ON THE RECORDS AND ZEROS OF A DETERMINISTIC RANDOM WALK

HENK BRUIN AND ROBBERT FOKKINK

Abstract. We settle two questions on sequence A120243 in the OEIS that were raised by Clark Kimberling and partly solve a conjecture of Van de Lune and Arias de Reyna. We extend Kimberling's questions to the framework of deterministic random walks, automatic sequences, and linear recurrences. Our results indicate that there may be a deeper connection between these structures. In particular, we conjecture that the records of deterministic random walks are ξ-Ostrowski automatic for a quadratic rotation number ξ.

Sequence A120243 in the OEIS consists of numbers n such that the fractional part $\{n\sqrt{2}\}$ is less than $\frac{1}{2}$. Its complementary sequence, denoted as $b(n)$, is A120749. The table below presents the first eighteen elements of these sequences:

$a(n)$	1	3	5	6	8	10	13	15	17	18	20	22	25	27	29	30	32	34
$b(n)$	2	4	7	9	11	12	14	16	19	21	23	24	26	28	31	33	36	38

Table 1. The first eighteen numbers from A120243 and A120749

These sequences were entered into the OEIS by Clark Kimberling, who posed the question of whether the difference $b(n) - a(n)$ is positive for all n and whether, for each integer k, there exist infinitely many values of n such that $b(n) - a(n) = k$. We confirm both of these properties. Kimberling's questions relate to the rotation of the unit circle by $\sqrt{2}$, which is often an initial case leading to broader mathematical results, such as in [4]. Our analysis of the sequences $a(n)$ and $b(n)$ extends to the more general framework of automatic sequences and ergodic theory.

We could also have described the sequences A120243 and A120749 by the parity of $\lfloor 2n\sqrt{2}\rfloor$. The sequence a corresponds to the even numbers, and b corresponds to the odd numbers. Or alternatively, if instead of parity we use signs by putting $(-1)^{\lfloor 2n\sqrt{2}\rfloor}$, then we can think of the sequences a and b as the steps in the positive direction and the negative direction, respectively, in what is known as, rather curiously, a *deterministic random walk*

$$S_n(\xi) = \sum_{j=1}^{n} (-1)^{\lfloor j\xi \rfloor}, \tag{1}$$

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for $\xi = 2\sqrt{2}$. If we move from fractional parts $\{n\xi\}$ to deterministic random walks $S_n(2\xi)$, then we double ξ. Kimberling's question whether $b(n) - a(n)$ is positive is equivalent to the question whether $S_n(2\sqrt{2})$ is non-negative. This problem closely resembles Problem B6 from the 81st William Lowell Putnam Mathematical Competition, which requires proving that $S_n(\sqrt{2}-1)$ is non-negative, see [5].

A number n is a *record* of the deterministic random walk if none of the previous partial sums (including zero) is equal to $S_n(\xi)$. Sequence A123737 in the OEIS contains the partial sums of the deterministic random walk for $\xi = \sqrt{2}$. In a comment on A123737, Václav Kotěšovec asked if the records satisfy a certain recurrence relation. Van de Lune and Arias de Reyna [13] conjecture that there is a system of recurrence relations for the records of the deterministic random walk $S_n(\xi)$ for all quadratic irrationals ξ. Their work extends earlier results of O'Bryant et al [15]. We confirm Kotěšovec's recurrence and settle some instances of the conjecture.

A number n is a *zero* of the deterministic random walk if $S_n(\xi) = 0$. Deterministic random walks $S_n(\xi) = \sum_{j=1}^{n}(-1)^{\lfloor x+j\xi\rfloor}$ (with offset $x \in [0,1)$) were introduced by Aaronson and Keane [1], and their primary interest was an estimate of the asymptotic number of zeros as n goes to infinity for a generic number x. More specifically, if N_n is the number of zeros up to n, then it can be interpreted as a random variable depending on a uniformly random x. The asymptotic mean and variance of N_n remain a subject of ongoing research [2, 7].

1. Irrational rotations and automata

The sequences a and b arise from an irrational rotation of the unit circle. The number $a(n)$ marks the n-th return to the semi-circle $[0, \frac{1}{2})$ under the rotation $x \mapsto x + \sqrt{2}$, starting from $x = 0$, and $b(n)$ is the n-th return to its complement $[\frac{1}{2}, 1)$. According to the ergodic theorem, both $\lim \frac{a(n)}{n}$ and $\lim \frac{b(n)}{n}$ equal two, which is suggested already by the first eighteen entries in Table 1. Given this, the next point of interest is the behavior of the differences $a(n) - 2n$ and $b(n) - 2n$. Such deviations from the mean are known as *discrepancies*, which is a significant topic in the study of sequences [8]. Kimberling's question regarding whether $b - a$ assumes every positive number infinitely often falls within this topic. His other question on the signature of $b - a$ is equivalent to asking whether $a(n) - 2n$ is negative and $b(n) - 2n$ is non-negative.

For $N \in \mathbb{N}$ an interval $I \subset (0,1)$ and $\xi \in (0,1)$, let $\ell(I)$ be the length of I and let $N(\xi, I)$ be the cardinality of $\{n\colon \{n\xi\} \in I,\ n \le N\}$. Kesten [12] famously proved that

$$\limsup_{N\to\infty} |N(\xi, I) - \ell(I)\cdot N| < \infty \tag{2}$$

if and only if $\ell(I) = \{m\xi\}$ for some $m \in \mathbb{N}$. If we apply this to $\xi = \{\sqrt{2}\}$ and $I = [0, \frac{1}{2})$, then we get that the difference between $N/2$ and the number of $\{n\xi\} < \frac{1}{2}$ up to N is unbounded. This implies that $a(n) - 2n$ and $b(n) - 2n$ are both unbounded.

