Notes on Topological Entropy.

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1 Topological entropy

Topological entropy was first defined in 1965 by Adler et al. [1], but the form that Bowen [2] and Dinaburg [4] redressed it in is commonly used nowadays.

We will start by start giving the original definition, because the idea of joints of covers easily relates to joints of partitions as used in measure-theoretic entropy. After that, we will give Bowen's approach, since it readily generalises to topological pressure as well.

1.1 The original definition

Let (X, d, T) be a continuous map on compact metric space (X, d). We say that $\mathcal{U} = \{U_i\}$ is an *open ε-cover* if all U_i are open sets of diameter $\leqslant \varepsilon$ and $X \subset \bigcup_i U_i$. Naturally, compactness of X guarantees that for every open cover, we can select a finite subcover. Thus, let $\mathcal{N}(\mathcal{U})$ the the minimal possible cardinality of subcovers of $\mathcal{U}$. We say that $\mathcal{U}$ *refines* $\mathcal{V}$ (notation $\mathcal{U} \succeq \mathcal{V}$) if every $U \in \mathcal{U}$ is contained in a $V \in \mathcal{V}$. If $\mathcal{U} \succeq \mathcal{V}$ then $\mathcal{N}(\mathcal{U}) \geq \mathcal{N}(\mathcal{V})$.

Given two cover $\mathcal{U}$ and $\mathcal{V}$, the joint

$$\mathcal{U} \vee \mathcal{V} := \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$$

is an open cover again, and one can verify that $\mathcal{N}(\mathcal{U} \vee \mathcal{V}) \leqslant \mathcal{N}(\mathcal{U})\mathcal{N}(\mathcal{V})$. Since T is continuous, $T^{-1}(\mathcal{U})$ is an open cover as well, although in this case it need not be an ε-cover; However, $\mathcal{U} \vee T^{-1}(\mathcal{U})$ is an ε-cover, and it refines $T^{-1}(\mathcal{U})$.

Define the *topological entropy* as

$$h_{top}(T) = \lim_{\varepsilon \to 0} \sup_{\mathcal{U}} \lim_{n} \frac{1}{n} \log \mathcal{N}(\mathcal{U}^n) \qquad \text{for } \mathcal{U}^n := \bigvee_{i=0}^{n-1} T^{-i}(\mathcal{U}), \tag{1}$$

where the supremum is taken over all open ε-covers $\mathcal{U}$. Because $\mathcal{N}(\mathcal{U}\vee\mathcal{V}) \leqslant \mathcal{N}(\mathcal{U})\mathcal{N}(\mathcal{V})$, the sequence $\log \mathcal{N}(\mathcal{U}^n)$ is subadditive, so the limit $\lim_n \frac{1}{n} \log \mathcal{N}(\mathcal{U}^n)$ exists.

1.2 Topological entropy of interval maps

If $X = [0,1]$ with the usual Euclidean metric, then there are various shortcuts to compute the entropy of a continuous map $T : [0,1] \to [0,1]$. Let us call any maximal interval on which T is monotone a *lap*; the number of laps is denoted as $\ell(T)$. Also, the *variation* of T is defined as

$$Var(T) = \sup_{0 \leqslant x_0 < \dots \ x_N \leqslant N} \sum_{i=1}^{N} |T(x_i) - T(x_{i-1})|,$$

where the supremum runs over all finite collections of points in $[0,1]$. The following result is due to Misiurewicz & Szlenk [5].

Proposition 1. *Let $T : [0,1] \to [0,1]$ have finitely many laps. Then*

$$\begin{aligned} h_{top}(T) &= \lim_{n\to\infty} \frac{1}{n} \log \ell(T^n) \\ &= \limsup_{n\to\infty} \frac{1}{n} \log \#\{\textit{clusters of } n\textit{-periodic points}\} \\ &= \max\{0, \lim_{n\to\infty} \frac{1}{n} \log \mathit{Var}(T^n)\}. \end{aligned}$$

where two n-periodic points are in the same cluster if they belong to the same lap of T^n.

