \frac{9}{10}]$. Suppose instead that $\tan(\tilde{W}) \subset (0, \frac{1}{10}]$ (we ignore the single $\xi = (0,1)$). In this case, we give an injection between the lattice points in the $\tilde{W}$-sector with coprime coordinates to the set of lattice points (with coprime coordinates and comparable norm) in a sector of comparable width, but near polar angle $\frac{1}{2}$. Indeed, set $\mathbb{Q}_{cp} = \{q/p : p.q \in \mathbb{N}, \gcd(p,q) = 1\} \cup \{0\}$ and $\mathbb{Z}_{cp} := \{(q,p) \in \mathbb{Z}^2 : p,q \geq 0, \gcd(p,q) = 1\}$, and define the Calkin-Wilf map $f : \mathbb{Q}_{cp} \to \mathbb{Q}_{cp}$ [6] as well as $g : \mathbb{Z}_{cp} \to \mathbb{Z}_{cp}$ by

$$f : x \mapsto \frac{1}{1 - x - 2\lfloor x \rfloor} \qquad g : (q,p) \mapsto (p, p - q + 2q\lfloor p/q \rfloor).$$

The f-orbit of 0 enumerates all non-negative lowest-term rationals, see [6] and g is just the same function expressed on the collection of lattice points. Since $f^2((0, \frac{1}{10}]) \subset (\frac{1}{2}, \frac{10}{21}]$ and $|g(\xi)| \leq 4|\xi|$, the second iterate g^2 provides the required injection. In case $\tan(\tilde{W}) \subset [\frac{9}{10}, 1)$ we use g^3. $\square$

B Distortion properties

Throughout, a uniform constant is a constant that is independent of ρ.

Let us recall some terminology and notations from [8]. Unstable curves generate dispersing wavefronts, which are evolved by the free flight, and then leave traces of unstable curves on the scatterer at the next collision. For wavefronts it is convenient to use the Jacobi coordinates $(d\xi, d\omega)$, and an important quantity[3] $\Omega = \frac{d\omega}{d\xi}$, the curvature of the wavefront. Ω^- and Ω^+ denote its value immediately before and after a particular collision, respectively.

On the scatterer, the traditional coordinates are (r, ϕ) yet, we prefer to use the ρ-independent (θ, ϕ) and take advantage of

$$\frac{d}{d\theta} = (2\pi\rho)\frac{d}{dr}.$$

First we relate Ω^- to the slope of the unstable curve:

$$(2\pi)^{-1}\frac{d\phi}{d\theta} = \rho\Omega^- \cos\phi + 1$$

differentiating with respect to θ gives

$$(2\pi)^{-1}\frac{d^2\phi}{d\theta^2} = \frac{d\Omega^-}{d\theta}\rho\cos\phi - \rho\Omega^- \sin\phi\frac{d\phi}{d\theta}. \tag{40}$$

Lemma B.1 *There exists a uniform constant $C > 0$ such that for any C^2 smooth unstable curve W there exists n_W such that for $n \geq n_W$ on all components of T^nW we have*

$$\left|\frac{d^2\phi}{d\theta^2}\right| \leq C\rho. \tag{41}$$

Thus we may restrict to the class of *regular* unstable curves for which (41) holds. Also, this shows that as $\rho \to 0$, the unstable curves limit in a C^2 sense to straight lines of slope 2π.

Proof. The properties of the free flight are not effected by shrinking the scatterers or using the θ-coordinate. Thus

$$0 \leq \Omega^- \leq (\tau_{min})^{-1}$$

[3]Usually called $\mathcal{B}$ in billiard literature such as [8], but we write Ω to avoid confusion with Banach spaces $\mathcal{B}$.

and, by (40), it is enough to show

$$\left|\frac{d\Omega^-}{d\theta}\right| \le C$$

to prove the lemma. Now $\frac{d\Omega^-}{d\theta} = (2\pi\rho)\frac{d\Omega^-}{dr}$, and the evolution of $\frac{d\Omega^-}{dr}$ is discussed in [8, section 4.6]. Following the notation there, introduce

$$\mathcal{E}_1 = \frac{d\Omega}{d\xi}; \qquad F_1 = \frac{\mathcal{E}_1}{\Omega^3}$$

and use superscripts $-$ and $+$ to denote pre- and post-collision values of these quantities, respectively. [8, Formula (4.37)] states

$$-F_1^+ = \left(\frac{\Omega^-}{\Omega^+}\right)^3 F_1^- + H_1,$$

where

$$H_1 = \frac{6\rho^{-2}\sin\phi + 6\rho^{-1}\Omega^-\cos\phi\sin\phi}{(2\rho^{-1}+\Omega^-\cos\phi)^3}$$

and by the analysis of [8, page 81]:

- F_1 remains constant between collisions
- there exists a uniform constant $\Theta < 1$ such that $\frac{\Omega^-}{\Omega^+} \le \Theta$,
- there exists a uniform constant $C_1 > 0$ such that $|H_1| \le C_1$. This remains valid for shrinking ρ as the denominator scales with ρ^{-3} while the numerator scales with ρ^{-2}.

Hence it follows that $|F_1(n+1)| \le \Theta^3|F_1(n)| + C$, where $F_1(n)$ is the value of F_1 between the n-th and the $(n+1)$st collision. This implies that there exists $C_2 > 0$ and n_W (depending on the curve W) such that for any $n \ge n_W$ we have $|F_1(n)| \le C_2$.

Now $|\mathcal{E}_1^-| = |F_1^-|\cdot(\Omega^-)^3 \le C_3$ for some uniform $C_3 > 0$, and finally [8, Formula (4.24)] states

$$\frac{d\Omega^-}{dr} = \mathcal{E}_1^-\cos\phi - (\Omega^-)^2\sin\phi$$

which thus implies that $\left|\frac{d\Omega^-}{dr}\right| \le C_4$ for some uniform constant $C_4 > 0$. This bound completes the proof of the lemma. $\square$

It follows that regular unstable curves can be parametrised by the coordinate θ, and for any smooth function $f : W \to \mathbb{R}$, $\frac{df}{d\theta} \asymp \frac{df}{dx}$, where x is (Euclidean) arc-length along the curve – $dx^2 = d\theta^2 + d\phi^2$ (not to be confused with the arc-length r along the scatterer).

Let us also recall that an unstable curve is homogeneous if it is regular and contained in one of the homogeneity strips $\mathbb{H}_k = \{(\theta,\phi)|\frac{\pi}{2} - k^{-r_0} < \phi < \frac{\pi}{2} - (k+1)^{-r_0}\}$. For such curves, analogous to [8, Formula (5.13)], we have

$$|W| \le C\cos^{\frac{r_0+1}{r_0}}\phi \tag{42}$$

for some uniform constant $C > 0$, where ϕ corresponds to some (and thus any) point of W. (This follows as the slope of the curve is uniformly bounded away from 0 and ∞.)

