Biconservative PMCV Surfaces in Robertson Walker Spacess

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(joint work with Rüya Yeğin Şen)

Differential Geometry Workshop 2024

September 04-06, 2024

Acknowledgement

In this talk, we are going to present some of results obtained during the TÜBİTAK project RWTSSubmanifolds (Project number: 121F352)

Section 1:

Preliminaries

Notation

Gauss and Weingarten Formulas

$$\begin{aligned}\tilde{\nabla}_X Y &= \nabla_X Y + h(X, Y),\\ \tilde{\nabla}_X \xi &= -A_\xi X + \nabla^\perp_X \xi,\end{aligned}$$

Notation

Gauss and Weingarten Formulas

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Gauss, Codazzi and Ricci Equations

$$\begin{aligned} (\tilde{R}(X,Y)Z)^T &= R(X,Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X, \\ (\tilde{R}(X,Y)Z)^\perp &= (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z), \\ (\tilde{R}(X,Y)\xi)^\perp &= R^D(X,Y)\xi + h(A_\xi X,Y) - h(X,A_\xi Y) \end{aligned}$$

Section 1.1:

Robertson-Walker Spacetimes

Robertson-Walker Spacetimes

- I is an open interval
- $f : I \to (0, \infty)$
- $\mathbb{Q}^n_c$: n-dimensional Riemannian space form with dimension n and constant curvature c, i.e.,

$$\mathbb{Q}^n_c = \begin{cases} \mathbb{S}^n & \text{if } c = 1, \\ \mathbb{H}^n & \text{if } c = -1, \\ \mathbb{E}^n & \text{if } c = 0, \end{cases}$$

with the metric tensor of g_c.

Definition

Robertson-Walker spacetime is the warped product manifold defined by $L^n_1(f, c) = I \times_f \mathbb{Q}^{n-1}_c$ with the metric tensor

$$\tilde{g} = -dt^2 + f(t)^2 g_c.$$

The Levi-Civita Connection

Let $\Pi^1 : I \times R_c^{n-1} \to I, \qquad \Pi^2 : I \times R_c^{n-1} \to R_c^{n-1}$ denote the canonical projections. For a given vector field X in $L_1^n(f,0)$, we define a function X_0 and a vector field $\bar{X}$ by the decomposition

$$X = X_0 \partial_t + \bar{X}.$$

[1] [Chen and Van der Veken, J. Math. Phys, 2007]

The Levi-Civita Connection

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$$X = X_0 \partial_t + \bar{X}.$$

We are going to use the following re-statement of Lemma 2.1 in [1]

Lemma

The Levi-Civita connection $\tilde{\nabla}$ of $L_1^n(f,c)$ is

$$\widetilde{\nabla}_X Y \quad = \quad \nabla^0_X Y + \frac{f'}{f}\left(\tilde{g}(\bar{X},\bar{Y})\partial_t + X_0 \bar{Y} + Y_0 \bar{X}\right)$$

whenever X and Y are tangent to $L_1^n(f,c)$, where ∇^0 denotes the Levi-Civita connection of the Cartesian product space $L_1^n(1,c) = I \times \mathbb{Q}_c^{n-1}$.

[1] [Chen and Van der Veken, J. Math. Phys, 2007]

Curvature Tensor

Curvature tensor of $L_1^n(f,c)$ (See [1])

Lemma

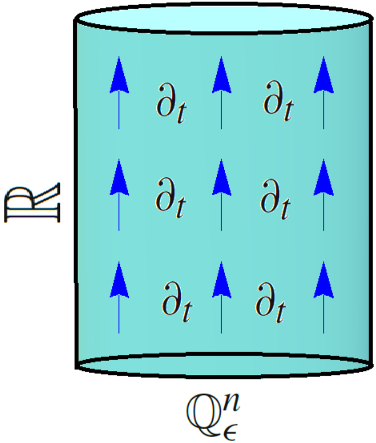
The curvature tensor $\tilde{R}$ of $L_1^n(f,c)$ satisfies

$$\tilde{R}(\partial_t,\bar{X})\partial_t = \frac{f''}{f}\bar{X}, \qquad \tilde{R}(\partial_t,\bar{X})\bar{Y} = \frac{f''}{f}\langle\bar{X},\bar{Y}\rangle\partial_t,$$