Sós [19] observed that the discrepancy $N(\xi, I) - \ell(I)\cdot N$ may be bounded on one side and unbounded on the other. Recall the (regular) continued fraction of a real number ξ

$$\xi = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$$

The coefficients a_i are the partial quotients and the finite expansions $\frac{p_n}{q_n}$ are the convergents, starting from $\frac{p_0}{q_0} = \frac{a_0}{1}$. Dupain and Sós [9] gave a necessary and sufficient condition on $I = [0, \beta)$ and ξ for one-sided boundedness, for ξ with bounded partial quotients (such as quadratic irrationals). Boshernitzan and Ralston [6] found an elegant condition for nonnegative discrepancy in terms of the convergents of ξ.

Theorem 1 (Boshernitzan and Ralston)**.** *Let $I = [0, \frac{h}{k})$ and let $\frac{p_n}{q_n}$ be the convergents of ξ. The discrepancy $N(\xi, I) - \ell(I) \cdot N$ is nonnegative if and only if $k \mid q_{2n+1}$ for all n.*

In particular, if the odd convergents q_{2n+1} of ξ are even then the number of $\{n\xi\}$ in $[0, \frac{1}{2})$ up to N exceeds, or is equal to, the number of $\{n\xi\}$ in $[1/2, 1)$. This is equivalent to the non-negativity of $S_n(2\xi)$. The denominators q_n of the convergents of $\sqrt{2}$, starting from $q_0 = 1$, are the Pell numbers $1, 2, 5, 12, \ldots$, entry A000129 in the OEIS. The q_n are even for odd n and so the discrepancy is nonnegative by Theorem 1. For each N at least half of the iterates $\{n\sqrt{2}\}$ are in $[0, \frac{1}{2})$. It follows that $a(n) - 2n$ is negative and $b(n) - 2n$ is nonnegative for all n, and therefore $b - a$ is positive. This settles one of the two questions of Kimberling.

Corollary 2. *The sequence $b - a$ is positive.*

The odd convergents are even if and only if the odd partial quotients a_{2n+1} are even. Problem B6 of the 2021 Putnam Competition asks one to prove that $S_n(\sqrt{2} - 1)$ is non-negative. The continued fraction expansion of $\frac{\sqrt{2}-1}{2}$ is $[0; \overline{4, 1}]$ where the bar indicates that these coefficients are repeated. The partial quotients a_{2n+1} are all equal to 4 so that Theorem 1 settles problem B6.

The Pell numbers P_n form the basis of a numeration system [18, Ch 3.4]. They satisfy the recurrence $P_{n+1} = 2P_n + P_{n-1}$ starting from $P_0 = 0,\ P_1 = 1$. Please note that there is a mismatch between the indexing of the Pell numbers and the denominators of the convergents of $\sqrt{2}$. The Pell numbers start at $P_0 = 0$ and the denominators start at $q_0 = 1$. Each number can be represented as a sum $n = \sum_{i=1}^{j} d_i P_i$ with digits $d_i \in \{0, 1, 2\}$. The representation is unique under the condition that $d_i = 0$ if $d_{i+1} = 2$. For example, $69 = 1 \cdot 1 + 0 \cdot 2 + 2 \cdot 5 + 0 \cdot 12 + 2 \cdot 29$. As in decimal notation, it is standard practice to write the digits with the most significant digit first (msd). The decimal number 69 is represented by 20201 in Pell numeration. A sequence is *Pell automatic* if there exists a deterministic finite-state automaton (DFA) that reads digits in Pell numeration and decides if a number is in the sequence. The DFA in Fig. 1 decides if a number is in A120749 or not. For example, 69 is entered as 20201, leading to the state transitions $0 \to 2 \to 4 \to 2 \to 4 \to 5$ ending in the accepting state 5. The number is in the sequence. In fact, inspection of the OEIS shows that it is the 34th element of the sequence, which implies that the 69th step of the deterministic random walk has value 1. The next step, seventy, is Pell number P_6, which has representation 100000. It ends in state

3, which is accepting. Therefore, the walk has value 0 at this point: P_6 is a zero of $S_n(2\sqrt{2})$. We will prove below that the zeros and records of this walk are Pell automatic.

The DFA in Fig. 1 is constructed from a method due to Schaeffer et al. [17], who showed how a Beatty sequence such as $\lfloor n\sqrt{2}\rfloor$ can be implemented in the automatic theorem prover `Walnut` [14]. In particular, it is possible to implement a command `beattysqrt2(n,r)` which accepts numbers $r = \lfloor n\sqrt{2}\rfloor$. Sequence A120749 contains the numbers n such that $\lfloor 2n\sqrt{2}\rfloor$ is even. In terms of first-order logic, a number n is in the sequence if