Distortion bounds are stated as follows. Let W be a homogeneous unstable curve, and assume that for some $N \ge 1$, $W_n = T^{-n}W$ is a homogeneous unstable curve for $n = 0, 1, \dots, N$. For $x \in W$, let $x_n = T^{-n}x \in W_n$. Let $J_W T^{-n}(x)$ and $J_{W_n}T^{-1}(x_n)$ denote the respective Jacobians.

Lemma B.2 *Consider W and N as above and $y, z \in W$ arbitrary. There exists a uniform constant $C_d > 0$ such that*

$$|\log J_W T^{-N}(y) - \log J_W T^{-N}(z)| \leq C_d |W|^{\frac{1}{r_0+1}}.$$

Proof. The lemma relies on Formula

$$\left| \frac{d}{dx_n} \log J_{W_n} T^{-1}(x_n) \right| \leq \frac{C}{\cos\phi_n} \tag{43}$$

for some uniform $C > 0$, cf. [8, Formula (5.8)].

Using this formula the argument in the proof of [8, Lemma 5.27] can be repeated literally:

$$\begin{aligned} |\log J_W T^{-N}(y) - \log J_W T^{-N}(z)| &\leq \sum_{n=0}^{N-1} |\log J_{W_n} T^{-1}(y_n) - \log J_{W_n} T^{-1}(z_n)| \\ &\leq \sum_{n=0}^{N-1} |W_n| \max \left| \frac{d}{dx_n} \log J_{W_n} T^{-1}(x_n) \right| \leq C \sum_{n=0}^{N-1} \frac{|W_n|}{\cos\phi_n} \\ &\leq C \sum_{n=0}^{N-1} |W_n|^{\frac{1}{r_0+1}} \leq C|W|^{\frac{1}{r_0+1}}, \end{aligned} \tag{44}$$

where we have used the chain rule, Formula (43), Formula (42) and the uniform hyperbolicity.

It remains to prove (43). Here we essentially follow [8, pp. 106–107]. We have

$$\begin{aligned} \log J_{W_n} T^{-1}(x_n) &= \log\cos\phi_n + \frac{1}{2}\log\left(4\pi^2\rho^2 + \left(\frac{d\phi_n}{d\theta_n}\right)^2\right) - \frac{1}{2}\log\left(4\pi^2\rho^2 + \left(\frac{d\phi_{n+1}}{d\theta_{n+1}}\right)^2\right) \\ &\quad - \log\left(2\rho^{-1}\tau_{n+1} + \cos\phi_{n+1}(1 + \tau_{n+1}\Omega^-_{n+1})\right). \end{aligned}$$

Let us consider the derivatives of these terms separately. As noted above, differentiation with respect to θ_n and x_n can be interchanged. By Lemma B.1, the derivative of the second term w.r. to θ_n is uniformly bounded. The same applies to the derivative of the third term with respect to θ_{n+1}, while

$$\frac{dx_{n+1}}{dx_n} = J_{W_n} T^{-1}(x_n)$$

is uniformly bounded from above. The first term gives the main contribution: as $\cos\phi_n$ is not bounded away from 0, the derivative of its logarithm is

$$\left| \frac{d(\log\cos\phi_n)}{dx_n} \right| \leq C \left| \frac{d(\log\cos\phi_n)}{d\theta_n} \right| \leq \frac{C}{\cos\phi_n}.$$

The fourth term is the logarithm of the quantity

$$2\rho^{-1}\tau_{n+1} + \cos\phi_{n+1}(1 + \tau_{n+1}\Omega^-_{n+1})$$

that is bounded from below, but not from above. It is thus (more than) enough to show that, when taking the derivative, all contributions to the numerator are uniformly bounded. This holds immediately by the previous discussion for all the terms except $2\rho^{-1}\frac{d\tau_{n+1}}{dx_n}$ which requires further investigation. Note

$$\tau_{n+1} = dist(P(x_n), P(x_{n+1}))$$

where $P(x_n)$ and $P(x_{n+1})$ are points on the billiard table (and thus on $\mathbb{R}^2$) associated to the points $x_n \in W_n$ and $x_{n+1} \in W_{n+1}$ on the two scatterers, respectively. In an appropriate reference frame $P(x_n) = (\rho\cos\theta_n, \rho\sin\theta_n)$ hence the θ_n-derivatives of both coordinates are $\ll \rho$, and the same holds for the θ_{n+1}-derivatives of the coordinates of $P(x_{n+1})$. Thus

$$\left|\frac{d\tau_{n+1}}{dx_n}\right| \leq C\rho$$

which is sufficient for our purposes.

Case $m \geq 1$**.** The main differences in this case come down to bringing the integrals containing non bounded terms such as κ'' and κ'''' to a form similar to (56) after having gained some exponential decay in m. It is exactly here where we shall use that $\mathcal{B}_{\Delta_\rho} \subset L^p(\mu_{\bar{\Delta}_\rho})$. □

C Decay of correlation for κ.

The main result of this section is the justification of (28), that is

Proposition C.1 *There exist $\hat{C}_\rho > 0$ and $\hat{\vartheta}_\rho < 1$ such that*

- *for any $j \geq 1$ we have*

$$\Big|\int_{\mathcal{M}_0} (e^{it\kappa(\rho)} - 1)\,(e^{it\kappa(\rho)} - 1) \circ T_\rho^j \,d\mu \;-\; \int_{\mathcal{M}_0} (e^{it\kappa(\rho)} - 1)\,d\mu \int_{\mathcal{M}_0} (e^{it\kappa(\rho)} - 1)\,d\mu\Big| \;\leq \\ \leq \hat{C}_\rho |t|^2 \hat{\vartheta}_\rho^j, \tag{45}$$

- *and for any $j, \ell \geq 1$ we have*

$$\begin{aligned}\Big|\int_{\mathcal{M}_0} &(e^{it\kappa(\rho)} - 1) R_\rho^\ell (e^{it\kappa(\rho)} - 1)\,(e^{it\kappa(\rho)} - 1) \circ T_\rho^j \,d\mu \\ &- \int_{\mathcal{M}_0} (e^{it\kappa(\rho)} - 1) R_\rho^\ell (e^{it\kappa(\rho)} - 1)\,d\mu \int_{\mathcal{M}_0} (e^{it\kappa(\rho)} - 1)\,d\mu \\ &- \Big(\int_{\mathcal{M}_0} (e^{it\kappa(\rho)} - 1)\,d\mu\Big) \int_{\mathcal{M}_0} (e^{it\kappa(\rho)} - 1)\,(e^{it\kappa(\rho)} - 1) \circ T_\rho^j \,d\mu \\ &+ \Big(\int_{\mathcal{M}_0} (e^{it\kappa(\rho)} - 1)\,d\mu\Big)^3\Big| \;\leq\; \hat{C}_\rho |t|^2 \hat{\vartheta}_\rho^{m+j}.\end{aligned} \tag{46}$$

The dependence of this exponential rate on ρ gives the main source of unknown dependence on ρ in the main results of our paper. During the proof we will point out the exact sources of unknown dependence of $\hat{\vartheta}_\rho < 1$ on ρ.