$$\tilde{R}(\bar{X},\bar{Y})\partial_t = 0, \qquad \tilde{R}(\bar{X},\bar{Y})\bar{Z} = \frac{f'^2+c}{f^2}\left(\langle\bar{Y},\bar{Z}\rangle\bar{X} - \langle\bar{X},\bar{Z}\rangle\bar{Y}\right)$$

whenever $\Pi_1^*(\bar{X}) = \Pi_1^*(\bar{Y}) = \Pi_1^*(\bar{Z}) = 0$

Comoving observer field



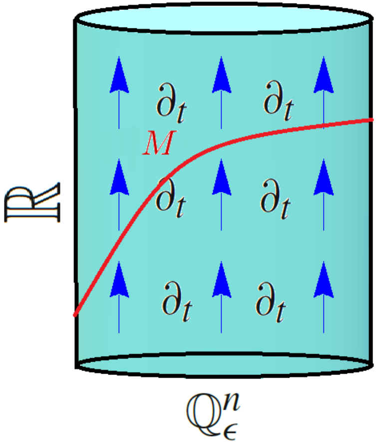
- Robertson-Walker spacetime: $L^n_1(f,c) = I \times_f \mathbb{Q}^{n-1}_c$
- Metric tensor: $\tilde{g} = -dt^2 + f(t)^2 g_c$.

Definition

The vector field ∂_t is called as "comoving observer field".

Note: When $f = 1$, ∂_t is parallel along $\mathbb{R} \times \mathbb{Q}^n_c$.

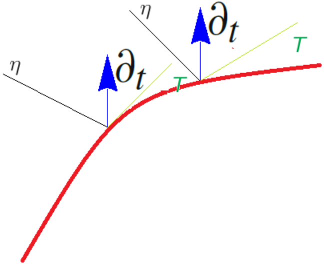
Submanifolds of $L_1^n(f,c)$



Let M be a submanifold of $L_1^n(f,c)$

- R: Curvature tensor of M
- ∇: LC connection of M
- A_ξ: Shape operator along $\xi \in T^\perp M$
- h: SFF of M
- $\nabla^\perp$: Normal Connection
- H: MCV of M

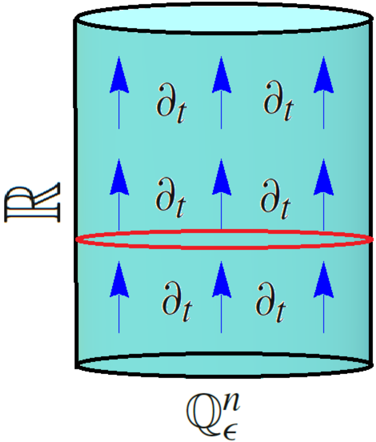
Vector Fields T, η



We define a tangent vector field T on M and a normal vector field η by decomposing ∂_t as

$$\partial_t = T + \eta.$$

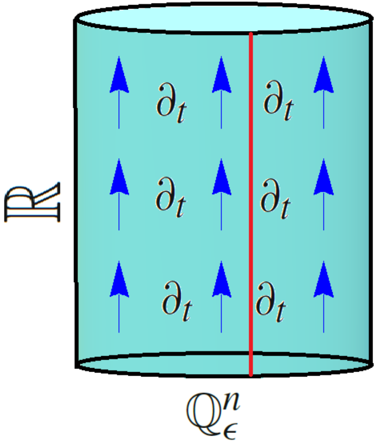
Some special cases



$\partial_t = T + \eta$

If $T = 0$, then M is a slice: $M \subset \mathbb{Q}^n_c \times \{t_0\}$

Some special cases



$$\partial_t = T + \eta$$

If $\eta = 0$, then M is a vertical cylinder: $M = N \times \mathbb{R}$ for some submanifold N of $\mathbb{Q}^n_c$.

Some References

- Surfaces in RW Spacetimes [B.-Y. Chen, and J. Van der Veken, 2007]
- Marginally trapped submanifolds in RW Spacetimes [H. Anciaux, N. Cipriani, 2020]
- Surfaces in Space Forms and RW Spacetimes [K. Dekimpe, J. Van der Veken, 2020]

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- Light-like Submanifolds of RW Spacetimes [X. Liu And Q. Pan, 2015]
- Light-like Submanifolds of GRW Spacetimes [T. H. Kang, 2014], [M. H. A. Hamed, F. Massamba, S. Ssekajja, 2019]

Section 1.2:

Biharmonic Maps

Biharmonic Maps

Let $\psi : (M, g) \to (N, \widetilde{g})$ be a smooth map between two semi-Riemannian manifolds.

