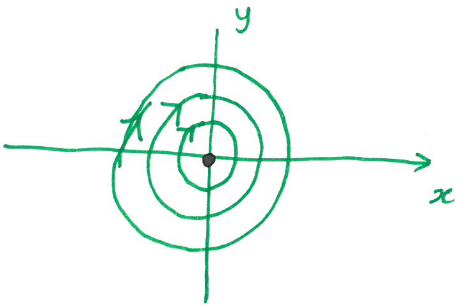
Oscillators and Resonance

harmonic oscillator : $\boxed{\ddot{x} + \omega^2 x = 0}$

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\omega^2 x \end{cases}$$

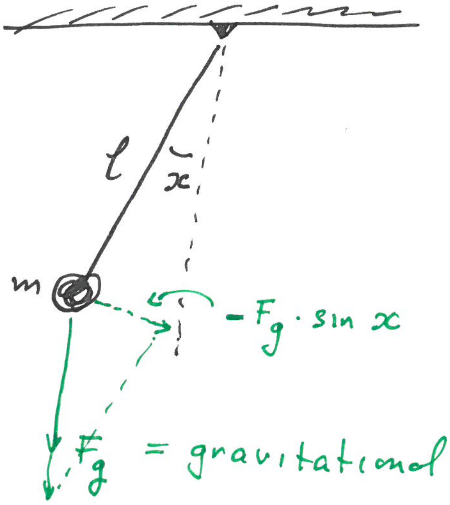
with solution
$$\begin{cases} x(t) = A\cos\omega t + B\sin\omega t \\ y(t) = -A\omega\sin\omega t + B\omega\cos\omega t. \end{cases}$$



All solutions are periodic except a stationary point of center type at the origin.

pendulum $\boxed{\ddot{x} + \omega^2 \sin x = 0}$

Newton's equation $\underset{\text{mass}}{\overset{\text{force}}{F}} = \underset{\uparrow\,\text{mass}}{m}\overset{\text{acceleration}}{a}$ becomes



F_g = gravitational force
= m$\underset{\uparrow}{g}$

gravitational constant
$\approx 9.8 \frac{m}{sec^2}$ on Earth

$$-F_g \sin x = m\ell\ddot{x}$$

Insert $F_g = mg$
and divide by $m\ell$:

$$\ddot{x} + \underbrace{\left(\frac{g}{\ell}\right)}_{= \omega^2} \sin x = 0$$

$= \omega^2$ = frequency squared.

Different derivation via preservation of energy:

$$E_{kin} = \frac{1}{2} m v^2 = \frac{1}{2} m (\ell \dot{x})^2$$ kinetic energy

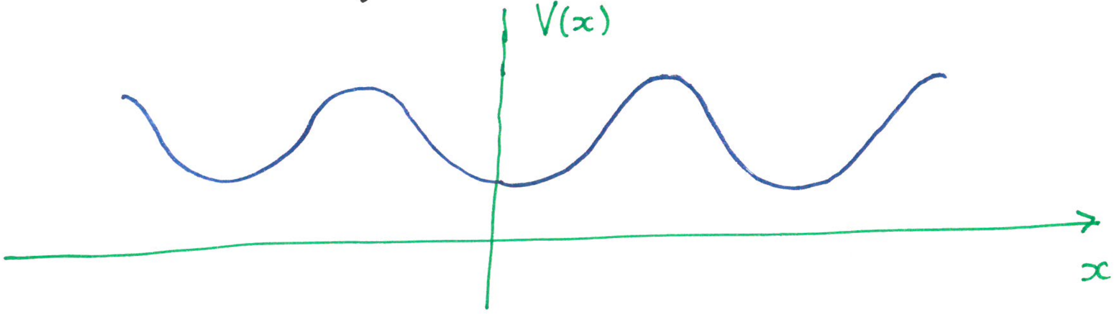
$$E_{pot} = \text{Const} + mgh = \overbrace{\text{Const} - mg\ell \cos x}^{V(x)} .$$ potential energy