The proof of this result consists in: a) reconsider [9, Proposition 9.1]; b) only for (46), work with a version of R_ρ with spectral gap in a Banach space embedded in some L^p space with $p > 1$. Item a) is needed in order to obtain the bound $|t|^2$ and the decay of correlation in j. Item b) is needed to obtain the joint decay in j and ℓ. Item b) is possible because for every $\rho > 0$, *there exists* a Young tower Δ_ρ and a tower map T_{Δ_ρ} associated with the billiard map T_ρ; this is ensured by the construction in [7, 24]. We emphasize that we will not exploit any fine dependence on ρ of T_{Δ_ρ} (the mere existence is enough), which is why this part of our arguments can be worked on the Young tower Δ_ρ.

Throughout, $C > 0$ $(C_1, C_2...)$ and $\vartheta < 1$ $(\vartheta_1, \vartheta_2, ...)$ denote constants independent of ρ and $f \asymp g$ means that there exists $C > 0$ such that $C^{-1} f \leq g \leq Cf$. In particular $d_\rho(\xi) \asymp |\xi|^{-1}$.

C.1 Standard pair argument

In this section we reconsider [9, Proposition 9.1]. Let us introduce truncation levels $H, \hat{H} > 0$ to be fixed later and

$$\kappa' = \hat{\kappa} \cdot \mathbf{1}_{|\kappa| \le H} \qquad \kappa'' = \kappa - \kappa';$$
$$\kappa''' = \kappa \cdot \mathbf{1}_{|\kappa| \le \hat{H}} \qquad \kappa'''' = \kappa - \kappa'''.$$

As $|\kappa| \asymp |\xi| m$ on $D_{\xi,m}$, the truncation κ' restricts κ to the cells $D_{\xi,m}$ with $m \le H|\xi|^{-1}$.

The result we will use in the proof of Proposition C.1 below is

Lemma C.2 *For any $c_0 > 2$ we have*

(i)' $\int |\kappa'| \cdot |\kappa''''| \circ T^j \, d\mu \le C H^2 \hat{H}^{-1} \rho^{-3}$,

(ii)' $\int |\kappa''| \cdot |\kappa| \circ T^j \, d\mu \le C |\log \rho| \cdot \left(H^{-\frac{1}{2} + \frac{1}{2r_0}} \log H \, \rho^{-3-\nu} + H^{2-c_0} \rho^{-2-c_0} \right)$.

Furthermore, for any $q \in \left(1, \frac{8}{7} - \frac{6}{7(7r_0-1)}\right)$ and

$$\frac{q+1}{2-q} < c < \frac{1 - \frac{1}{r_0}}{2q-2} - 1,$$

(i) $\int |\kappa'|^q \cdot |\kappa''''|^q \circ T^j \, d\mu \le C H^{q+1} \hat{H}^{q-2} \rho^{-3}$,

(ii) $\int |\kappa''|^q \cdot |\kappa|^q \circ T^j \, d\mu \le C \left(H^{-\frac{3}{2} + q + c(q-1) + \frac{1}{2r_0}} \rho^{c(q-1)-q-2-\nu} + H^{c(q-2)+q+1} \rho^{-1-q-c(2-q)} \right)$.

Remark C.3 *Let $q(r_0) = \frac{8}{7} - \frac{6}{7(7r_0-1)}$, the upper bound on q for r_0 fixed. Let, furthermore, $c_1(q) = \frac{q+1}{2-q}$ and $c_2(q) = \frac{1-\frac{1}{r_0}}{2q-2} - 1$, the lower and upper bounds on c for q fixed. Note that $c_1(q)$ is increasing in q, while $c_2(q)$ is decreasing in q, and $c_1(q(r_0)) = c_2(q(r_0))$. Also $c_1(1) = 2$ and $c_2(1) = \infty$, which is in accordance with the conditions on c_0. Note also that:*

- *The condition $c < c_2(q) = \frac{1-\frac{1}{r_0}}{2q-2} - 1$ is equivalent to $q + c(q-1) < \frac{3}{2} - \frac{1}{2r_0}$. This ensures that the power of H in the first term of (ii) is negative.*
- *Since $c > c_1(q) = \frac{q+1}{2-q}$, the power of H in the second term of (ii) is negative.*
- *Choosing $\hat{H} = H^c$, the power of H in (i) is also negative, again for $c > c_1(q) = \frac{q+1}{2-q}$.*

Standard pairs and families. Let us recall some terminology related to standard pairs, see also [9, page 29]. A *standard pair* $\ell = (W, h_W)$ is a regular unstable curve W that supports a dynamically log-Hölder continuous probability density h_W. As such, it can be regarded as a probability measure on the phase space $\mathcal{M}$, which will be denoted by ℓ, too.

A *standard family* is a collection of standard pairs $\mathcal{G} = \{\ell_a\}$, $a \in \mathcal{A}$ equipped with a probability factor measure $\lambda_{\mathcal{G}}$ on $\mathcal{A}$. This induces a probability measure $\mathbb{P}_{\mathcal{G}}$ on $\mathcal{M}$.

For a standard pair $\ell = (W, h_W)$ any $x \in W$ splits W into two subcurves, let $r_W(x)$ denote the length of the shorter, and let $\mathcal{Z}_\ell = \sup_{\varepsilon > 0} \varepsilon^{-1} \ell(r_W \le \varepsilon)$. By Hölder continuity of $\log h_W$, ℓ is equivalent to the normalized Lebesgue measure on W and thus $\mathcal{Z}_\ell \asymp |W|^{-1}$. This generalizes for the Z-function of a standard family $\mathcal{Z}_{\mathcal{G}} \asymp \int \frac{\lambda_{\mathcal{G}}(a)}{|W_a|} dm_W$.

The T-image of a standard pair is a countable collection of standard pairs. Hence, the image of a standard family is a standard family. Given a standard family $\mathcal{G}$, for $n \ge 1$, $\mathcal{G}_n$ denotes the

T^n-image of $\mathcal{G}$. It follows from the growth lemma (Proposition 3.1 for $\varsigma = 1$) that there exists $\vartheta < 1$ and $C_1, C_2 > 0$ such that

$$\mathcal{Z}_{\mathcal{G}_n} \le C_1 \vartheta^n \mathcal{Z}_{\mathcal{G}} + C_2 \delta_0^{-1}$$

where $\delta_0 \asymp \rho^\nu$, see (7) and also Remark 3.2, part (i)). As consequence, for any standard pair and for any $n \ge 1$

$$\mathcal{Z}_{\mathcal{G}_n} \le C \max(\mathcal{Z}_{\mathcal{G}_1}, \rho^{-\nu}). \tag{47}$$