- v_g represents the volume element of M
- $\tau(\psi) = \operatorname{trace} \nabla d\psi,$

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Bienergy Functional

$$E_2 : C^\infty(M, N) \to \mathbb{R}, \quad E_2(\psi) = \frac{1}{2} \int_M \widetilde{g}(\tau(\psi), \tau(\psi))\, v_g,$$

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Biharmonic map

A mapping ψ is said to be biharmonic if it is a critical point of the energy functional E_2.

Biharmonic Mappings

In [2] G.Y. Jiang studied the first and second variation formulas of E_2 in order to understand its critical points, called biharmonic maps (See also [3]).

[2] G. Y. Jiang, *2-harmonic maps and their first and second variational formulas*, 1986.

[3] G. Y. Jiang, *2-harmonic isometric immersions between Riemannian manifolds*, 1986.

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Euler-Lagrange Equation

ψ is biharmonic if and only if it satisfies the Euler-Lagrange equation

$$\tau_2(\psi) := \Delta\tau(\psi) - \operatorname{trace} \widetilde{R}(d\psi, \tau(\psi))\, d\psi = 0. \tag{1}$$

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Remark

It is obvious that a harmonic map is biharmonic.

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$$\tau_2(\phi) = 0 \Leftrightarrow \begin{cases} m\nabla \|H\|^2 + 4\,\mathrm{trace}\, A_{\nabla^{\perp}_{\cdot} H}(\cdot) + 4\,\mathrm{trace}\left(\tilde{R}(\cdot, H)\cdot\right)^T = 0, & (\mathrm{T}) \\ \mathrm{trace}\, h(A_H(\cdot), \cdot) - \Delta^{\perp} H + \mathrm{trace}\,(\tilde{R}(\cdot, H)\cdot)^{\perp} = 0 & (\perp), \end{cases}$$

Biconservative Immersions

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Biconservative Submanifolds

If $\psi : (M, g) \to (N, \tilde{g})$ is an isometric immersion, then it is biconservative if and only if

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Proposition

An immersion $\phi : (M, g) \hookrightarrow (N, \tilde{g})$ is biconservative if and only if the equation

$$m\nabla \|H\|^2 + 4\,\mathrm{trace}\, A_{\nabla^\perp_\cdot H}(\cdot) + 4\,\mathrm{trace}\left(\tilde{R}(\cdot, H)\,\cdot\,\right)^T = 0 \qquad \text{(T)}$$

is satisfied.

Section 2:

Class $\mathcal{A}$ Surfaces

Section 2.1:

Submanifolds of $\mathbb{R} \times \mathbb{Q}^n_c$

Class $\mathcal{A}$ immersions

Recall the expression

$$\partial_t = T + \eta$$

for a given isometric immersion $\phi : (\Omega, g) \to \mathbb{R} \times \mathbb{Q}^n_c$ and put $M = \phi(\Omega)$.

[4] [Mendonça B., Bull. Braz. Math. Soc., 2010]
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The following definition is given in for hypersurfaces in [4] and for submanifolds with arbitrary codimension in [5]

Definition

ϕ belongs to class $\mathcal{A}$ if T is a principle direction of all shape operators of ϕ.

[4] [Mendonça B., Bull. Braz. Math. Soc., 2010]

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Trivial examples

If $T = 0$ or $\eta = 0$, then we have the following trivial examples of $\mathcal{A}$ immersions:

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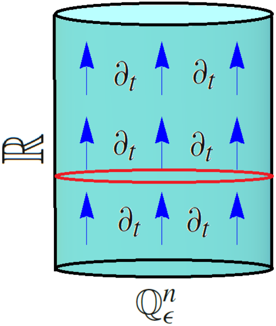
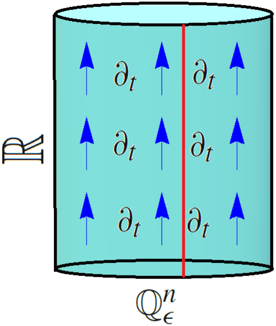


$T = 0$ Slices



$\eta = 0$ Vertical cylinders

Non-trivial example

- A non-trivial example of class $\mathcal{A}$ immersions into $\mathbb{R} \times \mathbb{Q}^n_c$ was constructed in [6] and [7].
- Also, the complete classification of class $\mathcal{A}$ submanifolds of $\mathbb{R} \times \mathbb{Q}^n_c$ was obtained in these papers.