Remark 1. *The identity map has one branch, consisting of (uncountably many) fixed point, that form one cluster. The map $x \mapsto x + (x/10)^2 \sin(\pi/x) \mod 1$ has also one branch, but with countably many fixed point, forming one cluster. For an expanding map, every branch can contain only one fixed point.*

Proof. Since the variation of a monotone function is given by $\sup T - \inf T$, and due to the definition of "cluster" of n-periodic points, the inequalities

$$\#\{\text{clusters of } n\text{-periodic points}\}, \mathrm{Var}(T^n) \leqslant \ell(T^n)$$

are immediate. For a lap I of T^n, let $\gamma := |T^n(I)|$ be its *height*. We state without proof (cf. [3, Chapter 9]):

$$\begin{gathered} \text{For every } \delta > 0, \text{ there is } \gamma > 0 \text{ such that} \\ \#\{J : J \text{ is a lap of } T^n,\ |T^n(J)| > \gamma\} \geq 1 - \delta)^n \ell(T_n). \end{gathered} \tag{2}$$

This means that $Var(T^n) \geqslant \gamma(1-\delta)^n \ell(T^n)$, and therefore

$$-2\delta + \lim_n \frac{1}{n} \log \ell(T^n) \leqslant \lim_n \frac{1}{n} \log \mathrm{Var}(T^n) \leqslant \lim_n \frac{1}{n} \log \ell(T^n).$$

Since δ is arbitrary, both above quantities are all equal.

Making the further assumption (without proof[1]) that there is $K = K(\gamma)$ such that $\cup_{i=0}^K T^i(J) = X$ for every interval of length $|J| \geqslant \gamma$, we also find that

$$\#\{\text{clusters of } n+i\text{-periodic points}, 0 \leqslant i \leqslant K\} \geqslant (1-\delta)^n \ell(T^n).$$

This implies that

$$-2\delta + \lim_n \frac{1}{n} \log \ell(T^n) \leqslant \limsup_n \frac{1}{n} \max_{0 \leqslant i \leqslant K} \log \#\{\text{clusters of } n+i\text{-periodic points}\}$$

so also $\lim_n \frac{1}{n} \log \ell(T^n) = \limsup_{n\to\infty} \frac{1}{n} \log \#\{\text{clusters of } n\text{-periodic points}\}$

If $\varepsilon > 0$ is so small that the width of every lap is greater than 2ε, then for every ε-cover $\mathcal{U}$, every subcover of $\mathcal{U}^n$ has at least one element in each lap of T^n. Therefore $\ell(T^n) \leqslant \mathcal{N}(\mathcal{U}^n)$ for every ε-cover, so $\lim_n \frac{1}{n} \log \ell(T^n) \leqslant h_{top}(T)$. □

1.3 Bowen's approach

Let T be map of a compact metric space (X, d). If my eyesight is not so good, I cannot distinguish two points $x, y \in X$ if they are at a distance $d(x, y) < \varepsilon$ from one another. I may still be able to distinguish there orbits, if $d(T^k x, T^k y) > \varepsilon$ for some $k \geqslant 0$. Hence, if I'm willing to wait $n-1$ iterations, I can distinguish x and y if

$$d_n(x, y) := \max\{d(T^k x, T^k y) : 0 \leqslant k < n\} > \varepsilon.$$

If this holds, then x and y are said to be **(n, ε)-separated**. Among all the subsets of X of which all points are mutually (n, ε)-separated, choose one, say $E_n(\varepsilon)$, of maximal cardinality. Then $s_n(\varepsilon) := \#E_n(\varepsilon)$ is the maximal number of n-orbits I can distinguish with ε-poor eyesight.

The **topological entropy** is defined as the limit (as $\varepsilon \to 0$) of the exponential growth-rate of $s_n(\varepsilon)$:

$$h_{top}(T) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log s_n(\varepsilon). \tag{3}$$

Note that $s_n(\varepsilon_1) \geqslant s_n(\varepsilon_2)$ if $\varepsilon_1 \leqslant \varepsilon_2$, so $\limsup_n \frac{1}{n} \log s_n(\varepsilon)$ is a decreasing function in ε, and the limit as $\varepsilon \to 0$ indeed exists.