$$\exists k\ \lfloor 2n\sqrt{2}\rfloor = 2k + 1.$$

In the `Walnut` environment this becomes

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"?msd_pell E k $beattysqrt2(2*n,2*k+1)":
```

which produces the depicted DFA.

2. The records of $S_n(2\sqrt{2})$

We say that r is a *record* if $S_r(\xi) = m$ and none of the $S_n(\xi)$ for $n < r$ are equal to m. The sequence R_n of records of $S_n(\sqrt{2})$ starts as

$$0,\ 1,\ 3,\ 8,\ 20,\ \ldots$$

Jan van de Lune [20] conjectured that the records of $S_n(\sqrt{2})$ satisfy a Pell-like recurrence, which was confirmed in [10].

Theorem 3 (Van de Lune)**.** *The sequence of records R_n of $S_n(\sqrt{2})$ satisfies the recurrence*

$$R_{n+1} = 2R_n + R_{n-1} + 1, \text{ with } R_0 = 0, R_1 = 1. \tag{3}$$

and the values of consecutive records have alternating signs.

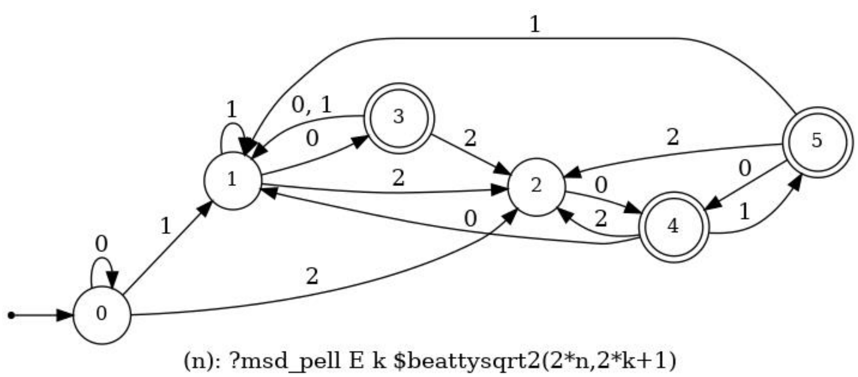

Figure 1. A finite state automaton that decides if a number n is in A120749 or in its complement A120243. The input is in msd format. Inputs that end in double circled states (acceptance) are in A120749. Inputs that end in state 1 (rejection) are in A120243. Inputs that end in 2 are not a valid Pell representation. The automaton is produced by the automatic theorem prover `Walnut` using the results in [17].

The records are sums of Pell numbers $R_n = \sum_{j=1}^{n} P_j$. In Pell numeration, their representation is $11\cdots 1$ and the sequence R_n is Pell automatic.

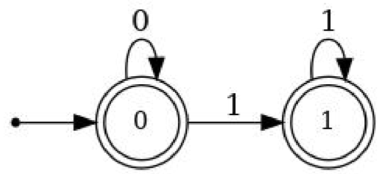

FIGURE 2. An automaton that accepts the records of $S_n(\sqrt{2})$ in Pell numeration in msd representation.

Václav Kotěšovec separated the records into positive and negative values. For a natural number m, let A_m be the first index such that $S_n(\sqrt{2}) = m$ and let B_m be the first index such that $S_n(\sqrt{2}) = -m$, both starting out from $A_0 = B_0 = 0$. Kotěšovec noticed that these are recurrence sequences and that A_n occurs as A001652 while B_n occurs as A001108 in the OEIS.

Corollary 4. *The sequences A_n and B_n satisfy the recurrences*

$$A_{n+1} = 6A_n - A_{n-1} + 2, \text{ with } A_0 = 0, A_1 = 3,$$

and

$$B_{n+1} = 6B_n - B_{n-1} + 2, \text{ with } B_0 = 0, B_1 = 1.$$

Proof. This follows from Van de Lune's recursion (3) and the observation that the first step is in the negative direction:

$$\begin{aligned} R_{n+2} &= 2\big(2R_n + R_{n-1} + 1\big) + R_n + 1 \\ &= 5R_n + 2R_{n-1} + 3 \\ &= 6R_n - R_{n-2} + 2. \end{aligned}$$

□

The first few records of $S_n(2\sqrt{2})$ are

$$0,\ 1,\ 6,\ 35,\ 204,\ \ldots$$

which happen to be the first few half Pell numbers $P_{2n}/2$. These numbers satisfy the recurrence relation $Q_{n+1} = 6Q_n - Q_{n-1}$, which is Kotěšovec's recursion up to a constant. To verify that the records Q_n are indeed half Pell numbers, we apply the algorithm from [10] to compute values of the deterministic random walk $S_n(\xi)$. It assumes that $0 < \xi < 1$ and depends on the denominators of the convergents q_n of $\xi/2$. In our case $\xi = 2\sqrt{2}$, we translate to $2\sqrt{2} - 2$ to place it in the unit interval and divide by two, to get $\sqrt{2} - 1$. The denominators of its convergents are the Pell numbers (starting from $q_0 = 1 = P_1$). For this particular case we have the following three rules. We write $q = q_{n+1}$, $q' = q_n$, and $q'' = q_{n-1}$ and we write S_n for $S_n(2\sqrt{2})$. The rules are, in this special case of $\xi = 2\sqrt{2}$:

Recursive rules for $S_n(2\sqrt{2})$

Rule A: $S_q = 1$ if q is odd and $S_q = 0$ if q is even.
Rule B: $S_{q-k} = S_{q'} + S_{k-1}$ if $1 \leq k \leq q''$.
Rule C: $S_{q'+k} = S_{q'} + S_k$ if $1 \leq k < q'$.

An explanation for these rules is that $q\xi$ is a close return to 0 mod 1. If q is even, there are equally many steps in the two semicircles $[0, 1/2)$ and $[1/2, 1)$. If q is odd, then it is the denominator of a convergent $p/q < \xi$. The return $q\xi$ is in $[0, 1/2)$ and therefore there is one more step in $[0, 1/2)$.

Rule B determines S_n for $n \in [q - q'', q)$ and Rule C determines S_n for $n \in (q', 2q')$. Since $q - q'' = 2q'$ these rules, together with Rule A, determine all values S_n by recursion. The parity of the Pell numbers alternates. If q is odd than q' is even and vice versa. If the interval $[q', q]$ is marked by an even q' and an odd q, then the contributions S'_q in rule B and rule C are zero. It follows that there is no record in $[q', q]$ in this case. Records can only occur if q' is odd and q is even. Notice that a Pell number P_n is even if and only if its index is even.