[6] [Mendonça, B., Bull. Braz. Math. Soc., 2010]
[7] [Mendonça, B. and Tojeiro, R., Canad. J. Math., 2014.]

Section 2.2:

Class $\mathcal{A}$ Surfaces in $L^4_1(f,0)$

Example 1

For some smooth functions x_1, x_2, consider the following spacelike surface in $L^4_1(f, 0)$

$$\phi(u, v) = (u, x_1(u), x_2(u), v) \tag{2}$$

with $-1 + f^2(x_1'^2(u) + x_2'^2(u)) > 0$.

Proposition

The surface given by (2) is class $\mathcal{A}$.

Example 2

- $\alpha : I_v \to \mathbb{S}^2$ is an arc-length parameterized curve with unit normal n and curvature κ.

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- Define $\phi_2(u, v)$ and $\phi_3(u, v)$ by

$$\phi_2(u, v) = \int_{u_0}^{u} R(\xi) \sin \tau(\xi, v) d\xi + \psi_1(v),$$
$$\phi_3(u, v) = \int_{u_0}^{u} R(\xi) \cos \tau(\xi, v) d\xi + \psi_2(v)$$

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- The following conditions hold:

$$\psi_1' = \kappa\psi_2 - \phi_1, \qquad \psi_2' = -\kappa\psi_1, \qquad \tau_v = \kappa$$

and

$$(-1 + f^2R^2)(\phi_1' - \phi_2) > 0$$

Consider the surface given by

$$\phi(u,v) = (u, \phi_1(v)\alpha(v) + \phi_2(u,v)\alpha'(v) + \phi_3(u,v)n(v)) \tag{3}$$

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Proposition

The surface given by (3) is class $\mathcal{A}$.

Local Classification Theorem

See [8]

Theorem

A spacelike surface in $L^4_1(f,0)$ is a class $\mathcal{A}$ surface if and only if it is locally congruent to one of the following surfaces:

(i) The cylinder described in Example 1,

(ii) The surface described in Example 2.

[8] [Bektaş Demirci, NCT, Yeğin Şen, arXiv:2408.00475]

Minimal Class $\mathcal{A}$ Surfaces

Theorem

Let M be a space-like surface in $L^4_1(f,0)$ which has no open part lying on a totally geodesic hypersurface of $L^4_1(f,0)$.

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Theorem

Let M be a space-like surface in $L_1^4(f,0)$ which has no open part lying on a totally geodesic hypersurface of $L_1^4(f,0)$. Then, M is minimal and class $\mathcal{A}$ surface if and only if it is locally congruent to the surface

$$\phi(u,v) = (u, \zeta_1(u)\cos v, \zeta_1(u)\sin v, \zeta_2(u)),$$

where ζ_1, ζ_2 satisfies

$$\begin{aligned} f\zeta_1'' &= f'\zeta_1' \left(2f^2 \left({\zeta_1'}^2 + {\zeta_2'}^2\right) - 3\right) + \sqrt{f^2 \left({\zeta_1'}^2 + {\zeta_2'}^2\right) - 1}, \\ f\zeta_2'' &= f'\zeta_2' \left(2f^2 \left({\zeta_1'}^2 + {\zeta_2'}^2\right) - 3\right). \end{aligned}$$

Further Results

We have obtained the following results about class $\mathcal{A}$ surfaces:

Proposition

A surfaces satisfying $\nabla^{\perp}\eta = 0$ in $L^4_1(f, c)$ is class $\mathcal{A}$.

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Proposition

A surface in $L^4_1(f,c)$ with positive relative nullity is either class $\mathcal{A}$ or it lays on $L^3_1(f,c) \subset L^4_1(f,c)$.

