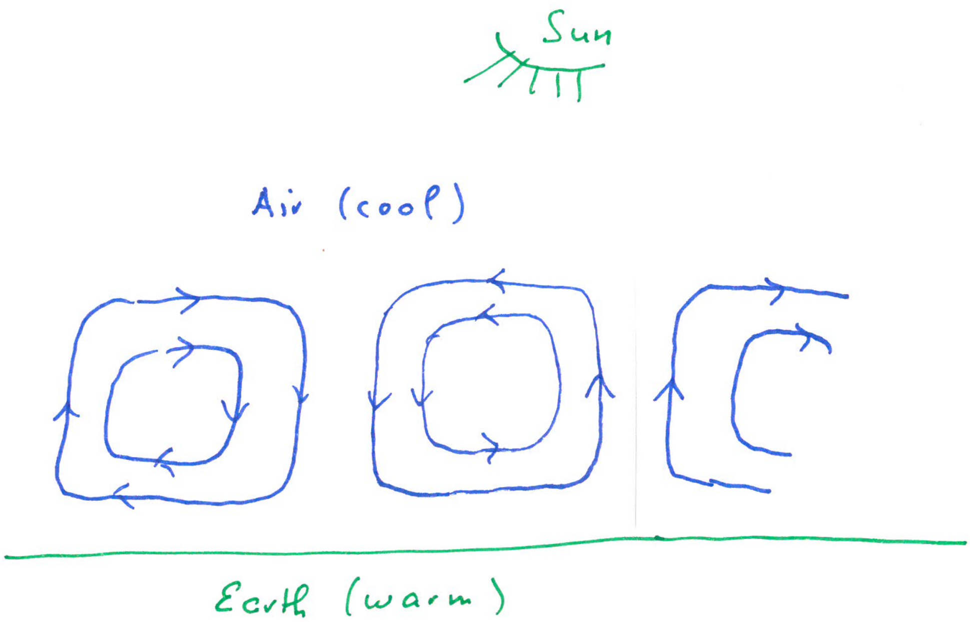
Lorenz Attractors

In the 1960s, the MIT meteorologist Edward Lorenz studied a model for convection rolls in the atmosphere with his computer



This is a model with PDEs, but by translating the problem to Fourier series and truncating to the first three Fourier modes of the solution (Galerkin approximation)

non-linear!

he reduced the problem to an ODE in $\mathbb{R}^3$, which he then fed to the computer