Theorem 5. *The records Q_n of $S_n(2\sqrt{2})$ are the half-Pell numbers $P_{2n}/2$.*

Note that the half-Pell number $P_{2n}/2$ is the sum of the first n odd Pell numbers. By rules A and C, each odd Pell number contributes 1 to the walk at step $P_{2n}/2$.

Proof. We can limit our attention to intervals $[q', q]$ such that q' is odd. These are the intervals $[P_{2n-1}, P_{2n}]$. For $n = 1$ we have $P_2 = 2$ and the unique record in $[1, 2]$ occurs indeed at $1 = P_2/2$. By rules B and C the value of S_n increases by $S_{q'} = 1$ for $n \in (q', q)$ compared to earlier values, so there is at most one record in this interval. By rule C $S_{q'+k}(\sqrt{2}) = 1 + S_k(\sqrt{2})$ for all $k \in [0, q')$ and therefore the record has to come from this rule. The new record in $[q', q]$ is $q' + r$ for the old record $r \in [0, q')$. By induction, records occur as sums of odd Pell numbers $P_1 + P_3 + \cdots + P_{2m-1} = P_{2m}/2$. □

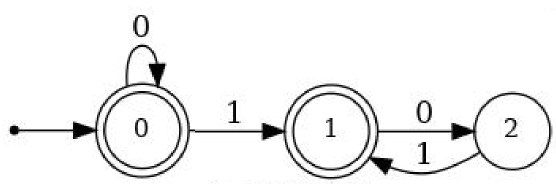

FIGURE 3. An automaton that accepts the records of $S_n(2\sqrt{2})$ in Pell numeration in msd representation.

If we ignore the first record at 0, then in Pell numeration the records of $S_n(2\sqrt{2})$ occur at 1, 101, 10101, ... and more generally $(10)^*1$, where the Kleene $*$ represents an arbitrary repetition. Again, they form a Pell automatic sequence. The Ostrowski numeration system can be defined for every real number ξ, see [18, ch 3.5]. It is equal to Pell numeration if $\xi = \sqrt{2}$.

Definition 6. *Given a positive real irrational number $\xi = [a_0, a_1, \dots]$ with continued fraction convergents $p_n/q_n = [a_0, a_1, \dots, a_n]$, we can write every integer $N \geq 0$ uniquely as*

$$N = \sum_{0 \leq i \leq j} b_i q_i$$

where the digits $(b_i)_{i \geq 0}$ satisfy the conditions

(a) $0 \leq b_0 < a_1$.

(b) $0 \le b_i \le a_{i+1}$, *for* $i \ge 1$.
(c) *For* $i \ge 1$, *if* $b_i = a_{i+1}$, *then* $b_{i-1} = 0$.

A sequence is ξ-Ostrowski automatic if there exists a finite state automaton that decides if a number is in the sequence. The following conjecture is a variation of the conjecture of van de Lune and Arias de Reyna, which we mentioned earlier.

Conjecture 7. *The records of $S_n(2\xi)$ are ξ-Ostrowski automatic for quadratic irrational ξ.*

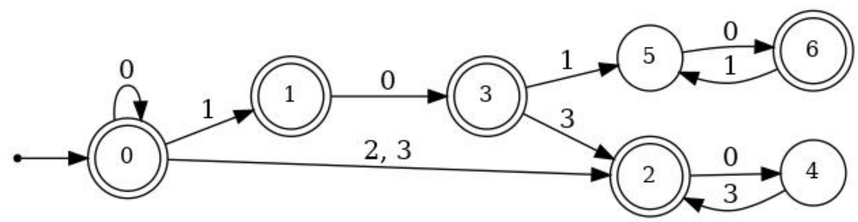

FIGURE 4. An automaton that accepts the records of $S_n(\sqrt{3})$ in $\sqrt{3}/2$-Ostrowski numeration, provided that the system of recurrences conjectured by Van de Lune and Arias de Reyna holds. The continued fraction of $\sqrt{3}/2$ has partial coefficients $[0; 1, \overline{6, 2}]$, where the bar marks that these coefficients repeat. The denominators of its convergents are $1, 7, 15, 97, \ldots$.

To back up their conjecture, Van de Lune and Arias de Reyna provide a specific recurrence relation for the records of $\xi = \sqrt{3}$, with initial values $t_j = 0$ for negative indices (where we corrected a typo for t_{4n+1}).

$$\begin{aligned} t_{4n+1} &= 2t_{4n} + t_{4n-1} + 1 \\ t_{4n+2} &= t_{4n+1} + 2t_{4n} + 1 \\ t_{4n+3} &= t_{4n+2} + 2t_{4n} + 1 \\ t_{4n+4} &= 2t_{4n+3} + t_{4n} + 1. \end{aligned}$$

This recurrence was found experimentally. It generates the sequence

$$1,\ 2,\ 3,\ 7,\ 18,\ 33,\ 48,\ 104,\ 257,\ 466,\ 675,\ 1455,\ 3586,\ \ldots$$

which unfortunately does not yet match any sequence in the OEIS. It is possible to construct an automaton, shown in Fig. 4, for this system of recurrences using the automatic theorem prover `Walnut`.

3. The zeros of $S_n(2\sqrt{2})$

An index n is a zero of a deterministic random walk if $S_n(\xi) = 0$. The study of the asymptotic number of zeros of deterministic random walks with offsets was initiated in [1] and remains a topic of ongoing research. For our walk $S_n(2\sqrt{2})$ the first few zeros are

$$0,\ 2,\ 4,\ 12,\ 14,\ 16,\ 24,\ 26,\ 28,\ 70,\ 72,\ 74,\ 82,\ 84,\ 86,\ \ldots$$

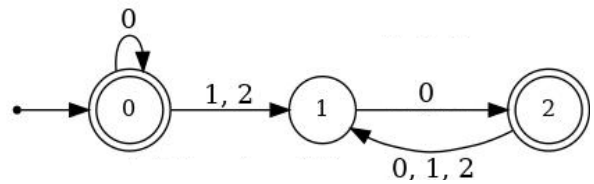

FIGURE 5. An automaton that accepts the zeros of $S_n(2\sqrt{2})$ in Pell numeration in msd representation.

which is entry A194368 of the OEIS. In Pell numeration these numbers are

$$\epsilon,\ 10,\ 20,\ 1000,\ 1010,\ 1020,\ 2000,\ 2010,\ 2020,\ 100000,\ \ldots$$

where ϵ is the empty word. Like its records, the zeros of $S_n(\sqrt{2})$ are easy to spot in Pell numeration.

Theorem 8. *The zeros of $S_n(2\sqrt{2})$ occur at $n = 0$ or $(10|20)(00|10|20)^*$ in Pell numeration.*