Section 3:

Biconservative Surfaces in RW Spaces

Some References

- Biharmonic maps between warped product manifolds [A. Balmuș, S. Montaldo, C. Oniciuc, J. Geom. Phys. 57 (2007)]
- CMC biconservative surfaces in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ [Fetcu, D., Oniciuc, C., Pinheiro, A. L., J. Math. Anal. Appl., 2015]
- PMCV Biconservative submanifolds in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ [Manfio F., NCT, Upadhyay, U., J. Geom. Anal., 2019]

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- Biharmonic submanifolds in nonflat Lorentz 3-space forms [Sasahara, T., Bull. Aust. Math. Soc. 85, 2012]

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Biconservative PMCV Submanifolds

Lemma

A PMCV submanifold M in $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$ is biconservative if and only if

$$\langle H, \eta \rangle T = 0.$$

$$E_0(H) = \{X \in TM | A_H X = 0\}$$

Remark

Assume that $\dim E_0(H) = 1$. Then, a PMCV biconservative surface belongs to class $\mathcal{A}$.

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- CMC biconservative surfaces in $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$ [Fetcu, Oniciuc, Pinheiro, J. Math.Anal.Appl., 2015]
- 3-dimensional PMCV biconservative submanifolds in $\mathbb{Q}^4_\varepsilon \times \mathbb{R}$ [Manfio, NCT, Upadhyay, Journal of Geometric Analysis, 2019]
  The case $\mathrm{dim} E_0(H) = 2$. A PMCV biconservative surface still belongs to class $\mathcal{A}$.

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- CMC biconservative surfaces in $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$ [Fetcu, Oniciuc, Pinheiro, J. Math.Anal.Appl., 2015]
- 3-dimensional PMCV biconservative submanifolds in $\mathbb{Q}^4_\varepsilon \times \mathbb{R}$ [Manfio, NCT, Upadhyay, Journal of Geometric Analysis, 2019]
  The case $\mathrm{dim} E_0(H) = 2$. A PMCV biconservative surface still belongs to class $\mathcal{A}$.
- PMCV biconservative surfaces in $L^n_1(f, c)$ [NCT, Yeğin Şen, arXiv:2409.00132]
  A PMCV biconservative surface belongs to class $\mathcal{A}$.

Section 3.1:

Biconservative Surfaces in $L_1^n(f, c)$

Biconservative Surfaces in $L_1^n(f, c)$

Let M be a space-like surface in $L_1^n(f, c)$. Recall that M is biconservative if and only if

$$2\nabla\|H\|^2 + 4\,\mathrm{trace}\,A_{\nabla^\perp_\cdot H}(\cdot) + 4\,\mathrm{trace}\left(\tilde{R}(\cdot, H)\,\cdot\,\right)^T = 0 \qquad \text{(T)}$$

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$$\partial_t = T + \eta$$

Lemma

$$\mathrm{trace}\,(\widetilde{R}(\cdot,H)\cdot)^T = \left(\frac{f''}{f} - \frac{f'^2+c}{f^2}\right)\langle H,\eta\rangle T.$$

Biconservative Surfaces in $L^n_1(f, c)$

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$$2\nabla\|H\|^2 + 4\,\mathrm{trace}\, A_{\nabla^\perp_\cdot H}(\cdot) + 4\,\mathrm{trace}\left(\tilde{R}(\cdot, H)\,\cdot\,\right)^T = 0 \qquad \text{(T)}$$

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Lemma

$$\mathrm{trace}\,(\widetilde{R}(\cdot, H)\cdot)^T = \left(\frac{f''}{f} - \frac{f'^2 + c}{f^2}\right)\langle H, \eta\rangle T.$$

Lemma

M is biconservative and PMCV if and only if

$$\left(\frac{f''}{f} - \frac{f'^2 + c}{f^2}\right)\langle H, \eta\rangle T = 0.$$