Instead of (n, ε)-separated sets, we can also work with **(n, ε)-spanning** sets, that is, sets that contain, for every $x \in X$, a y such that $d_n(x, y) \leqslant \varepsilon$. Note that, due to its

[1] In fact, it is not entirely true if T has an invariant subset attracting an open neighbourhood. But it suffices to restrict T to its *nonwandering set*, that is, the set $\Omega(T) = \{x \in X : x \in \cup_{n\geqslant 1} T^n(U))$ for every neighbourhood $U \ni x\}$, because $h_{top}(T) = h_{top}(T|_{\Omega(T)})$.

maximality, $E_n(\varepsilon)$ is always (n,ε)-spanning, and no proper subset of $E_n(\varepsilon)$ is (n,ε)-spanning. Each $y \in E_n(\varepsilon)$ must have a point of an $(n,\varepsilon/2)$-spanning set within an $\varepsilon/2$-ball (in d_n-metric) around it, and by the triangle inequality, this $\varepsilon/2$-ball is disjoint from $\varepsilon/2$-ball centred around all other points in $E_n(\varepsilon)$. Therefore, if $r_n(\varepsilon)$ denotes the minimal cardinality among all (n,ε)-spanning sets, then

$$r_n(\varepsilon) \leqslant s_n(\varepsilon) \leqslant r_n(\varepsilon/2). \tag{4}$$

Thus we can equally well define

$$h_{top}(T) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log r_n(\varepsilon). \tag{5}$$

Examples: Consider the β-transformation $T_\beta : [0,) \to [0,1)$, $x \mapsto \beta x \pmod 1$ for some $\beta > 1$. Take $\varepsilon < 1/(2\beta^2)$, and $G_n = \{\frac{k}{\beta^n} \ : \ 0 \leqslant k < \beta^n\}$. Then G_n is (n,ε)-separating, so $s_n(\varepsilon) \geqslant \beta^n$. On the other hand, $G'_n = \{\frac{2k\varepsilon}{\beta^n} \ : \ 0 \leqslant k < \beta^n/(2\varepsilon)\}$ is (n,ε)-spanning, so $r_n(\varepsilon) \leqslant \beta^n/(2\varepsilon)$. Therefore

$$\log\beta = \limsup_n \frac{1}{n} \log \beta^n \leqslant h_{top}(T_\beta) \leqslant \limsup_n \log \beta^n/(2\varepsilon) = \log\beta.$$

Circle rotations, or in general isometries, T have zero topological entropy. Indeed, if $E(\varepsilon)$ is an ε-separated set (or ε-spanning set), it will also be (n,ε)-separated (or (n,ε)-spanning) for every $n \geqslant 1$. Hence $s_n(\varepsilon)$ and $r_n(\varepsilon)$ are bounded in n, and their exponential growth rates are equal to zero.

Finally, let (X,σ) be the full shifts on N symbols. Let $\varepsilon > 0$ be arbitrary, and take m such that $2^{-m} < \varepsilon$. If we select a point from each $n+m$-cylinder, this gives an (n,ε)-spanning set, whereas selecting a point from each n-cylinder gives an (n,ε)-separated set. Therefore

$$\begin{aligned}
\log N = \limsup_n \frac{1}{n} \log N^n &\leqslant \limsup_n \frac{1}{n} \log s_n(\varepsilon) \leqslant h_{top}(T_\beta) \\
&\leqslant \limsup_n \frac{1}{n} \log r_n(\varepsilon) \leqslant \limsup_n \log N^{n+m} = \log N.
\end{aligned}$$

Proposition 2. *For a continuous map T on a compact metric space (X,d), the three definitions* (1)*,* (3) *and* (5) *give the same outcome.*

Proof. The equality of the limits (3) and (5) follows directly from (4).