Cells. For $\xi \in \mathbb{Z}^2$ such that the corridor is opened up, and for $m \in \mathbb{Z}$ let $D_{\xi,m} \subset \mathcal{M}$ denote the set of points for which $\kappa = m\xi + \xi'$. The geometric properties of $D_{\xi,m}$ and its image $TD_{\xi,m}$ will play an important role in the argument. $TD_{\xi,m}$ is depicted in Figure 2. A similar description applies to $D_{\xi,m}$; it is delimited by a long singularity curve, decreasing in the (θ, φ) coordinates, which is connected to the boundary of $\mathcal{M}$ by two shorter decreasing singularity curves, of length $\asymp (|\xi|\rho m)^{-1/2}$, running at a distance $\asymp (|\xi|m)^{-2}$ from each other. Further properties:

- $\mu(D_{\xi,m}) = \mu(TD_{\xi,m}) \asymp \rho^{-1}|\xi|^{-3}m^{-3}$ (due to the factor $\cos\phi$ in the measure).
- an unstable curve may intersect $D_{\xi,m}$ in a subcurve of length $\le C(|\xi|m)^{-2}$
- $TD_{\xi,m}$ intersects homogeneity strips of index $k \ge C(\rho|\xi|m)^{\frac{1}{2r_0}}$

If $\ell = (W, h_W)$ is a standard pair, then it can intersect $D_{\xi,m}$ in a subcurve of length $\le C(|\xi|m)^{-2}$, thus the intersection has probability bounded above by $C(|\xi|m)^{-2}|W|^{-1} \asymp \mathcal{Z}_\ell(|\xi|m)^{-2}$. It follows that for a standard family $\mathcal{G}$ we have

$$\mathbb{P}_{\mathcal{G}}(D_{\xi,m}) \le C(|\xi|m)^{-2}\mathcal{Z}_{\mathcal{G}}. \tag{48}$$

Our argument below follows the proof of [9, Proposition 9.1] taking into account that the corridor structure depends on ρ.

Proof of Lemma C.2. For item (i), using $\mu(T^{-j}D_{\hat{\xi},\hat{m}}) \ll \rho^{-1}\hat{m}^{-3}|\hat{\xi}|^{-3}$ as well as Lemma A.4 several times, we get

$$\begin{aligned}
\int |\kappa'|^q \cdot |\kappa'''' \circ T^j|^q \, d\mu &\le C \sum_\xi \sum_{\hat{\xi}} |\xi|^q |\hat{\xi}|^q \sum_{m=1}^{\frac{H}{|\xi|}} \sum_{\hat{m}=\frac{\hat{H}}{|\hat{\xi}|}}^{\infty} m^q \hat{m}^q \mu(D_{\xi,m} \cap T^{-n} D_{\hat{\xi},\hat{m}}) \\
&\le C\rho^{-1} \sum_\xi \sum_{\hat{\xi}} |\xi|^q |\hat{\xi}|^q \sum_{m=1}^{\frac{H}{|\xi|}} m^q \sum_{\hat{m}=\frac{\hat{H}}{|\hat{\xi}|}}^{\infty} \hat{m}^{q-3} |\hat{\xi}|^{-3} \\
&\le C\rho^{-1} \sum_\xi H^{q+1} |\xi|^{-1} \sum_{\hat{\xi}} \hat{H}^{q-2} |\hat{\xi}|^{-1} \le C H^{q+1} \hat{H}^{q-2} \rho^{-3}.
\end{aligned}$$

We will take $\hat{H} = H^c$ for $c > 0$ to be determined. To get a negative power of H, we need $q < 2$ and $c > \frac{q+1}{2-q}$.

For the proof of (ii), we need to estimate

$$\int |\kappa''|^q \cdot |\kappa \circ T^j|^q \, d\mu \le C \sum_\xi \sum_{\hat{\xi}} |\xi|^q |\hat{\xi}|^q \sum_{m=\frac{H}{|\xi|}}^{\infty} m^q \sum_{\hat{m}=1}^{\infty} \hat{m}^q \mu(D_{\xi,m} \cap T^{-j} D_{\hat{\xi},\hat{m}}). \tag{49}$$

For different ranges of the indices, we will use two different estimates to bound $\mu(D_{\xi,m}\cap T^{-j}D_{\hat{\xi},\hat{m}})$. On the one hand, as before, we have

$$\mu(D_{\xi,m}\cap T^{-n}D_{\hat{\xi},\hat{m}})\le \mu(D_{\hat{\xi},\hat{m}})\le C\rho^{-1}|\hat{\xi}|^{-3}\hat{m}^{-3}. \tag{50}$$

For the other estimate, foliate $D_{\xi,m}$ with unstable curves $|W|$ of length $\asymp (|\xi|m)^{-2}$. The image of any such curve stretches along $TD_{\xi,m}$, crossing homogeneity strips with indices $k\ge C(\rho|\xi|m)^{\frac{1}{2r_0}}$. The piece of TW in the k-th homogeneity strip will be denoted by TW_k, it has length $\asymp k^{-r_0-1}$, and its preimage has length

$$|W_k|\asymp k^{-r_0-1}\frac{\rho}{|\xi|mk^{r_0}}=\frac{\rho}{|\xi|mk^{2r_0+1}}$$

as the expansion factor of T on W_k is $\asymp \rho^{-1}|\xi|mk^{r_0}$. Equipped with the conditional measure induced by μ, W is a standard pair $\ell=(W,h_W)$, and its image is a standard family $T\ell$ associated to the curves TW_k. To obtain the Z function, we use that the weight of $|TW_k|$ within this family is $\frac{|W_k|}{|W|}$, thus

$$\begin{aligned}\mathcal{Z}_{T\ell}&\asymp \sum_{k\ge C(\rho|\xi|m)^{\frac{1}{2r_0}}}\frac{|W_k|}{|W|}|TW_k|^{-1}\asymp \sum_{k\ge C(\rho|\xi|m)^{\frac{1}{2r_0}}}\frac{\rho|\xi|^2m^2}{|\xi|mk^{2r_0+1}}k^{r_0+1}\\ &\asymp \rho m|\xi|\sum_{k\ge C(\rho|\xi|m)^{\frac{1}{2r_0}}}k^{-r_0}\asymp(\rho m|\xi|)^{\frac{1}{2}+\frac{1}{2r_0}}.\end{aligned}$$

This analysis applies to all the curves in the foliation. Accordingly, μ conditioned on $D_{\xi,m}$ can be regarded as a standard family $\mathcal{G}$, and the Z function of its T-image satisfies

$$\mathcal{Z}_{\mathcal{G}_1}\asymp C(\rho m|\xi|)^{\frac{1}{2}+\frac{1}{2r_0}}.$$

For further iterates, it follows form (47) that

$$\mathcal{Z}_{\mathcal{G}_n}\le C\rho^{-\nu}(m|\xi|)^{\frac{1}{2}+\frac{1}{2r_0}}.$$