Assumptions

$$\partial_t = T + \eta$$

Lemma

The curvature tensor $\tilde{R}$ of $L_1^n(f, c)$ satisfies

$$\tilde{R}(\partial_t, \bar{X})\partial_t = \frac{f''}{f}\bar{X}, \qquad \tilde{R}(\partial_t, \bar{X})\bar{Y} = \frac{f''}{f}\langle \bar{X}, \bar{Y}\rangle \partial_t,$$

$$\tilde{R}(\bar{X}, \bar{Y})\partial_t = 0, \qquad \tilde{R}(\bar{X}, \bar{Y})\bar{Z} = \frac{f'^2 + c}{f^2}\left(\langle \bar{Y}, \bar{Z}\rangle \bar{X} - \langle \bar{X}, \bar{Z}\rangle \bar{Y}\right)$$

whenever $\Pi_1^*(\bar{X}) = \Pi_1^*(\bar{Y}) = \Pi_1^*(\bar{Z}) = 0$

Assumptions

- $\dfrac{f''(t)}{f(t)} - \dfrac{f'(t)^2 + c}{f(t)^2} \neq 0$ for any $t \in I$,
- T does not vanish on M,
- $\{e_1, e_2\}$ is an orthonormal basis for the tangent bundle of M, where e_1 is proportional to T,
- All surfaces are connected and all vector fields are smooth.

Integrability Conditions

Since $(\tilde{R}(X, Y)\xi)^{\perp} = 0$, we have

$$R^{\perp}(X, Y)\xi = h(X, A_\xi Y) - h(A_\xi X, Y).$$

Therefore,

Lemma

Let M be a submanifold in $L^n_1(f, c)$. Then,

- M is space-like:
  M has flat normal bundle $\Leftrightarrow$ All shape operators can be simultaneously diagonalized.
- M is time-like:
  M has flat normal bundle $\Leftrightarrow$ All shape operators have same "casual" character.

Integrability Conditions

Codazzi equations:

$$\begin{aligned} 0 &= \left(\nabla^\perp_{e_1} h\right)(e_2, e_1) - \left(\nabla^\perp_{e_2} h\right)(e_1, e_1), \\ \sinh\theta\left(-\frac{f''}{f} + \frac{f'^2 + c}{f^2}\right)\eta &= \left(\nabla^\perp_{e_1} h\right)(e_2, e_2) - \left(\nabla^\perp_{e_2} h\right)(e_1, e_2). \end{aligned}$$

Integrability Conditions

$$\left.\frac{\partial}{\partial t}\right|_M = \sinh\theta\, e_1 + \cosh\theta\, e_3. \tag{4}$$

Remark

In addition to the Gauss, Codazzi, and Ricci equations, the following equations are satisfied on a submanifold in Robertson-Walker space:

$$\tilde{\nabla}_X \partial_t = (\nabla_X T - A_\eta X) + (h(X,T) + \nabla^\perp_X \eta)$$

Note that the LHS of this equation is not zero unless $f = 1$.

Integrability Conditions

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Remark

In addition to the Gauss, Codazzi, and Ricci equations, the following equations are satisfied on a submanifold in Robertson-Walker space:

$$\tilde{\nabla}_X \partial_t = (\nabla_X T - A_\eta X) + (h(X,T) + \nabla^\perp_X \eta)$$

Note that the LHS of this equation is not zero unless $f = 1$.

Lemma

Let M be a space-like surface in $L^n_1(f,c)$, where $n \geq 4$. Then, the equations

$$\begin{aligned}
e_1(\theta)\cosh\theta\, e_1 + \sinh\theta \nabla_{e_1} e_1 - \cosh\theta A_{e_3} e_1 &= \frac{f'}{f}\cosh^2\theta e_1,\\
e_2(\theta)\cosh\theta\, e_1 + \sinh\theta \nabla_{e_2} e_1 - \cosh\theta A_{e_3} e_2 &= \frac{f'}{f} e_2,\\
e_1(\theta)\sinh\theta\, e_3 + \sinh\theta h(e_1,e_1) + \cosh\theta \nabla^\perp_{e_1} e_3 &= \frac{f'}{f}\cosh\theta\sinh\theta e_3,\\
e_2(\theta)\sinh\theta\, e_3 + \sinh\theta h(e_1,e_2) + \cosh\theta \nabla^\perp_{e_2} e_3 &= 0,
\end{aligned}$$

Space-like Surfaces in $L_1^n(f, c)$

Let M be a space-like PMCV surface in $L_1^n(f, c)$.

Space-like Surfaces in $L_1^n(f, c)$

Let M be a space-like PMCV surface in $L_1^n(f, c)$. M is biconservative if and only if

$$\langle H, \eta \rangle = 0.$$

Corollary

There are no marginally trapped biconservative PMCV surface in $L_1^n(f, c)$.