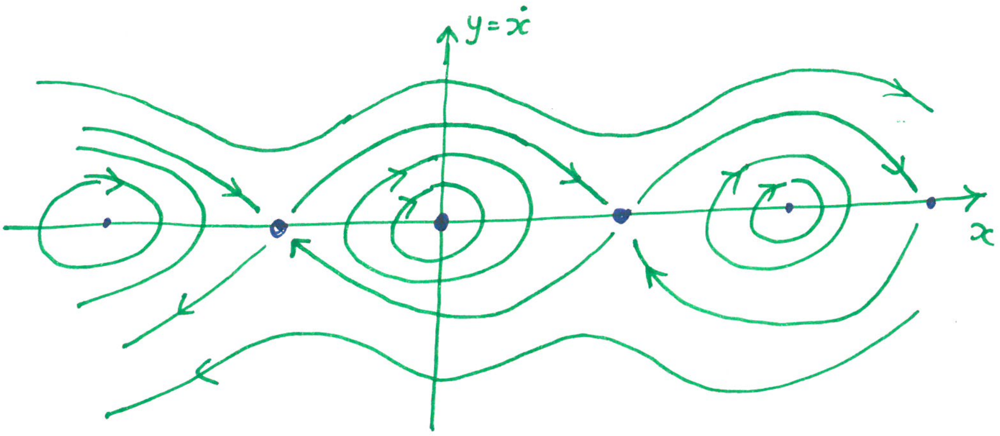
$$0 = \dot{E} = \frac{d}{dt}\left(E_{kin} + E_{pot} \right)$$

$$= m \ell^2 \dot{x} \ddot{x} + mg\ell \sin x \dot{x}$$

Divide by $m\ell^2 \dot{x}$

$$0 = \ddot{x} + \frac{g}{\ell} \sin x$$





phase portrait of pendulum

no explicit formulas for the solutions known

damped pendulum

$$\boxed{\ddot{x} + \underbrace{r\,\dot{x}}_{\text{friction term},\ r>0} + \omega^2 \sin x = 0}$$

For the damped pendulum, energy dissipates:

$$\dot{E} = \frac{d}{dt}\left(E_{kin} + E_{pot}\right) = \frac{d}{dt}\left(\tfrac{1}{2}m(\ell\dot{x})^2 + (\text{const} - mg\ell\cos x\right)$$

$$= m\ell^2\dot{x}\,\ddot{x} + mg\ell\sin x\,\dot{x}$$

$$= m\ell^2\dot{x}\left(\ddot{x} + \underbrace{\frac{g}{\ell}}_{=\omega^2}\sin x\right)$$

$$= m\ell^2\dot{x}\,(-r\dot{x}) = -m\ell^2 g\, r\,\dot{x}^2 \leq 0$$

NB The damped harmonic oscillator can be solved explicitly.

$$\boxed{\ddot{x} + r\dot{x} + \omega^2 x = 0}$$

Ansatz:

$$\boxed{x(t) = e^{\lambda t}} \qquad \times\,\omega^2$$

$$\dot{x}(t) = \lambda e^{\lambda t} \qquad \times\, r$$

$$\ddot{x}(t) = \lambda^2 e^{\lambda t} \qquad \times\, 1 \qquad +$$

$$(\omega^2 + r\lambda + \lambda^2)\,e^{\lambda t} = 0$$

Hence $\lambda = -\frac{r}{2} \pm \sqrt{\left(\frac{r}{2}\right)^2 - \omega^2}$, with solution

$$x(t) = A\,e^{-\frac{r}{2} + i\sqrt{\omega^2 - \left(\frac{r}{2}\right)^2}\,t} + B\,e^{-\frac{r}{2} - i\sqrt{\omega^2 - \left(\frac{r}{2}\right)^2}\,t}$$

$$= e^{-\frac{r}{2}t}\left(A'\cos\sqrt{\omega^2 - \left(\tfrac{r}{2}\right)^2}\,t + B'\sin\sqrt{\omega^2 - \left(\tfrac{r}{2}\right)^2}\,t\right)$$

for small friction: $\frac{r}{2} < |\omega|$

Driven oscillators

$$\boxed{\ddot{x} + \underbrace{r\dot{x}}_{\text{friction term}} + \omega^2 x = \underbrace{A\cos t}_{\text{driving force (also oscillating)}}}$$

describes the motion of a child on a swing pushed by its parent, but also the behaviour of a radio antenna.

For simplicity, we give it a proper linear term (i.e. not $\omega^2 \sin x$), so this is a 2nd order inhomogeneous linear ODE.

The homogeneous solution (with RHS = 0) was just given. To solve the entire ODE we have to find a particular solution and add it to the homogeneous solution.