If $\mathcal{U}$ is an ε-cover, every $A \in \mathcal{U}^n$ can contain at most one point in an (n,ε)-separated set, so $s(n,\varepsilon) < \mathcal{N}(\mathcal{U}^n)$, whence $\limsup_n \frac{1}{n} \log s(n,\varepsilon) \leqslant \lim_n \frac{1}{n} \log \mathcal{N}(\mathcal{U}^n)$.

Finally, in a compact metric space, every open cover $\mathcal{U}$ has a number (called its *Lebesgue number*) such that for every $x \in X$, there is $U \in \mathcal{U}$ such that $B_\delta(x) \subset U$. Clearly $\delta < \varepsilon$

if $\mathcal{U}$ is an ε-cover. Now if an open ε-cover $\mathcal{U}$ has Lebesgue number δ, and E is an (n,δ)-spanning set of cardinality $\#E = r(n,\delta)$, then $X \subset \cup_{x\in E} \cap_{i=0}^{n-1} T^{-i}(B_\delta(T^i x))$. Since each $B_\delta(T^i(x))$ is contained in some $U \in \mathcal{U}$, we have $\mathcal{N}(\mathcal{U}^n) \leqslant r(n,\delta)$. Since $\delta \to 0$ as $\varepsilon \to 0$, also

$$\lim_{\varepsilon\to 0}\lim_n \frac{1}{n}\log \mathcal{N}(\mathcal{U}^n) \leqslant \lim_{\delta\to 0}\limsup_n \frac{1}{n}\log r(n,\delta).$$

□

1.4 Properties of entropy

Lemma 1. *If f is an isometry, then $h_{top}(f) = 0$.*

Proof. Since f preserves distances, $B_\varepsilon(x) = B_{n,\varepsilon}(x)$ for all $n \in \mathbb{N}$. It follows that $s_n(\varepsilon)$ is independent of n, say $s_n(\varepsilon) = s(\varepsilon)$. Therefore

$$h_{top}(f) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log s_n(\varepsilon) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log s(\varepsilon) = \lim_{\varepsilon\to 0} 0 = 0.$$

□

The same proof sows that every contraction has zero entropy as well, as do homeomorphisms of the interval or the circle.

Lemma 2. *For each $N \geq 0$ we have $h_{top}(f^N) = Nh_{top}(f)$. If f is invertible, then also $h_{top}(f^{-N}) = Nh_{top}(f)$, so in particular $h_{top}(f) = h_{top}(f^{-1})$.*

Proof. If $N = 0$, then f^N is the identity map, obviously with zero topological entropy. Choose $N \in \mathbb{N}$. Since f^j is continuous on the compact metric space X, it is uniformly continuous, so for every $\varepsilon > 0$ we can choose $\delta \in (0,\delta)$ such that $f^j(B_\delta(x)) \subset B_\varepsilon(f^j(x))$ for $j = 0,\dots,N-1$. Now if S is an (n,ε)-separating set for f^N, then it is also (nN,δ)-separating for f, and an (nN,ε)-separating set for f is also (n,ε)-separating for f^N. Therefore $s_{nN}(f,\delta) \geq s_n(f^N,\varepsilon) \geq s_{nN}(f,\varepsilon)$, and

$$\limsup_{n\to\infty}\frac{1}{n}\log s_{nN}(f,\delta) \geq \limsup_{n\to\infty}\frac{1}{n}\log s_n(f^N,\varepsilon) \geq \limsup_{n\to\infty}\frac{1}{n}\log s_{nN}(f,\varepsilon).$$

Finally, take the limit $\varepsilon \to 0$ which also means $\delta \to 0$.

For the second statement, if f is invertible, then an (n,ε)-ball $B_{n,\varepsilon}(x)$ for f is an (n,ε)-ball $B_{n,\varepsilon}(f^{n-1}(x))$ for f^{-1}. Therefore the cardinalities of (n,ε)-separating (or (n,ε)-spanning) sets are the same for f and f^{-1}, and hence so are their entropies.