Proof. There cannot be a zero in $[q', q]$ if q' is an odd Pell number because rules B and C add $S_{q'}(\sqrt{2}) = 1$ to previous values, and these are non-negative. Zeros occur in $[q', q]$ for even Pell numbers q', starting at q' following rule A. Note that rules B and C partition this interval into numbers with initial digits 1 and 2 in Pell numeration.

We assume by induction that the earlier zeros all are of the required form. If rule C applies, then $S_{q'+k}(\sqrt{2}) = 0$ if and only if $S_k(\sqrt{2}) = 0$, so indeed all zeros in this part of $[q', q]$ are of the form $(10)(00|10|20)^*$. For rule B we use the identity

$$P_{2m} - 1 = \sum_{i=1}^{m-1} 2P_{2i}.$$

We rewrite rule B as $S_{(q-1)-k)}(\sqrt{2}) = S_k(\sqrt{2})$, using that q' is an even Pell number. In particular, $q - 1 - k$ is a zero if and only if k is a zero. Now $q - 1$ is equal to $(20)^m$ in Pell numeration and $k = (00|10|20)^{m-1}$ by our inductive assumption. It follows that $(q - 1) - k = (20)(00|10|20)^{m-1}$ and we are done. □

Surprisingly, the set of zeros of $S_n(\sqrt{2})$ does not seem to be Pell automatic, or if it is, its automaton needs many states. The one-sidedness of $S_n(2\sqrt{2})$ seems to be essential. We say that ξ is a *BR-number* (after Boshernitzan and Ralston) if its odd convergents q_{2n+1} are even, or, equivalently, if all its odd partial quotients a_{2n+1} are even. According to Theorem 1, the walk $S_n(2\xi)$ is nonnegativeif and only if ξ is a BR-number. Note that $q_0 = 1$ and q_1 is even for a BR-number. By the recursion $q_{n+1} = a_n q_n + q_{n-1}$, the parity of the convergents alternates for BR-numbers.

We write $q = q_{n+1}$, $q' = q_n$, and $q'' = q_{n-1}$ so that $q = aq' + q''$ for the partial quotient $a = a_{n+1}$. The rules B and C from [10] allow some overlap, but we state them in such a form that they apply to separate parts of $[q', q)$.

Recursive rules for $S_n(2\xi)$ for a BR-number $\xi \in (0, 1)$
Rule A: $S_q = 1$ if q is odd and $S_q = 0$ if q is even.
Rule B: $S_{q-k} = S_{q'} + S_{k-1}$ if $1 \le k \le q/2$.
Rule C: $S_{q'+k} = S_{q'} + S_k$ if $1 \le k < q/2 - q'$.

Theorem 9. *Let $\xi \in (0,1)$ be a BR-number and let $N = \sum_{i=0}^{n} b_i q_i$ be the ξ-Ostrowski representation. Then N is a zero of $S_n(2\xi)$ if and only if $b_i = 0$ for all even i.*

Proof. By rules B and C, for each $n \in [q', q)$ there is some $k < n$ such that $S_n = S_{q'} + S_k$. If q' is odd, then $S_{q'} = 1$ and $S_k \geq 0$ since the walk is one-sided. There are no zeros in $[q', q)$ if q' is odd. If N is a zero, it must be in $[q', q)$ for an even q'.

We need to prove that the odd digits of N are zero in Ostrowski numeration. First suppose that $N < q/2$. Rule C applies and $S_N = S_{q'} + S_k = S_k$ for $N = q' + k$. Therefore, k is a zero and by induction has odd digits zero. Note that $k < q/2 - q' < q'$ and that q' gives a digit 1 at an even position in Ostrowski numeration. The odd digits of N are zero if $N < q/2$. If $N \geq q/2$ then $S_N = S_{q'} + S_{k-1} = S_{k-1}$ for $N = q - k$. By induction, N is a zero if and only if $N = q - k$ for a zero $k - 1 < N$. Now we use the identity

$$q_{2j+2} = \sum_{i=0}^{j} a_{2i+2} q_{2i+1} + 1 \tag{4}$$

to write

$$N = \sum_{i=0}^{j} a_{2i+2} q_{2i+1} - (k-1)$$

for a zero $k-1$. By induction $k - 1 = \sum_{i=0}^{j} b_{2i+1} q_{2i+1}$ for digits $b_{2i+1} \leq a_{2i+2}$. We conclude that all even digits in the expansion of N are zero. □

It is easy to construct an automaton that decides if all digits on even positions are zero.

Corollary 10. *The zeros of $S_n(2\xi)$ form a ξ-automatic sequence if ξ is a BR-number.*

The following generalizes Theorem 5 from $\xi = \sqrt{2}$ to BR-numbers.

Theorem 11. *Let $\xi \in (0,1)$ be a BR-number and let $N = \sum_{i=0}^{n} b_i q_i$ be the ξ-Ostrowski representation. Then N is a record of $S_n(2\xi)$ if and only if n is even, $b_i = 0$ for all odd i, $b_i = a_{i+1}/2$ for all even $i < n$, and $b_n \leq a_{n+1}/2$.*