Now we apply (48) to get

$$\begin{aligned}\mu(D_{\xi,m}\cap T^{-n}D_{\hat{\xi},\hat{m}})&=\mu(D_{\xi,m})\mathbb{P}_{\mathcal{G}_n}(D_{\hat{\xi},\hat{m}})\le C\mu(D_{\xi,m})\mathcal{Z}_{\mathcal{G}_n}|\hat{\xi}|^{-2}\hat{m}^{-2}\\ &\le C|\hat{\xi}|^{-2}\hat{m}^{-2}|\xi|^{-\frac{5}{2}+\frac{1}{2r_0}}m^{-\frac{5}{2}+\frac{1}{2r_0}}\rho^{-1-\nu}.\end{aligned} \tag{51}$$

We split (49) into two parts. If $\hat{m}\le m^c$ (for some $c>0$ to be determined), we use (51) and get

$$\begin{aligned}&\sum_{\xi}\sum_{\hat{\xi}}|\xi|^q|\hat{\xi}|^q\sum_{m=\frac{H}{|\xi|}}^{\infty}m^q\sum_{\hat{m}=1}^{m^c}\hat{m}^q\mu(D_{\xi,m}\cap T^{-n}D_{\hat{\xi},\hat{m}})\\ &\le C\rho^{-1-\nu}\sum_{\xi}\sum_{\hat{\xi}}|\xi|^{-\frac{5}{2}+q+\frac{1}{2r_0}}|\hat{\xi}|^{q-2}\sum_{m=\frac{H}{|\xi|}}^{\infty}m^{-\frac{5}{2}+q+\frac{1}{2r_0}}m^{c(q-1)}\\ &\le C\rho^{-1-\nu}H^{-\frac{3}{2}+q+c(q-1)+\frac{1}{2r_0}}\left(\sum_{\xi}|\xi|^{-1-c(q-1)}\right)\left(\sum_{\hat{\xi}}|\hat{\xi}|^{q-2}\right)\\ &\le CH^{-\frac{3}{2}+q+c(q-1)+\frac{1}{2r_0}}\rho^{c(q-1)-q-2-\nu},\end{aligned}$$

where we have used that by $q+c(q-1)<\frac{3}{2}-\frac{1}{2r_0}$ the contribution of m is summable (this condition is equivalent to $c<c_2(q)=\frac{1-\frac{1}{r_0}}{2q-2}$, cf. Remark C.3). Note that if $q=1$ then this contribution is independent of c; however, there is an additional factor of $|\log\rho|\cdot\log H$.

If $m>m^c$ we use (50) and get

$$
\begin{aligned}
&\sum_{\xi}\sum_{\hat\xi}|\xi|^q|\hat\xi|^q\sum_{m=\frac{H}{|\xi|}}^{\infty}m^q\sum_{\hat m=m^c}^{\infty}\hat m^q\mu(D_{\xi,m}\cap T^{-n}D_{\hat\xi,\hat m})\\
&\qquad\le C\rho^{-1}\sum_{\xi}\sum_{\hat\xi}|\xi|^q|\hat\xi|^{q-3}\sum_{m=\frac{H}{|\xi|}}^{\infty}m^q\sum_{\hat m=m^c}^{\infty}\hat m^{q-3}\\
&\qquad\le C\rho^{-1}\sum_{\xi}\sum_{\hat\xi}|\xi|^q|\hat\xi|^{q-3}\sum_{m=\frac{H}{|\xi|}}^{\infty}m^{c(q-2)+q}\\
&\qquad\le CH^{c(q-2)+q+1}\rho^{-1}\left(\sum_{\xi}|\xi|^{c(2-q)-1}\right)\left(\sum_{\hat\xi}|\hat\xi|^{q-3}\right)\le CH^{c(q-2)+q+1}\rho^{-1-q-c(2-q)},
\end{aligned}
$$

and in case $q=1$ we still have an additional $|\log\rho|$ factor. The condition of summability $c(q-2)+q<-1$ is satisfied as $c>\frac{q+1}{2-q}$.

Summarizing, we need

$$1\le q<2,\qquad q+c(q-1)<\frac{3}{2}-\frac{1}{2r_0},\qquad \frac{q+1}{2-q}<c.$$

First we may fix q such that

$$\frac{3}{2}-\frac{1}{2r_0}>q+\frac{q+1}{2-q}(q-1)=\frac{2q-1}{2-q}\quad\Leftrightarrow\quad q<2-\frac{6}{7-\frac{1}{r_0}}$$

and then we can fix c slightly larger than $\frac{q+1}{2-q}$, such that the conditions are still met. The range of allowed q depends on r_0, it can never exceed $\frac{8}{7}$; for the traditional $r_0=2$ the upper bound is $\frac{14}{13}$, while for $r_0=4$ the upper bound is $\frac{10}{9}$. $\square$

C.2 Exploiting the existence of a Young tower for T_ρ

Let $(\bar\Delta_\rho,T_{\bar\Delta_\rho},\mu_{\bar\Delta_\rho})$ be the corresponding one sided Young tower and let $R_{\bar\Delta_\rho}$ be the transfer operator of $T_{\bar\Delta_\rho}$. Let $\hat\kappa(\rho)$ be the version of $\kappa(\rho)$ on $\bar\Delta_\rho$. Since κ is constant on stable leaves, we have for any $j,\ell\ge 0$,

$$
\begin{aligned}
&\int_{\mathcal{M}_0}(e^{it\kappa(\rho)}-1)\,R_\rho^\ell(e^{it\kappa(\rho)}-1)\,(e^{it\kappa(\rho)}-1)\circ T_\rho^j\,d\mu\\
&=\int_{\bar\Delta_\rho}(e^{it\hat\kappa(\rho)}-1)\,R_{\bar\Delta_\rho}^\ell(e^{it\hat\kappa(\rho)}-1)\,(e^{it\hat\kappa(\rho)}-1)\circ T_{\bar\Delta_\rho}^j\,d\mu_{\bar\Delta_\rho}.
\end{aligned}\tag{52}
$$

We recall that for every $\rho>0$, $R_{\bar\Delta_\rho}$ when viewed as an operator acting on the Young Banach space $\mathcal{B}_{\bar\Delta_\rho}\subset L^p(\mu_{\bar\Delta_\rho})$ has a spectral gap (see [7, 24]).

Proof of Proposition C.1. We first prove the statement for the case when $\ell = 0$ and point out the required modifications when $\ell \geq 1$.

Case $\ell = 0$**.** Given (52), in this case we need to show that

$$\left| \int_{\bar{\Delta}_\rho} (e^{it\hat{\kappa}(\rho)} - 1)\, (e^{it\hat{\kappa}(\rho)} - 1) \circ T^j_{\bar{\Delta}_\rho}\, \mu_{\bar{\Delta}_\rho} - \Big(\int_{\bar{\Delta}_\rho} (e^{it\hat{\kappa}(\rho)} - 1)\, \mu_{\bar{\Delta}_\rho} \Big)^2 \right| \leq \hat{C}_\rho |t|^2 \hat{\vartheta}^j_\rho. \tag{53}$$

for some $\hat{\vartheta}_\rho < 1$ and some uniform $\hat{C}_\rho$.

Throughout this proof, we let $\kappa', \kappa'', \kappa''', \kappa''''$ also denote their corresponding versions on the tower Δ_ρ and the context in which they appear will make it clear which version we are referring to.

