The Poincaré - Bendixson Theorem

This is about limit behaviour of flows φ^t (i.e. solutions of ODEs) in the plane $\mathbb{R}^2$

Recall the omega-limit / alpha-limit sets:

$$\omega(x) = \{ y : \exists t_n \to +\infty \quad \varphi^{t_n}(x) \to y \}$$

$$\alpha(x) = \{ y : \exists t_n \to -\infty \quad \varphi^{t_n}(x) \to y \}$$

what holds for $\omega(x)$ holds for $\alpha(x)$ in reverse time:

$\omega(x)$ is closed, invariant and if $\{ \varphi^t(x) : t \geq 0 \}$ is bounded, then $\omega(x)$ is compact and connected.

<u>Example</u>: For planar flows, $\omega(x)$ can be

1. A single point, e.g. a stationary point p that is stable or a saddle and $x \in W^s(p)$

2. A periodic solution, e.g. a limit cykel as in the Van der Pol equation, or one of a foliation of periodic solutions as in the case of the harmonic oscillator or the Lotka-Volterra system.

3. A combination of 1. & 2.

The Poincaré - Bendixson Theorem says that in the plane, there are no other possibilities

$\dot{x} = F(x)$ when vector field F has only isolated zeroes

Theorem Let φ^t be the flow of a planar ODE. Suppose $x \in \mathbb{R}^2$ is such that $\{\varphi^t(x)\}_{t \geq 0}$ is a bounded set. Then $\omega(x)$ is one of the foll.

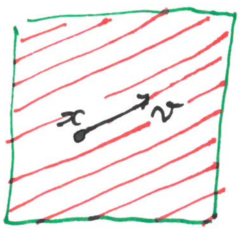
1) a stationary point

2) a regular periodic solution

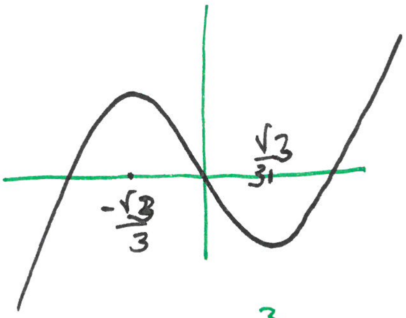
3) a finit union of stationary points $\{y_j\}_{j=1}^N$ and non-closed orbits $\gamma(y)$ such that $\omega(y), \alpha(y) \in \{y_j\}_{j=1}^N$.

Remark The Poincaré - Bendixson does not apply to 2-D manifolds other than (subsets of) $\mathbb{R}^2$ or the 2-sphere.

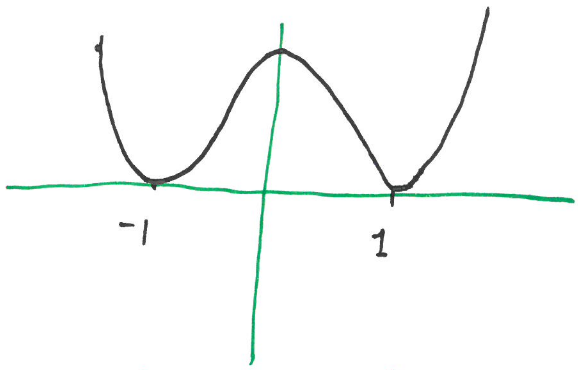
For instance $\dot{x} = v$ for $v \in \mathbb{R}^2$ $\frac{v_1}{v_2} \notin \mathbb{Q}$ and $x(t)$ on the 2-torus $\mathbb{T}^2$, $\varphi^t(x) = x + tv \mod 1$ is dense in $\mathbb{T}^2$ and $\omega(x) = \mathbb{T}^2$.