Proof. We argue by induction on the index n of the most significant digit. Records can only occur in $[q', q)$ if q' is odd, so n is even. The first partial quotient a_1 is even and the initial record of the walk occurs immediately at the initial steps $a_1/2$, which shows that the statement is true for $n = 0$. Assume that it is true for $n - 2$. The final record with most significant digit $n - 2$ is equal to $(q_{n-1})/2$ by Equation (4). We write q_{n+1}, q_n, q_{n-1} as q, q', q'' and $q = aq' + q''$. The final record before q'' is $r = (q'' - 1)/2$. Rule C implies that the records in $[q', q/2)$ are $bq' + r$ up to $q/2$. Since $(a/2)q' + r = (q - 1)/2$ the digit b runs up to $a/2$. Rule B implies that the only possible record in the subinterval $[q/2, q)$ is $q - k$ if $k - 1$ is the final record before $q/2$. This record is $(q - 1)/2$. Since q is odd, the minimal number in $[q/2, q)$ is $(q + 1)/2$. If we write it as $q - k$, then we get $k = (q - 1)/2$ and $k - 1$ is below the final record. All records are of the prescribed form. □

Corollary 12. *Both Conjecture 7 and the conjecture of Van de Lune and Arias de Reyna hold if ξ is a BR-number.*

Proof. We construct an lsd automaton. It checks that $b_i = 0$ if i is odd and that $b_i \leq a_{i+1}/2$ if i is even. If this inequality is strict, then all further digits need to be zero. Therefore, the odd transitions are 0. The even transitions are $a_{i+1}/2$, unless the transition is $< a_{i+1}/2$, which moves to a separate state for which all further transitions are 0. Since the partial quotients are eventually periodic, the automaton eventually loops back and the number of states is finite. For a BR number, the period is even, so this does not conflict with the check that odd digits are zero. This settles Conjecture 7 for BR-numbers.

To find a system of linear recurrences, observe that the difference between consecutive records equals q_j for some even index $j = 2k$. Each q_j occurs $a_{j+1}/2$ times and therefore the number of occurrences is eventually periodic. The convergents q_{2k} form a recurrence sequence and therefore the difference sequence of consecutive records satisfies a system of recurrences, and so does the sequence of records. □

4. The difference sequence $b - a$

We establish Kimberling's second observation that $b - a$ assumes all positive integers infinitely often. We focus on the walk $S_n(2\sqrt{2})$ and for simplicity, we abbreviate our notation to $S_n = S_n(2\sqrt{2})$. The walk exhibits two key symmetries, described by Rules B and C. Rule B is reflexive, meaning that the step from S_{q-k} to S_{q-k+1} mirrors the step from S_{k-1} to S_{k-2}. Rule C, on the other hand, is translational: it states that from the q-th step onward, the next q steps replicate the initial q steps. To fully address Kimberling's second observation, we need an additional symmetry, ensuring that each value $k > 0$ appears infinitely often in $b - a$.

Lemma 13. *Let $q = q_{2n-1}$ be an even denominator. Then*

$$S_{q/2+k} = S_{q/2} - S_k$$

for $0 \leq k \leq q/2$.

Proof. This follows from the rules of $S_n(2\sqrt{2})$. The intuition behind this equation is that after $q/2$ steps the parity of the rounded exponents $2(j + q/2)\sqrt{2}$ is opposite to that of $2j\sqrt{2}$ since $q\sqrt{2} \approx p$ where p/q is the convergent and p is odd.

The half-Pell number $q/2$ is a sum of odd convergents $q_{2n-1}/2 = q_{2n-2} + q_{2n-4} + \cdots + q_0$. We present k in Pell numeration by a word of length $2n - 1$, padding with initial zeros if necessary. Since $k \leq q/2$ its Pell presentation is $(10)^j 0w$ for some $j \geq 0$ and a word w of length $2(n - j - 1)$. In particular

$$k = q_{2n-2} + \ldots + q_{2n-2j} + k'$$

for $k' < q_{2n-2j-2}$. Rule C implies that $S_k = j + S_{k'}$. If we write $r = q_{2n-2j-1}$ then $q/2 - k = r/2 - k'$. By rule B and by induction

$$S_{q/2+k} = 1 + S_{q/2-k-1} = 1 + S_{r/2-k'-1} = S_{r/2+k'} = S_{r/2} - S_{k'}.$$

Now $q/2$ is the n-th record and $r/2$ is the $(n-j)$-th record. Therefore $S_{q/2} = j + S_{r/2}$ and we find

$$S_{r/2} - S_{k'} = S_{q/2} - S_k$$

as required. □

Lemma 14. *Let $q = q_{2n-1}$ be an even denominator.*

$$a(q/2 + j) = q + a(j) \text{ and } b(q/2 + j) = q + b(j) \tag{5}$$

for $j \leq q/2$

Proof. Since q is even, it is a zero, and there are equally many forward and backward steps to q. By definition, $a(q/2 + j)$ is the index of the $(q/2 + j)$-th forward step. There are $q/2$ forward steps up to index q, after which the walk repeats itself by rule C for the next $q/2$ steps. Therefore, $a(q/2 + j) = q + a(j)$. The argument for $b(q/2 + j)$ is the same. □

Lemma 15. *For any $k > 0$, if $b(n) - a(n) = k$ for some n, then there are infinitely many m such that $b(m) - a(m) = k$.*

Proof. It follows from Equation (5) that each difference $b(n) - a(n)$ repeats after $q/2$ steps for a sufficiently large q. □