Write

$$\begin{aligned}
&\int_{\bar{\Delta}_\rho} (e^{it\hat{\kappa}(\rho)} - 1)\, (e^{it\hat{\kappa}(\rho)} - 1) \circ T^j_{\bar{\Delta}_\rho}\, d\mu_{\bar{\Delta}_\rho} = \int_{\bar{\Delta}_\rho} (e^{i\hat{\kappa}t} - e^{i\kappa' t}) \cdot (e^{i\hat{\kappa}t} - 1) \circ T^j_{\bar{\Delta}_\rho}\, d\mu_{\bar{\Delta}_\rho} \\
&+ \int_{\bar{\Delta}_\rho} (e^{i\kappa' t} - 1) \cdot (e^{i\hat{\kappa}t} - e^{i\kappa''' t}) \circ T^j_{\bar{\Delta}_\rho}\, d\mu_{\bar{\Delta}_\rho} + \int_{\bar{\Delta}_\rho} (e^{i\kappa' t} - 1) \cdot (e^{i\kappa''' t} - 1) \circ T^j_{\bar{\Delta}_\rho}\, d\mu_{\bar{\Delta}_\rho} \\
&= \int_{\bar{\Delta}_\rho} e^{i\kappa' t} \cdot (e^{i\kappa'' t} - 1) \cdot (e^{i\hat{\kappa}t} - 1) \circ T^j_{\bar{\Delta}_\rho}\, d\mu_{\bar{\Delta}_\rho} + \int_{\bar{\Delta}_\rho} (e^{i\kappa' t} - 1) \cdot e^{i\kappa''' t} \circ T^j_{\bar{\Delta}_\rho} \cdot (e^{i\kappa'''' t} - 1) \circ T^j_{\bar{\Delta}_\rho}\, d\mu_{\bar{\Delta}_\rho} \\
&+ \int_{\bar{\Delta}_\rho} (e^{i\kappa' t} - 1) \cdot (e^{i\kappa''' t} - 1) \circ T^j_{\bar{\Delta}_\rho}\, d\mu_{\bar{\Delta}_\rho} = I_1(t,\rho) + I_2(t,\rho) + I_3(t,\rho).
\end{aligned}$$

For $I_3(t,\rho)$ we use the exponential decay of correlation. This gives the only source of unknown dependence on ρ in the case $m = 0$. More precisely, for every $\rho > 0$, there exists $\hat{\theta} < 1$ and some uniform constant $C_\rho > 0$ so that

$$\begin{aligned}
&\left| I_3(t,\rho) - \int_{\bar{\Delta}_\rho} (e^{i\kappa' t} - 1)\, d\mu_{\bar{\Delta}_\rho} \int_{\bar{\Delta}_\rho} (e^{i\kappa''' t} - 1)\, d\mu_{\bar{\Delta}_\rho} \right| \\
&\leq C_\rho\, \hat{\theta}^j_\rho\, \|e^{it\kappa' t} - 1\|_{\mathcal{B}_{\Delta_\rho}}\, \|e^{it\kappa''' t} - 1\|_{\mathcal{B}_{\Delta_\rho}} \leq C_\rho\, \hat{\theta}^j_\rho H\, \hat{H}\, |t|^2.
\end{aligned} \tag{54}$$

Thus,

$$\begin{aligned}
&\left| I_3(t,\rho) - \Big(\int_{\bar{\Delta}_\rho} (e^{it\hat{\kappa}(\rho)} - 1)\, \mu_{\bar{\Delta}_\rho} \Big)^2 \right| \leq C_\rho\, \hat{\theta}^j_\rho H\, \hat{H}\, |t|^2 \\
&+ \Big| \int_{\bar{\Delta}_\rho} (e^{i\kappa' t} - 1)\, d\mu_{\bar{\Delta}_\rho} \int_{\bar{\Delta}_\rho} (e^{i\kappa''' t} - 1)\, d\mu_{\bar{\Delta}_\rho} - \int_{\bar{\Delta}_\rho} (e^{i\hat{\kappa}t} - 1)\, d\mu_{\bar{\Delta}_\rho} \int_{\bar{\Delta}_\rho} (e^{i\hat{\kappa}t} - 1)\, d\mu_{\bar{\Delta}_\rho} \Big| \\
&= C_\rho\, \hat{\theta}^j_\rho H\, \hat{H}\, |t|^2 + |J(t,\rho)|.
\end{aligned}$$

By definition,

$$|J(t,\rho)| = \Big| \int_{\mathcal{M}_0} (e^{i\kappa' t} - 1)\, d\mu \int_{\mathcal{M}_0} (e^{i\kappa''' t} - 1)\, d\mu - \int_{\mathcal{M}_0} (e^{i\hat{\kappa}t} - 1)\, d\mu \int_{\mathcal{M}_0} (e^{i\hat{\kappa}t} - 1)\, d\mu \Big|$$

and we note that $J(t,\rho)$ is bounded by the sum of

$$\int_{\mathcal{M}_0} |e^{i\kappa' t} \cdot (e^{it\kappa''} - 1)|\, d\mu \int_{\mathcal{M}_0} |e^{it\kappa''' t} - 1|\, d\mu \leq |t|^2 \int |\kappa| 1_{\{\kappa > H\}}\, d\mu \int_{\mathcal{M}_0} |\kappa'''|\, d\mu$$

and a similar term with $\hat{H}$ instead of H. Using the Hölder inequality (with exponents $\frac{2}{1+\delta}$ and $\frac{2}{1-\delta}$), the tail behaviour of κ and Lemma A.5, we obtain that

$$\int_{\mathcal{M}_0} |\kappa| 1_{\{|\kappa| > H\}}\, d\mu \leq \|\kappa\|_{L^{2/(1+\delta)}}\, \mu(|\kappa| > H)^{(1-\delta)/2} \ll \rho^{-1} H^{-(1-\delta)}.$$

Also $\int_{\mathcal{M}_0} |\kappa'''| d\mu \le \|\kappa\|_{L^1(\mu)} \ll \rho^{-1}$. Hence,

$$|J(t,\rho)| \ll |t|^2 \rho^{-2} \left(H^{-(1-\delta)} + \hat{H}^{-(1-\delta)} \right). \tag{55}$$

Finally, note that

$$\begin{aligned} |I_1(t,\rho) + I_2(t,\rho)| &\le |t|^2 \int_{\bar{\Delta}_\rho} |\kappa''| \cdot |\kappa| \circ T^j_{\bar{\Delta}_\rho} \, d\mu_{\bar{\Delta}_\rho} + |t|^2 \int_{\bar{\Delta}_\rho} |\kappa'| \cdot |\kappa''''| \circ T^j_{\bar{\Delta}_\rho} \, d\mu_{\bar{\Delta}_\rho} \\ &= |t|^2 \left(\int_{\mathcal{M}_0} |\kappa''| \cdot |\kappa| \circ T^j_{\bar{\Delta}_\rho} \, d\mu + \int_{\mathcal{M}_0} |\kappa'| \cdot |\kappa''''| \circ T^j_{\bar{\Delta}_\rho} \, d\mu \right). \end{aligned} \tag{56}$$