Examples of case 3). Potential $V: \mathbb{R} \to \mathbb{R}$



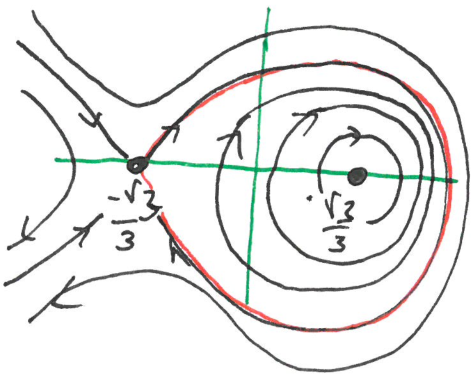
$V(x) = x^3 - x$

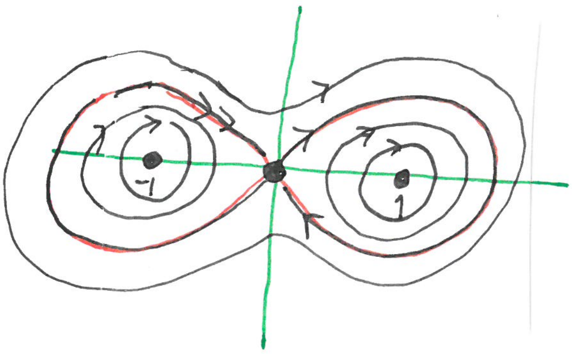


$V(x) = (x^2-1)^2$

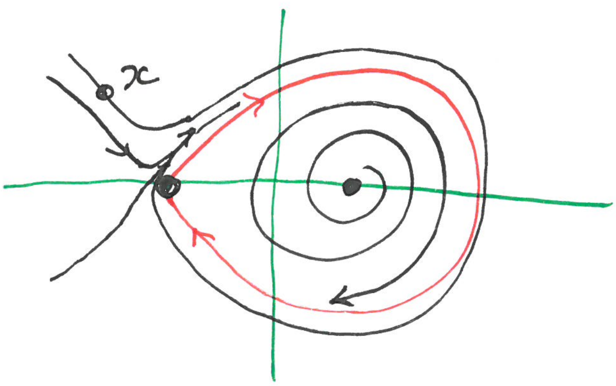
$$\ddot{x} = V'(x) \Rightarrow \begin{array}{l} \dot{x} = y \\ \dot{y} = -V'(x) \end{array}$$

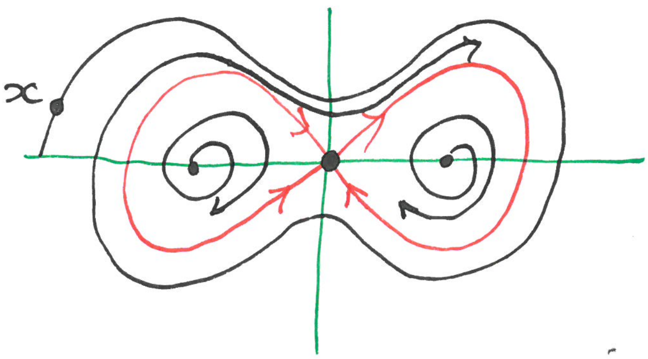
stationary points at extrema of V.





Add "friction term" $f(x, \dot{x})$ that is $= 0$ on indicated solutions $\ddot{x} + f(x, \dot{x}) + V'(x) = 0$