If $q = q_{2m-1}$ is the m-th even denominator, then $q/2$ is the m-th record, and there is a surplus of m forward steps among the first $q/2$ steps.

Lemma 16. *For all $m > 1$, we have*

$$b\left(\frac{q_{2m-1} + 2m}{4}\right) - a\left(\frac{q_{2m-1} + 2m}{4}\right) = a(m).$$

Proof. We write $q = q_{2m-1}$. There is a surplus of m forward steps; therefore, $(q + 2m)/4$ steps are forward and $(q - 2m)/4$ are backward. The final step of these first $q/2$ steps is forward, since it produces a record. By Lemma 14, after step $q/2$ the walk repeats itself, but in the opposite direction, for the next $q/2$ steps. Since $a(m) < 2m < q/2$ it follows that

$$b\left(\frac{q - 2m}{4} + j\right) = \frac{q}{2} + a(j) \text{ for } 1 \leq j \leq m.$$

In particular $b\left(\frac{q+2m}{4}\right) = \frac{q}{2} + a(m)$ and $a\left(\frac{q+2m}{4}\right) = \frac{q}{2}$. □

Theorem 17. *For every $k > 0$ there are infinitely many j such that $b(j) - a(j) = k$.*

Proof. The previous lemmas take care of the case that k is in the sequence a. We need to find a solution for k in b, say $k = b(n)$. Take a sequence of n odd denominators of convergents $c_n > c_{n-1} > \dots c_1$ such that $c_1/2 > b(n)$. These denominators do not need to be consecutive, which is why we write c instead of q, to avoid confusion. Let $d = c_n + c_{n-1} + \cdots c_1$. Rule C implies that $S_d = n$ and that the d-th step is forward. There is a surplus of n forward steps which implies that $a\left(\frac{n+d}{2}\right) = d$. By Rule C the deficit of n backward steps is compensated for at index $d + b(n)$. We conclude that $b\left(\frac{n+d}{2}\right) - a\left(\frac{n+d}{2}\right) = b(n)$. □

5. Self-similarity of noble mean rotations

A quadratic number $\xi^2 = m\xi + 1$ with constant continued fraction expansion $\xi = [m; m, m, m, \ldots]$ is called a *metallic mean* or *noble mean* [3]. For $m = 1$ it is the golden mean and for $m = 2$ it is the silver mean. A noble mean is a BR-number if m is even. Our standing example $S_n(2\sqrt{2})$ is generated by the rotation over the silver mean. It is well-known that the irrational rotation $\rho\colon x \mapsto x + \xi \mod 1$ has a self-similarity for noble means. We shall see that this explains why the deterministic random walk $S_n(2\xi)$ is non-negative, for noble means with even $m = 2k$. Only the fractional part of ξ matters for the rotation, which is why we subtract m from the noble mean, adjusting it to $\xi_m = [0; m, m, m, \ldots]$.

The rotation ρ is an *interval exchange transformation* [11], which adds ξ to $x \in [0, 1-\xi)$ and subtracts $1-\xi$ from $x \in [1-\xi, 1)$. Recall that the return of x to $I \subset [0, 1)$ is the iterated image $\rho^n(x) \in I$ such that no $\rho^j(x)$ for $0 < j < n$ is in I.

Lemma 18. *Let $\xi_m = \frac{\sqrt{m^2+4}-m}{2}$ be the adjusted noble mean. The return map to $[0, 1-m\xi)$ is a rescaling of the original rotation on $[0, 1)$.*

Proof. The first two convergents of ξ_m are

$$\frac{m}{m^2+1} < \xi_m < \frac{1}{m}$$

and the closest returns to 0 among its first $m^2 + 1$ rotations are $m\xi_m - 1 < 0 < (m^2+1)\xi_m - m$. We suppress the index and write ξ instead of ξ_m. The first rotation $\{j\xi\}$ that ends up in $[0, 1-m\xi)$ is $(m^2 + 1)\xi - m$. Let R be the return map to $[0, 1-m\xi)$. A straightforward computation gives

$$R(x) = \begin{cases} x + (m^2+1)\xi - m & \text{if } x < (m+1) - (m^2+m+1)\xi, \\ x + (m^2+m+1)\xi - m - 1 & \text{if } x \geq (m+1) - (m^2+m+1)\xi. \end{cases}$$

If we rescale this interval exchange to unit length, we get a rotation over

$$\frac{(m^2+1)\xi - m}{1 - m\xi} = \xi.$$

□

We partition the circle into three labeled intervals

$$\{a, b, c\} = \{[0, 1/2), [1/2, 1-\xi), [1-\xi, 1)\}.$$

The itinerary of $x \in [0, 1/2)$ up to its return $S(x)$ is a for $x \in I_1$ because the return is immediate. It is $ab^{k+1}c$ for $x \in I_2$ and ab^kc for $x \in I_3$.

Theorem 19. *Let ξ_m be the fractional part of the noble mean for an even $m = 2k$. The sequence $(a_n)_{n\geq 0}$ for $a_n = 1_{[0,1/2)}(\{n\xi_m\})$ equals $\lim_{k\to\infty} \tau \circ \sigma^k(a)$ for the substitution and coding*

$$\sigma : \begin{cases} a \to a^{k+1}b^{k-1}c\,(a^kb^{k-1}c)^{p-1}, \\ b \to a^kb^kc\,(a^kb^{k-1}c)^{p-1}, \\ c \to a^kb^kc\,(a^kb^{k-1}c)^{p}, \end{cases} \qquad \text{and} \qquad \tau : \begin{cases} a \to 1, \\ b \to 0, \\ c \to 0. \end{cases}$$