Space-like Surfaces in $L^n_1(f, c)$

$$\langle H, \eta \rangle = 0.$$

Proposition

Then, M is a biconservative PMCV surface if and only if there exists a non-zero constant H_0 and a unit normal vector field e_4 such that

$$\begin{aligned} \nabla^{\perp} e_4 &= 0, \qquad \langle e_4, \eta \rangle = 0, \\ A_{e_4} &= \begin{pmatrix} 0 & 0 \\ 0 & 2H_0 \end{pmatrix}, \\ A_{\xi} &= \begin{pmatrix} \gamma_\xi & 0 \\ 0 & -\gamma_\xi \end{pmatrix} \qquad \text{whenever } \langle e_4, \xi \rangle = 0, \end{aligned}$$

where $\gamma_\xi \in C^{\infty}(M)$.

Space-like Surfaces in $L_1^n(f, c)$

$$\langle H, \eta \rangle = 0.$$

Lemma

Let M be a space-like biconservative PMCV surface in $L_1^n(f, c)$ and $p \in M$. Then the vector fields $e_1,\ e_2$ and e_3 satisfy

$$\begin{aligned} h(e_1, e_2) &= 0, \\ \nabla_{e_1} e_1 = \nabla_{e_1} e_2 = \nabla_{e_2} e_1 = \nabla_{e_2} e_2 &= 0, \\ \frac{f'}{f} = \cosh\theta \gamma_{e_3}, \qquad & e_2(\theta) = 0. \end{aligned}$$

Consequently, there exists a local coordinate system $(\mathcal{N}_p, (u, v))$ such that $\mathcal{N}_p \ni p$ and

$$e_1|_{\mathcal{N}_p} = -\sinh\theta\partial_u, \qquad e_2|_{\mathcal{N}_p} = \partial_v.$$

Section 3.2:

Biconservative Surfaces in $L_1^4(f,0)$

PMCV Surfaces in $L^4_1(f,0)$

Theorem

The Robertson-Walker space-time $L^4_1(f,0)$ admits a space-like, biconservative PMCV surface M with mean curvature H_0 if and only if f satisfies

$$\left(a^2-4H_0^2\right)f^3f''-\left(f'^2-\left(a^2-4H_0^2\right)f^2\right)^2-f'^4=0$$

for a constant a such that $a^2-4H_0^2>0$.

PMCV Surfaces in $L^4_1(f,0)$

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for a constant a such that $a^2-4H_0^2>0$. In this case, M is locally congruent to the rotational surface

$$\phi(u,v)=\left(u,\frac{1}{af(u)}\sin av,\frac{1}{af(u)}\cos av,-\frac{2H_0}{a^2c_2f(u)}\right),$$

where c_2 is a constant.

Sketch Proof

Consider a space-like surface $M^2 \subset L_1^4(f,c)$. Choose an orthonormal frame $\{e_1, e_2; e_3, e_4\}$ such that

$$\eta = \cosh\theta e_3, \qquad H = H_0 e_4$$

Note that we have

$$A_{e_4} = \begin{pmatrix} 0 & 0 \\ 0 & 2H_0 \end{pmatrix}, \qquad A_{e_3} = \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix}.$$

Sketch Proof

Consider a space-like surface $M^2 \subset L^4_1(f,c)$. Choose an orthonormal frame $\{e_1, e_2; e_3, e_4\}$ such that

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Note that we have

$$A_{e_4} = \begin{pmatrix} 0 & 0 \\ 0 & 2H_0 \end{pmatrix}, \qquad A_{e_3} = \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix}.$$

Lemma

Let M be a space-like surface in $L^4_1(f,c)$ and $\{e_1, e_2; e_3, e_4\}$ be an orthonormal frame field. If M is PMCV and biconservative, then

$$\begin{aligned} \widetilde{\nabla}_{e_1} e_1 &= -\gamma e_3, & \widetilde{\nabla}_{e_2} e_1 &= 0, \\ \widetilde{\nabla}_{e_1} e_2 &= 0, & \widetilde{\nabla}_{e_2} e_2 &= \gamma e_3 + 2H_0 e_4, \\ \widetilde{\nabla}_{e_1} e_3 &= -\gamma e_1, & \widetilde{\nabla}_{e_2} e_3 &= \gamma e_2, \\ \widetilde{\nabla}_{e_1} e_4 &= 0, & \widetilde{\nabla}_{e_2} e_4 &= -2H_0 e_2. \end{aligned}$$