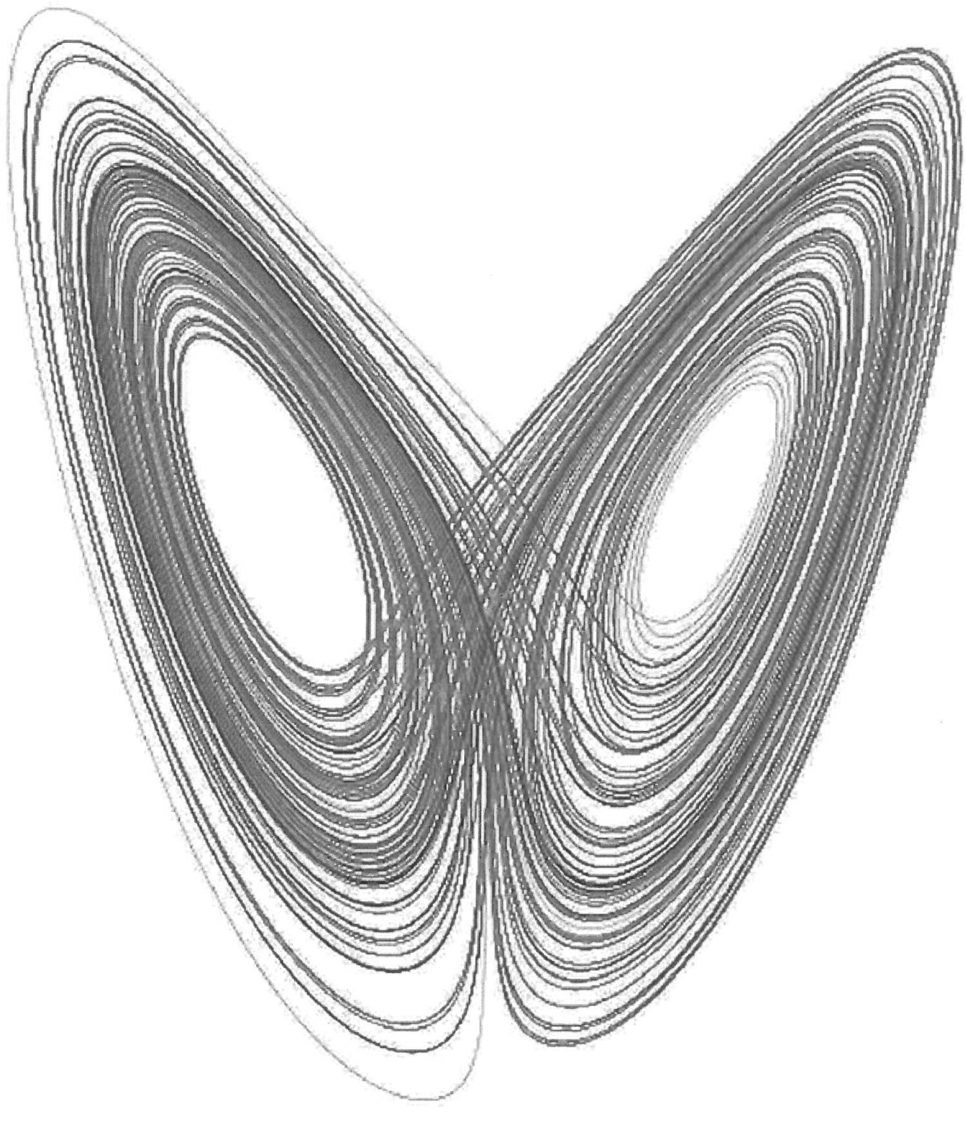


FIGURE 2. The Lorenz attractor

The Lorenz equations

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \sigma(y-x) \\ rx - y - \underline{xz} \\ -bz + \underline{xy} \end{pmatrix}$$

non-linear terms.

$\sigma = 10$ Prandtl number

$b = 8/3$ Rayleigh number

$r = 28$ $\longleftarrow$ we will vary this parameter

Note: The symmetry $(x,y,z) \longmapsto (-x,-y,z)$ (reflection in z-axis) sends solutions to solutions.

The z-axis itself consists of solutions.

There is a stationary point at the origin

$$DF\Big|_{(0,0,0)} = \begin{pmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{pmatrix}\Bigg|_{(0,0,0)} = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}$$

with eigenvalues $-b$ and $-\frac{\sigma+1}{2} \pm \sqrt{\left(\frac{\sigma+1}{2}\right)^2 + \sigma(r-1)}$

$$\Rightarrow \begin{cases} \text{sink for } r < 1 \\ \text{saddle for } r > 1 \end{cases}$$

with two attracting directions and one repelling direction

Further stationary points when

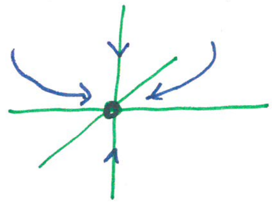
$$\begin{cases} x = y \\ r - \frac{y}{x} = z \\ bz = xy \end{cases} \implies \begin{cases} x = y \\ r-1 = z \\ x = \pm\sqrt{b(r-1)} \end{cases}$$

i.e. $P_{\pm} = \left(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1 \right)$ provided $r \geq 1$.

Bifurcation Analysis

(i) $0 \leq r < 1$ Sink at $(0,0,0)$

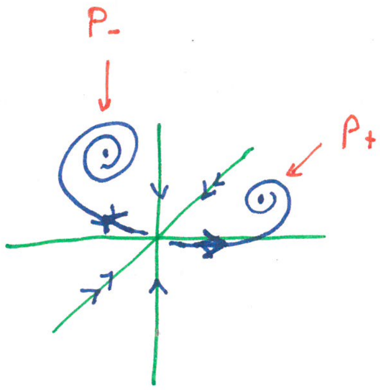
Attracts every orbit



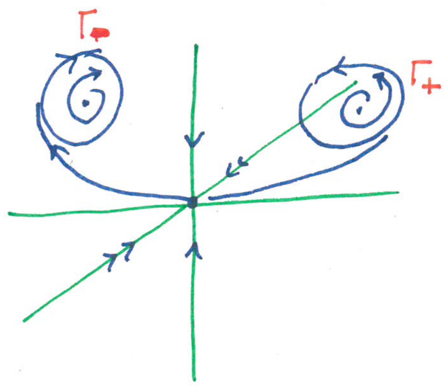
(ii) $r = 1$ Pitchfork bifurcation

$(0,0,0)$ becomes saddle

Two new stationary points emerge



(iii) $r \approx \frac{470}{19}$ $P_{\pm}$ undergo Hopf bifurcations, creating two attracting periodic orbits $\Gamma_{\pm}$



(iv) $r \approx 28$ $\Gamma_{\pm}$ "merge", creating a two-winged "strange attractor"