Proof of the Poincaré - Bendixson Theorem

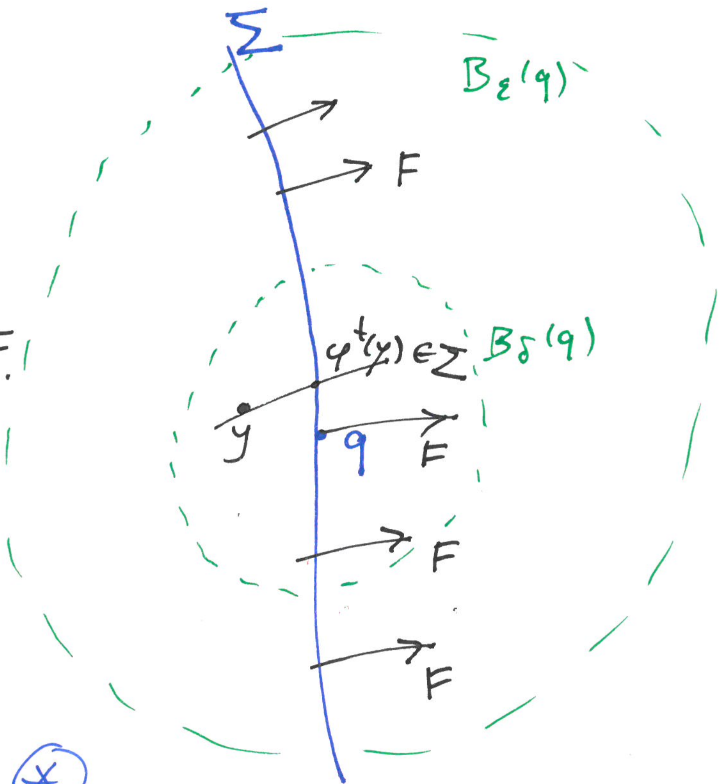
› Because $\{\varphi^t(x)\}_{t\geq 0}$ is bounded, i.e. contained in a compact region, there are accumulation points: $\omega(x) \neq 0$.

› If $\omega(x)$ consists of only stationary points (which are isolated!) connectedness of $\omega(x)$ implies that $\omega(x)$ is a single stationary point.

› Assume $q \in \omega(x)$ is not stationary, so $F(q) \neq 0$ and $\exists\, \varepsilon > 0$ s.t. $F(y) \neq 0$ for $y \in B_\varepsilon(q)$.

Let $\Sigma \subset B_\varepsilon(q)$ be a one-dimensional section that is transversal to the vector field F.

Then $\exists\, \delta < \varepsilon$ such that for every $y \in B_\delta(q)$ there is t close to 0 such that $\varphi^t(y) \in \Sigma$ (*)

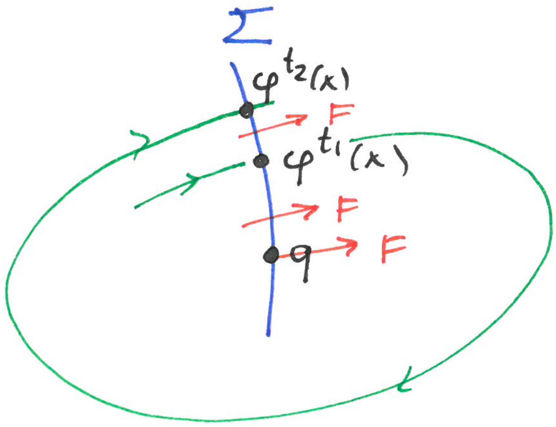


Since $q \in \omega(x)$, there is $t_1 > 0$ such that $\varphi^{t_1}(x) \in B_\delta(q)$

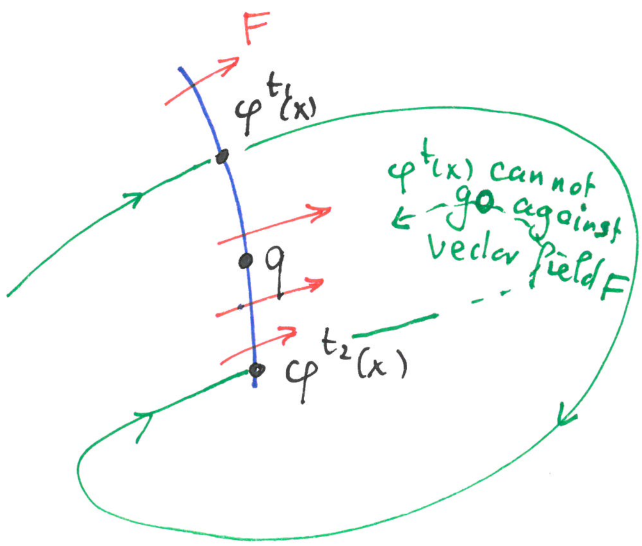
By ($*$) we may assume that $\varphi^{t_1}(x) \in \Sigma$.