For this case $\ell = 0$ case if we fix any $r_0 \ge 2$ (taking into account that $\hat{H} = H^{c_0}$), then we may bound the coefficients of $|t|^2$ in $|J(t,\rho)|$ from (55), $|I_1(t,\rho)|$ and $|I_2(t,\rho)|$ from (56), respectively by

$$\rho^{-2} H^{-(1-\delta)}; \qquad H^{-\frac{1}{5}} \rho^{-4} + H^{2-c_0} \rho^{-\frac{11}{5} - c_0}, \qquad H^{2-c_0} \rho^{-3},$$

where in the bound for $|I_1(t,\rho)|$ the exponents of H and ρ have been slightly decreased to bound the logarithmic factors. Fixing $c_0 = \frac{11}{5}$ and $\delta = \frac{4}{5}$, all these are dominated by $H^{-\frac{1}{5}} \rho^{-\frac{22}{5}}$. On the other hand the coefficient of $|t|^2$ in $|I_3(t,\rho)|$ is $C_\rho \, \hat{\theta}^j_\rho H^{c_0+1} = C_\rho \, \hat{\theta}^j_\rho H^{\frac{16}{5}}$. Thus letting $H = \left(C_\rho^{-1} \, \hat{\theta}_\rho^{-j} \rho^{-\frac{22}{5}} \right)^{\frac{5}{17}}$ we conclude that all terms are dominated by

$$\rho^{-\frac{352}{85}} C_\rho^{\frac{1}{17}} \, (\hat{\theta}_\rho^{\frac{1}{17}})^j.$$

I have not checked the $\ell \ge 1$ case yet but I think we can argue by continuity that the same exponents work there, too (choosing q sufficiently close to 1 and thus p large accordingly). Well the price we pay that we have to replace C_ρ by C^0_ρ from (58) – which, for large p, can be large – but since we have no control on it anyway, I do not think we should bother with that too much.

Case $\ell \ge 1$. The main differences in this case come down to dealing with integrals containing unbounded terms κ'' and κ'''' in such a way that can gain exponential decay in ℓ and then proceed as in the case $\ell = 0$ treated above. To do this, we exploit that $\mathcal{B}_{\bar{\Delta}_\rho} \subset L^p(\mu_{\bar{\Delta}_\rho})$.

Using (52), we need to estimate

$$\begin{aligned} J(t,\rho) := &\int_{\bar{\Delta}_\rho} (e^{it\hat{\kappa}(\rho)} - 1) \, R^\ell_{\bar{\Delta}_\rho} (e^{it\kappa(\rho)} - 1) \, (e^{it\hat{\kappa}(\rho)} - 1) \circ T^j_{\bar{\Delta}_\rho} \, d\mu_{\bar{\Delta}_\rho} \\ &- \int_{\bar{\Delta}_\rho} (e^{it\hat{\kappa}(\rho)} - 1) R^\ell_{\bar{\Delta}_\rho} (e^{it\hat{\kappa}(\rho)} - 1) \, d\mu_{\bar{\Delta}_\rho} \int_{\bar{\Delta}_\rho} (e^{it\hat{\kappa}(\rho)} - 1) \, d\mu_{\bar{\Delta}_\rho} \\ &- \int_{\bar{\Delta}_\rho} (e^{it\hat{\kappa}(\rho)} - 1) \, d\mu_{\bar{\Delta}_\rho} \int_{\bar{\Delta}_\rho} (e^{it\hat{\kappa}(\rho)} - 1) \, (e^{it\hat{\kappa}(\rho)} - 1) \circ T^j_\rho \, d\mu_{\bar{\Delta}_\rho} \\ &+ \left(\int_{\bar{\Delta}_\rho} (e^{it\hat{\kappa}(\rho)} - 1) \, d\mu_{\bar{\Delta}_\rho} \right)^3. \end{aligned}$$

For every $\rho > 0$, the transfer operator $R_{\bar{\Delta}_\rho}(0)$ has a spectral gap in $\mathcal{B}_{\bar{\Delta}_\rho}$. Thus, for every $\ell \ge 1$, $R^\ell_{\bar{\Delta}_\rho} (e^{it\hat{\kappa}(\rho)} - 1) - \int_{\bar{\Delta}_\rho} (e^{it\kappa(\rho)} - 1) \, d\mu_{\bar{\Delta}_\rho} = Q^\ell_{\bar{\Delta}_\rho} (e^{it\kappa(\rho)} - 1)$ where

$$\|Q^\ell_{\bar{\Delta}_\rho} (e^{it\hat{\kappa}(\rho)} - 1)\|_{\mathcal{B}_{\bar{\Delta}_\rho}} \le C_\rho \, \hat{\theta}^\ell_\rho, \tag{57}$$

for some non-uniform C_ρ and some $\hat{\theta}_\rho < 1$. This is the first source of unknown dependence on ρ. Since $\mathcal{B}_{\bar{\Delta}_\rho} \subset L^p(\mu_{\bar{\Delta}_\rho})$,

$$\|Q^\ell_{\bar{\Delta}_\rho} (e^{it\hat{\kappa}(\rho)} - 1)\|_{L^p(\mu_{\bar{\Delta}_\rho})} \le C^0_\rho \, \hat{\theta}^\ell_\rho, \tag{58}$$

for some non-uniform C^0_ρ. This is the second source of unknown dependence on ρ.

With these specified, we can write

$$\begin{aligned}
J(t,\rho) &= \int_{\bar\Delta_\rho} (e^{it\hat\kappa(\rho)}-1)\, Q^\ell_{\bar\Delta_\rho}(e^{it\hat\kappa(\rho)}-1)\,(e^{it\hat\kappa(\rho)}-1)\circ T^j_{\bar\Delta_\rho}\, d\mu_{\bar\Delta_\rho} \\
&\quad - \int_{\bar\Delta_\rho} (e^{it\hat\kappa(\rho)}-1) Q^\ell_{\bar\Delta_\rho}(e^{it\hat\kappa(\rho)}-1)\, d\mu_{\bar\Delta_\rho} \int_{\bar\Delta_\rho} (e^{it\hat\kappa(\rho)}-1)\, d\mu_{\bar\Delta_\rho} \\
&= E(t,\rho) - G(t,\rho)
\end{aligned}$$