Combining this with the first part, we get $h_{top}(f^{-N}) = Nh_{top}(f^{-1}) = Nh_{top}(f)$ as well. □

Theorem 1. *If $f : X \to X$ and $g : Y \to Y$ are conjugate maps on compact metric spaces, then they have the same topological entropy.*

Proof. Let h a conjugacy between f and g, so $h \circ f = g \circ h$. Sine h is continuous and X is compact, h is also uniformly continuous: for each $\varepsilon > 0$ there is a $\delta > 0$ such that $h(B_\delta(x)) \subset B_\varepsilon(h(x))$, and also $h(B_{n,\delta}(x)) \subset B_{n,\varepsilon}(n(x))$ for all $x \in X$ and $n \in \mathbb{N}$. In particular, if S is (n,δ)-spanning, then $h(S)$ is an (n,ε)-spanning set as well, but potentially not a minimal (n,ε)-spanning set, even if S is a minimal (n,δ)-spanning set. Hence $r_n(g,\varepsilon) \le r_n(f,\delta)$. This inequality remains true after taking logarithms, dividing by n and taking $\limsup_{n\to\infty}$. Finally, because $\sup_x h(B_{n,\delta}(x)) \to 0$ as $\delta \to 0$, we can assume that the limit $\delta \to 0$ corresponds to $\varepsilon \to 0$. Hence, after taking these limits, we find $h_{top}(g) \le h_{top}(f)$.

Reversing the roles of f and g gives the other inequality $h_{top}(f) \le h_{top}(g)$ □

Remark 2. *This proof also shows that if f and g are only semiconjugate, so there is a continuous surjection h so that $h \circ f = g \circ h$, then $h_{top}(f) \ge h_{top}(g)$.*

1.5 Topological pressure

The topological pressure $P_{top}(T,\psi)$ combines entropy with a potential function $\psi : X \to \mathbb{R}$. By definition, $h_{top}(T) = P_{top}(T,\psi)$ if $\psi(x) \equiv 0$. Denote the n-th ergodic sum of ψ by

$$S_n\psi(x) = \sum_{k=0}^{n-1} \psi \circ T^k(x).$$

Next set

$$\begin{cases} K_n(T,\psi,\varepsilon) = \sup\{\sum_{x\in E} e^{S_n\psi(x)} \ : \ E \text{ is } (n,\varepsilon)\text{-separated}\}, \\ L_n(T,\psi,\varepsilon) = \inf\{\sum_{x\in E} e^{S_n\psi(x)} \ : \ E \text{ is } (n,\varepsilon)\text{-spanning}\}. \end{cases} \tag{6}$$

For reasonable choices of potentials, the quantities $\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log K_n(T,\psi,\varepsilon)$ and $\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log L_n(T,\psi,\varepsilon)$ are the same, and this quantity is called the **topological pressure**. To give an example of an unreasonable potential, take X_0 be a dense T-invariant subset of X such that $X \setminus X_0$ is also dense. Let

$$\psi(x) = \begin{cases} 100 & \text{if } x \in X_0, \\ 0 & \text{if } x \notin X_0. \end{cases}$$

Then $L_n(T,\psi,\varepsilon) = r_n(\varepsilon)$ whilst $K_n(T,\psi,\varepsilon) = e^{100n}s_n(\varepsilon)$, and their exponential growth rates differ by a factor 100. Hence, some amount of continuity of ψ is necessary to make it work.

Lemma 3. *If $\varepsilon > 0$ is such that $d(x,y) < \varepsilon$ implies that $|\psi(x) - \psi(y)| < \delta/2$, then*

$$e^{-n\delta} K_n(T, \psi, \varepsilon) \leqslant L_n(T, \psi, \varepsilon/2) \leqslant K_n(T, \psi, \varepsilon/2).$$

Exercise 1. *Prove Lemma 3. In fact, the second inequality holds regardless of what ψ is.*

Theorem 2. *If $T : X \to X$ and $\psi : X \to \mathbb{R}$ are continuous on a compact metric space, then the topological pressure is well-defined by*

$$P_{top}(T, \psi) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log K_n(T, \psi, \varepsilon) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log L_n(T, \psi, \varepsilon).$$

Exercise 2. *Show that $P_{top}(T^R, S_R\psi) = R \cdot P_{top}(T, \psi)$.*

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