Let $t_2 = \min\{t > t_2 : \varphi^t(x) \in \Sigma\}$.

Since $q \in \omega(x)$ and by ($*$), t_2 exists.



If $\varphi^{t_1}(x)$ lies between q and $\varphi^{t_2}(x)$, then $\Sigma \cup \{\varphi^t(x)\}_{t=t_1}^{t_2}$ contains a closed curve surrounding q and shielding off $\{\varphi^t(x)\}_{t \geq t_2}$ from $q \Rightarrow$ $\boxed{q \notin \omega(x) \ ↯}$



If q lies between $\varphi^{t_1}(x)$ and $\varphi^{t_2}(x)$, then $\Sigma \cup \{\varphi^t(x)\}_{t=t_1}^{t_2}$ contains a closed curve with q on it, but shielding off $\{\varphi^t(x)\}_{t \geq t_2}$ from q:

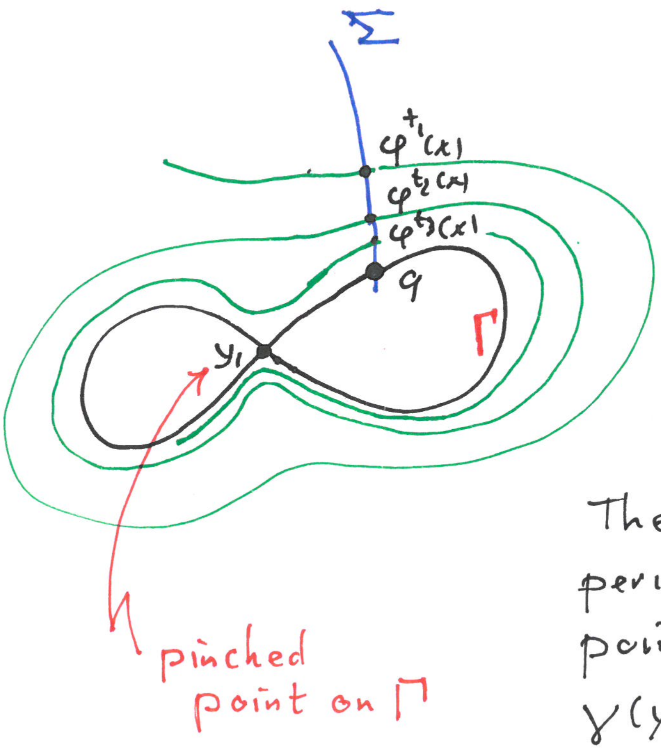
$\Rightarrow$ $\boxed{q \notin \omega(x) \quad ↯}$

Proof of Poincaré-Bendixson continued

> If $\varphi^{t_1}(x) = \varphi^{t_2}(x) = q$, then
$\omega(x) = \{\varphi^t(q)\}_{t \geq q}$ = periodic solution
so 2) holds.

> The remaining case is that
$\varphi^{t_k}(x) \longrightarrow q$ monotonically
(repeat the argument for $t_3 = \min\{t > t_2 : \varphi^t(x) \in \Sigma\}$ etc.)

But then $\{\varphi^t(x)\}_{t=t_k}^{t_{k+1}}$ converges to a (possibly pinched) closed curve Γ



The pinched points are stationary points, but only finitely many of them, because F has only isolated zeroes.

The non-pinched parts form a periodic solution (if $\nexists$ pinched points) or non-closed orbits $\gamma(y)$ with $\omega(y), \alpha(y) \in \{y_j\}_{j=1}^N$