and rearranging as in the case $m=0$,

$$\begin{aligned}
E(t,\rho) &= \int_{\bar\Delta_\rho} (e^{i\hat\kappa t}-e^{i\hat\kappa' t})\, Q^\ell_{\bar\Delta_\rho}(e^{it\kappa(\rho)}-1)\ (e^{i\hat\kappa t}-1)\circ T^j_{\bar\Delta_\rho}\, d\mu_{\bar\Delta_\rho} \\
&\quad + \int_{\bar\Delta_\rho} (e^{i\kappa' t}-1)\, Q^\ell_{\bar\Delta_\rho}(e^{it\hat\kappa(\rho)}-1)\,(e^{i\hat\kappa t}-e^{i\kappa''' t})\circ T^j_{\bar\Delta_\rho}\, d\mu_{\bar\Delta_\rho} \\
&\quad + \int_{\bar\Delta_\rho} (e^{i\kappa' t}-1)\, Q^\ell_{\bar\Delta_\rho}\,(e^{it\hat\kappa(\rho)}-1)\,(e^{i\kappa''' t}-1)\circ T^j_{\bar\Delta_\rho}\, d\mu_{\bar\Delta_\rho} \\
&= \int_{\bar\Delta_\rho} e^{i\kappa' t}\, Q^\ell_{\bar\Delta_\rho}(e^{it\hat\kappa(\rho)}-1)\,(e^{i\kappa'' t}-1)\cdot(e^{i\hat\kappa t}-1)\circ T^j_{\bar\Delta_\rho}\, d\mu_{\bar\Delta_\rho} \\
&\quad + \int_{\bar\Delta_\rho} (e^{i\kappa' t}-1)\, Q^\ell_{\bar\Delta_\rho}(e^{it\hat\kappa(\rho)}-1)\, e^{i\kappa''' t}\circ T^j_{\bar\Delta_\rho}\cdot(e^{i\kappa'''' t}-1)\circ T^j_{\bar\Delta_\rho}\, d\mu_{\bar\Delta_\rho} \\
&\quad + \int_{\bar\Delta_\rho} (e^{i\kappa' t}-1)\, Q^\ell_{\bar\Delta_\rho}(e^{it\hat\kappa(\rho)}-1)\,(e^{i\kappa''' t}-1)\circ T^j_{\bar\Delta_\rho}\, d\mu_{\bar\Delta_\rho} = E_1(t,\rho)+E_2(t,\rho)+E_3(t,\rho).
\end{aligned}$$

Let $q\in(1,\frac{8}{7}-\frac{6}{7r_0-1})$ so that Lemma C.2 holds. Using Hölder inequality with $1/p+1/q=1$ and (58),

$$\begin{aligned}
|E_1(t,\rho)+E_2(t,\rho)| &\le \|Q^\ell_{\bar\Delta_\rho}(e^{it\hat\kappa(\rho)}-1)\|_{L^p(\mu_{\bar\Delta_\rho})}\,|t|^2\||\kappa''|\cdot|\hat\kappa|\circ T^j_{\bar\Delta_\rho}\|_{L^q(\mu_{\bar\Delta_\rho})} \\
&\quad + \|Q^\ell_{\bar\Delta_\rho}(e^{it\hat\kappa(\rho)}-1)\|_{L^p(\mu_{\bar\Delta_\rho})}\,|t|^2\||\kappa'|\cdot|\kappa''''|\circ T^j_{\bar\Delta_\rho}\|_{L^q(\mu_{\bar\Delta_\rho})} \\
&\le C^0_\rho\,\hat\theta^\ell_\rho\,|t|^2\left(\||\kappa''|\cdot|\hat\kappa|\circ T^j_{\bar\Delta_\rho}\|_{L^q(\mu_{\bar\Delta_\rho})}+\||\kappa'|\cdot|\kappa''''|\circ T^j_{\bar\Delta_\rho}\|_{L^q(\mu_{\bar\Delta_\rho})}\right).
\end{aligned}$$

Similar to estimating (56), using Lemma C.2 and Remark C.3 and without trying for optimal bounds, we can pick q close to 1 and $c_0<\frac{5}{2}$ such that $c_0(q-2)+q+1=-\frac{1}{5}$. For these values,

$$|E_1(t,\rho)+E_2(t,\rho)| \le C\,C^0_\rho\,\hat\theta^\ell_\rho\,|t|^2\,H^{-\frac{1}{5q}}\rho^{\frac{-5}{q}}. \tag{59}$$

Next, let

$$\begin{aligned}
L_1(t,\rho) &= \int_{\bar\Delta_\rho} (e^{i\kappa' t}-1)\, Q^\ell_{\bar\Delta_\rho}(e^{it\hat\kappa(\rho)}-1)\,(e^{i\kappa''' t}-1)\circ T^j_{\bar\Delta_\rho}\, d\mu_{\bar\Delta_\rho} \\
&\quad - \int_{\bar\Delta_\rho} (e^{it\kappa'}-1) Q^\ell_{\bar\Delta_\rho}(e^{it\hat\kappa(\rho)}-1)\, d\mu_{\bar\Delta_\rho} \int_{\bar\Delta_\rho} (e^{it\kappa'''}-1)\, d\mu_{\bar\Delta_\rho}
\end{aligned}$$

and note that

$$\begin{aligned}
E_3(t,\rho)-G(t,\rho) &= L_1(t,\rho) - \int_{\bar\Delta_\rho} (e^{it\hat\kappa(\rho)}-e^{it\kappa'}) Q^\ell_{\bar\Delta_\rho}(e^{it\hat\kappa(\rho)}-1)\, d\mu_{\bar\Delta_\rho} \int_{\bar\Delta_\rho} (e^{it\hat\kappa(\rho)}-1)\, d\mu_{\bar\Delta_\rho} \\
&\quad - \int_{\bar\Delta_\rho} (e^{it\hat\kappa(\rho)}-1) Q^\ell_{\bar\Delta_\rho}(e^{it\hat\kappa(\rho)}-1)\, d\mu_{\bar\Delta_\rho} \int_{\bar\Delta_\rho} (e^{it\hat\kappa(\rho)}-e^{it\kappa'''})\, d\mu_{\bar\Delta_\rho} \\
&= L_1(t,\rho) - L_2(t,\rho) - L_3(t,\rho).
\end{aligned}$$

By the exponential decay of correlations as in (54) as well as (58)

$$|L_1(t,\rho)| \le C_\rho\, \hat{\theta}_\rho^j H\, \hat{H}\, |t|^2 \|Q^\ell_{\bar{\Delta}_\rho}(e^{it\hat{\kappa}(\rho)}-1)\|_{L^p(\mu_{\bar{\Delta}_\rho})} \le C_\rho\, C^0_\rho\, \hat{\theta}^\ell_\rho\, |t|^2\, H^{1+c_0},$$

where as before $c_0 < \frac{5}{2}$. Finally, by the equation before (55), we have

$$|L_2(t,\rho)| \le |t^2|\, \rho^{-1} H^{-(1-\delta)} \|Q^\ell_{\bar{\Delta}_\rho}(e^{it\hat{\kappa}(\rho)}-1)\|_{L^p(\mu_{\bar{\Delta}_\rho})} \le C_\rho\; C^0_\rho\, \hat{\theta}^\ell_\rho |t^2|\, \rho^{-1} H^{-(1-\delta)}$$

A similar argument applies to $L_3(t,\rho)$.

The conclusion follows with a similar choice of H as in the case $\ell = 0$ treated above. $\square$

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