Ansatz

$$\begin{aligned} x_{par}(t) &= p\cos t + q\sin t && \times\,\omega^2 \\ \dot{x}_{par}(t) &= -p\sin t + q\cos t && \times\, r \\ \ddot{x}_{par}(t) &= -p\cos t - q\sin t && \times\, 1 \end{aligned}$$

$$A\cos t = (\omega^2 p + rq - p)\cos t + (\omega^2 q - rp - q)\sin t$$

$$\Rightarrow \begin{cases} q = \dfrac{rp}{\omega^2 - 1} \\[2ex] p = \dfrac{A - rq}{\omega^2 - 1} \end{cases}$$

Driven oscillator continued

$$\boxed{\ddot{x} + r\dot{x} + \omega^2 x = A\cos t}$$

$$\begin{cases} q = \dfrac{rp}{\omega^2 - 1} \\ p = \dfrac{A - rq}{\omega^2 - 1} \end{cases} \Rightarrow \quad p = \frac{A}{\omega^2 - 1 + r^2/(\omega^2-1)}$$

Hence the amplitude of the particular solution x_{par} is very large if both friction $r \approx 0$ and $\boxed{\omega^2 \approx 1}$ ← resonance.

In the extreme case of a frictionless driven oscillator: $\boxed{\ddot{x} + \omega^2 x = A\cos t}$ the same method with adjusted Ansatz gives:

Ansatz:

$$\begin{aligned} x_{par}(t) &= p\sin t + q t \sin t & \times\,\omega^2 \\ \dot{x}_{par}(t) &= p\cos t + q\sin t + q t\cos t & \times\,0 \\ \ddot{x}_{par}(t) &= -p\sin t + 2q\cos t - q t\sin t & \times\,1 \end{aligned}$$

$$A\cos t = 2q\cos t$$

so $$x_{par}(t) = \frac{At}{2}\cos t$$

but this solution is unbounded over time!

Coupled oscillators

$$\boxed{\begin{aligned}\ddot{x}_1 + \omega_1^2 x_1 &= \varepsilon f_1(x_2)\\ \ddot{x}_2 + \omega_2^2 x_2 &= \varepsilon f_2(x_1)\end{aligned}}$$

Naturally there can be more than two oscillators and friction terms are left out for simplicity

If there is no coupling i.e. $\varepsilon = 0$ then the solitions are

$$\begin{cases} x_1(t) = A_1 \cos(t + t_1) \\ x_2(t) = A_2 \cos(t + t_2) \end{cases}$$

Consider a Poincaré section $\Sigma = \{x_1 = A_1\}$,

then $(x_1(t), x_2(t)) \in \Sigma$ if $t = \frac{2\pi k}{\omega_1} - t_1 \quad k \in \mathbb{Z}$

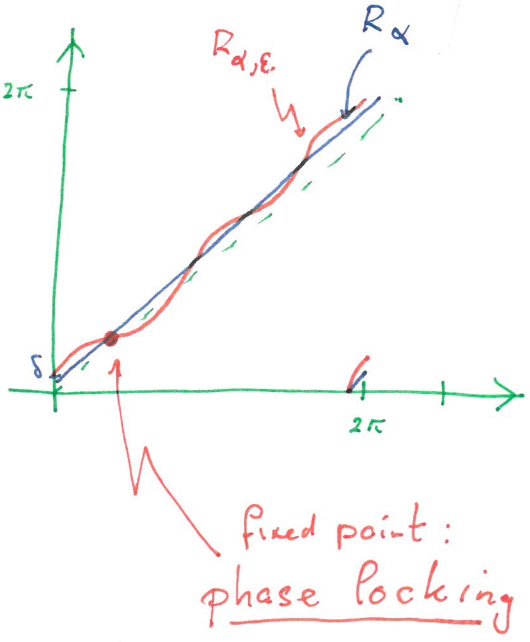
and then $x_2(t) = A_2 \cos\big(\underbrace{2\pi k \frac{\omega_2}{\omega_1} - \omega_2 t_1 + t_2}_{u_k}\big)$

i.e. $x_2(t) = A_2 \cos(u_k)$ for $u_k = R_\alpha^k(u_0) \qquad u_0 = t_2 - \omega_2 t_1$

and R_α is the rotation with angle $\alpha = \frac{\omega_2}{\omega_1}$ (usually $\notin \mathbb{Q}$)

Also assume $\omega_2 \approx \omega_1$ so

$\frac{\omega_2}{\omega_1} \bmod 1 \equiv \delta \approx 0$



If there is coupling, i.e. $\varepsilon > 0$, then R_α becomes $R_{\alpha,\varepsilon}$, some nonlinear perturbation of R_α. If δ is small compared to ε, then $R_{\alpha,\varepsilon}$ is likely to have a fixed point

Oscillators with parametric resonance

time dependent, usually periodic

$$\boxed{\ddot{x} + \omega^2(t) \sin x = 0}$$

This models the behaviour of a person on a swing keeping it swinging by shifting his body back and forth.

This can make the stationary point $x(t) \equiv 0$ unstable and the stationary point $x(t) \equiv \pi$ stable.