For (i) $\boxed{r<1}$

Try Lyapunov function $L = \frac{1}{2}\left(r x^2 + \sigma y^2 + \sigma z^2\right)$

Then $\dot{L} = \nabla L \bullet F = \begin{pmatrix} r x \\ \sigma y \\ \sigma z \end{pmatrix} \bullet \begin{pmatrix} \sigma(y-x) \\ r x - y - xz \\ -bz + xy \end{pmatrix}$

$$= r\sigma(xy - x^2) + r\sigma xy - \sigma y^2 - \sigma xyz$$
$$- \sigma b z^2 + \sigma xyz$$
$$= -\sigma\left(r(x-y)^2 + (1-r)y^2 + bz^2\right) \leq 0.$$

So L is indeed a strict Lyapunov function, and every orbit is asymptotic to the sink at $(0,0,0)$, provided $r<1$.

In general, so also $\boxed{r \geq 1}$, we can try

$$L = \frac{1}{2}\left(r x^2 + \sigma y^2 + \sigma(z-2r)^2\right)$$

Then $\dot{L} = \nabla L \bullet F = \begin{pmatrix} r x \\ \sigma y \\ \sigma(z-2r) \end{pmatrix} \bullet \begin{pmatrix} \sigma(y-x) \\ r x - y - xz \\ -bz + xy \end{pmatrix}$

$$= -\sigma\left(r x^2 + y^2 + b(z+r)^2 - b r^2\right)$$

and this is negative <u>outside</u> the ellipsoid
filled

$E:\ r x^2 + y^2 + b(z+r)^2 \leq b r^2$

Hence all orbit converge to and are then confined to this ellipsoid.

Over time this ellipsoid decreases in volume because

$$\frac{d}{dt} \mathrm{Vol}(\varphi^t(E)) = \int_{\varphi^t(E)} \mathrm{div}\, F \; d\mathrm{Vol}$$

flow

and for the vector field F of the Lorenz equations $\mathrm{div}\, F = -\sigma - 1 - b < 0$,

so $\mathrm{Vol}(\varphi^t(E)) = \mathrm{Vol}(E)\, e^{-(\sigma+1+b)t}$.

It follows that the <u>Lorenz attractor</u>

$$A = \bigcap_{t \geq 0} \varphi^t(E)$$

has zero Lebesgue measure.

However A is much more complicated than (the union of) stationary points and limit cycles.

for large values of r

For (iii), the Hopf bifurcation.

By the symmetry $(x,y,z) \mapsto (-x,-y,z)$ it suffices to look only at the stationary point $P_+ = (x, x, r-1)$ ← $x = \sqrt{b(r-1)}$

The characteristic equation of

$$DF\Big|_{P_+} = \begin{pmatrix} -\sigma-\lambda & \sigma & 0 \\ r-z & -1-\lambda & -x \\ x & x & -b-\lambda \end{pmatrix}\Bigg|_{P_+}$$

is

$$\lambda^3 + \lambda^2(\sigma+1+b) + \lambda b(\sigma+r) + 2b\sigma(r-1) = 0$$

At a Hopf bifurcation, there are purely imaginary roots, so we try $\lambda = \pm i\omega$.

$$\begin{cases} 0 = \text{real part} = -\omega^2(\sigma+1+b) + 2b\sigma(r-1) \\ 0 = \text{imaginary part} = -\omega^3 + \omega b(\sigma+r) \end{cases}$$