Proof. The return map to $[0, 1-m\xi)$ is a rescaling of the rotation, in which the partition $\{a, b, c\}$ is scaled to

$$\{a', b', c'\} = \{[0, 1/2 - k\xi), [1/2 - k\xi, (1-\xi)(1-m\xi)), [(1-\xi)(1-m\xi), 1-m\xi)\}.$$

The return time is $m^2 + 1$ on $a' \cup b'$ and $m^2 + m + 1$ on c'. The substitution σ gives the itinerary of elements of $\{a', b', c'\}$ through $\{a, b, c\}$ until their return. Note that the return times correspond to the lengths of the substitution.

After m rotations, $[0, 1-m\xi)$ is in $[m\xi, 1)$ and after one more rotation it is in $[\xi, 1-(m-1)\xi)$. For an element of a', the itinerary if $a^{k+1}b^{k-1}c$ after m rotations, including its initial position in a. For the other elements, the itinerary is a^kb^kc. That explains the prefixes of the coding σ. After k rotations, the initial point of b'

reaches $\frac{1}{2}$, which marks the transition from coding 1 to coding 0. This is where we need that m is even.

For each $x \in [\xi, 1-(m-1)\xi)$, the itinerary of its initial position and a subsequent $m-1$ rotations is $a^k b^{k-1} c$. This itinerary of blocks of m remains the same until the return to $[0, 1-m\xi)$. It follows that σ describes the return map, if we associate $\{a', b', c'\}$ to $\{a, b, c\}$. The coding τ corresponds to the indicator function. □

As a consequence, we get another proof that deterministic random walks are one-sided for noble means that are BR-numbers.

Corollary 20. *The deterministic random walk $S_n(2\xi)$ is nonnegativefor noble means with even m.*

Proof. If we put $+1$ for a and -1 for b and c then the running sum of

$$a^{k+1} b^{k-1} c\, (a^k b^{k-1} c)^{m-1},$$

which is the substitution word of a, is positive. The running sum of the other two substitution words,

$$a^k b^k c\, (a^k b^{k-1} c)^{m-1} \qquad \text{and} \qquad a^k b^k c\, (a^k b^{k-1} c)^{m},$$

is ≥ -1. The deterministic random walk $S_n(\xi)$ is a running sum of substitution words. If it were negative at some point, then it must have a surplus of b and c over a at an earlier index, which is nonsense. □

Lemma 18 holds for all noble means and Theorem 19 can be extended to all noble means as well, but it gets more elaborate. For a noble mean ξ_m with even m, the mid-point of the interval $[0, 1-m\xi)$ is in the backward orbit of $\frac{1}{2}$ and that is why the partition $\{a, b, c\}$ carries over under rescaling. If m is odd, we need to take a smaller interval.

If $p_n/q_n < \xi$ is a convergent, then the return map to $[0, p_n - q_n\xi)$ is a rescaling of the rotation. We need q_n to be even to preserve the partition, and we need n to be odd so that $p_n - q_n\xi > 0$. For example, for the golden mean $\xi_1 = \frac{1}{2}(\sqrt{5}-1)$, we can choose the interval $[0, 5 - 8\xi_1)$. The corresponding substitution and coding are

$$\sigma : \begin{cases} a \to acacbacaccacb \\ b \to acacbacaccacbacaccacb \\ c \to acaccacaccacbacaccacb \end{cases} \qquad \text{and} \qquad \tau : \begin{cases} a \to 1 \\ b \to 1 \\ c \to 0 \end{cases}$$

The running sums of the substitution words for a and b are non-negative. The minimum of the running sum of the word for c is -2. This indicates that, as we know from Theorem 1, the walk is unbounded in both directions.

6. Concluding questions and remarks

We established a connection between deterministic random walks and automata in certain cases, but much remains to be explored. The general deterministic random walk is defined by

$$\sum_{j=1}^{n} (-1)^{\lfloor j\xi + \gamma \rfloor}.$$

We only considered the homogeneous case with offset $\gamma = 0$ and ξ as a BR-number. Do our results extend to general walks and arbitrary quadratic irrationals? Van de Lune and Arias de Reyna found that the records of general walks for quadratic

ξ appear to satisfy a recurrence for certain γ. Is it true that the records of this general walk form a ξ-automatic set if γ is rational?

We found that the zeros of $S_n(2\xi)$ are ξ-automatic for BR-numbers. To extend this result to the general walk, it seems that a generalization of Theorem 1 is needed. Under what conditions on ξ and γ is the general walk one-sided? A general result of Peres [16] states that for every (arbitrary irrational) ξ, there exists a γ such that the walk is one-sided. Is it true that for each quadratic ξ, there exists a rational γ such that the walk is one-sided?

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Email address: `henk.bruin@univie.ac.at and r.j.fokkink@tudelft.nl`