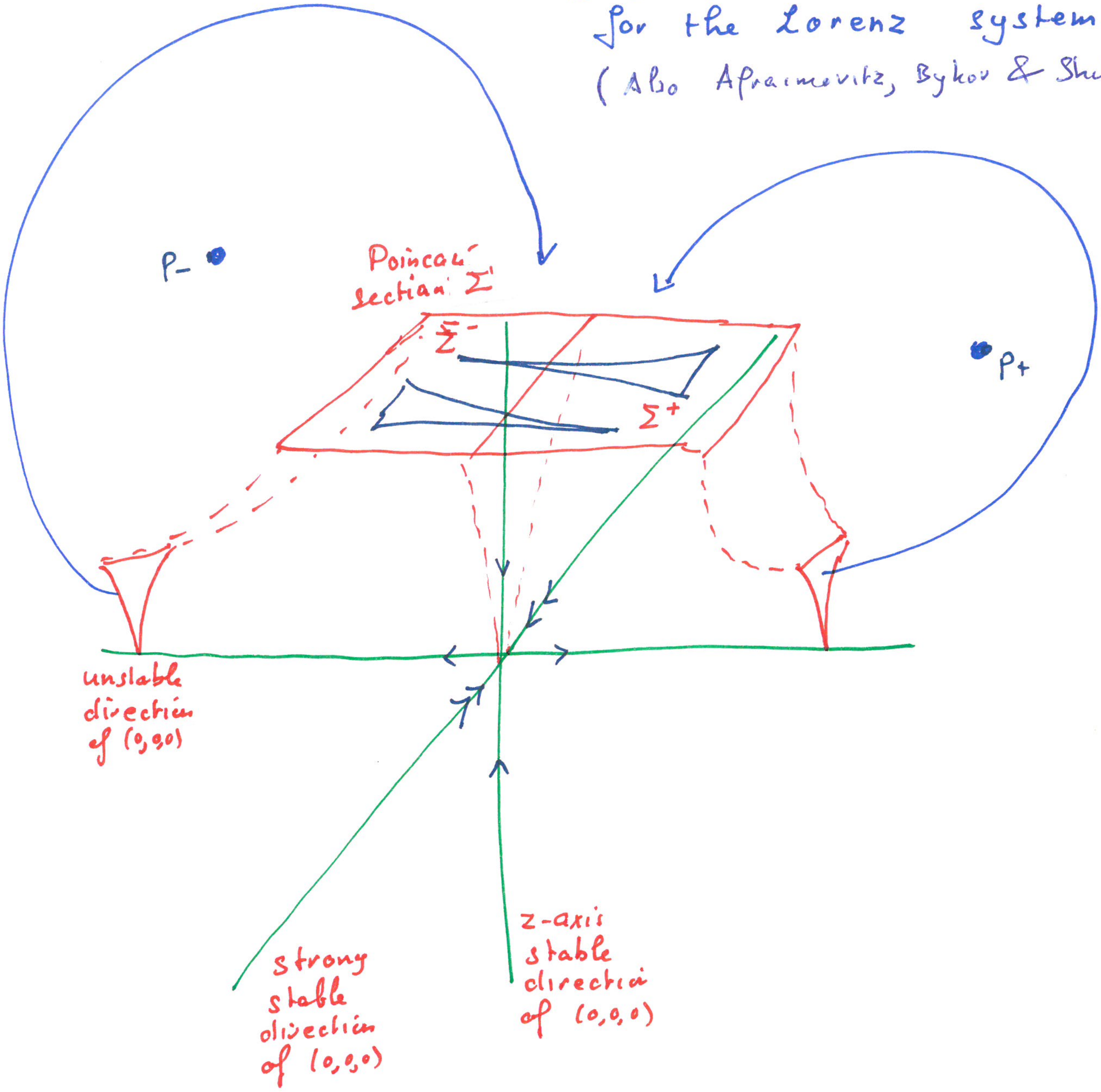
Hence $\omega^2 = \underbrace{b(\sigma+r) = \dfrac{2b\sigma(r-1)}{\sigma+1+b}}_{\text{make } r \text{ subject}}$

$$r = \frac{\sigma(3+b+\sigma)}{\sigma-1-b} > 0$$

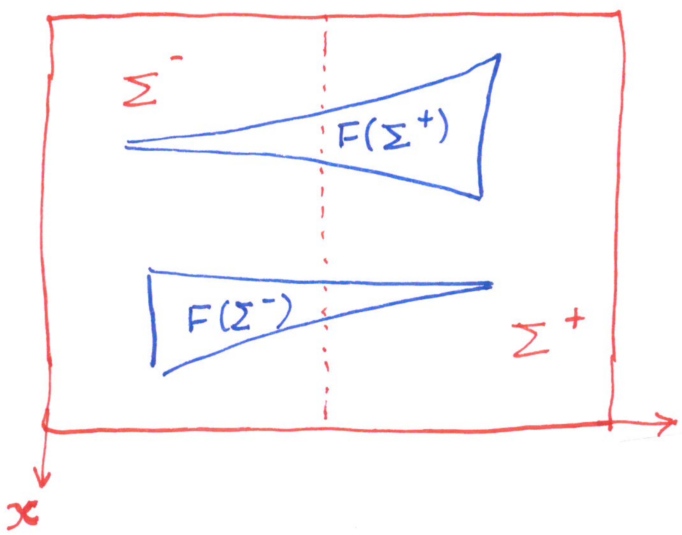
$\sigma = 10$
$b = 8/3$

$$\boxed{r_c = \frac{470}{19}}$$

Guckenheimer - Williams model for the Lorenz system.
(Also Afraimovitz, Bykov & Shilnikov)



Poincaré map $F: \Sigma \to \Sigma$



contracting in x-direction

repelling in y-direction: The y-direction is responsible for the sensitive dependence on initial condition, i.e. the chaos