Sketch Proof

Next, we obtain

Lemma

If M is PMCV and biconservative, then

$$\begin{aligned} \phi(u,v) &= \left(u, \frac{1}{af(u)}\sin av, \frac{1}{af(u)}\cos av, y(u)\right), \qquad (5) \\ e_4 &= \frac{1}{f}\left(0, -\frac{2H_0}{a}\sin av, -\frac{2H_0}{a}\cos av, c_2\right) \end{aligned}$$

for some constants a, c_2 satisfying

$$4H_0^2 + c_2^2 a^2 = a^2, \qquad c_2 > 0,$$

where H_0 is the mean curvature of M and ϕ is the position vector of M.

Sketch Proof

Next, we obtain

Lemma

If M is PMCV and biconservative, then

$$\begin{aligned} \phi(u,v) &= \left(u, \frac{1}{af(u)}\sin av, \frac{1}{af(u)}\cos av, y(u)\right), \\ e_4 &= \frac{1}{f}\left(0, -\frac{2H_0}{a}\sin av, -\frac{2H_0}{a}\cos av, c_2\right) \end{aligned} \tag{5}$$

for some constants a, c_2 satisfying

$$4H_0^2 + c_2^2 a^2 = a^2, \qquad c_2 > 0,$$

where H_0 is the mean curvature of M and ϕ is the position vector of M.

Note that the MCV of (5) is

$$H = -\frac{\left(a^2 - 4H_0^2\right) f^3 f'' + 2\left(a^2 - 4H_0^2\right) f^2 f'^2 - \left(a^2 - 4H_0^2\right)^2 f^4 - 2f'^4}{2f\left(f'^2 - \left(a^2 - 4H_0^2\right) f^2\right)^{3/2}} e_3 + H_0 e_4.$$

By a direct computation, we obtain the result.

Section 3.3:

Biconservative Surfaces in $L^5_1(f, 0)$

PMCV Surfaces in $L^5_1(f,0)$

Theorem

Let y, f be some functions satisfying the system given by

$$\begin{aligned}
&-a^6b^2c_3^2f^7y'y'' - a^4c_3^2c_4f^3f'f'' - 2a^6c_2c_3^2H_0f^5\left(f''y' + f'y''\right) \\
&-2a^2c_4f^2f'^3\left(a^2c_3^2 - 8c_2H_0f'y'\right) - 4a^4b^2f^6f'y'^2\left(a^2c_3^2 - 4c_2H_0f'y'\right) \\
&+a^2f^4f'\left(-12a^4c_2c_3^2H_0f'y' + 4\left(a^4c_3^2 - 12a^2\left(c_3^2-1\right)H_0^2 - 48H_0^4\right)f'^2y'^2\right) \\
&\qquad +2a^4b^4f^8f'y'^4 + a^8c_3^4f^4f' + 2c_4^2f'^5 = 0 \\
&a^4c_3^2f^3\left(f'y'' - f''y'\right) - a^2b^4f^6y'^3 + a^2b^2f^4y'\left(a^2c_3^2 - 6c_2H_0f'y'\right) \\
&+f^2f'\left(2a^4c_2c_3^2H_0 + \left(a^4c_3^2 + 12a^2\left(c_3^2-1\right)H_0^2 + 48H_0^4\right)f'y'\right) - 2c_2c_4H_0f'^3 = 0
\end{aligned} \tag{6}$$

PMCV Surfaces in $L^5_1(f, 0)$

Theorem

Let y, f be some functions satisfying the system given by

$$\begin{aligned}
&-a^6b^2c_3^2f^7y'y'' - a^4c_3^2c_4f^3f'f'' - 2a^6c_2c_3^2H_0f^5\left(f''y' + f'y''\right)\\
&-2a^2c_4f^2f'^3\left(a^2c_3^2 - 8c_2H_0f'y'\right) - 4a^4b^2f^6f'y'^2\left(a^2c_3^2 - 4c_2H_0f'y'\right)\\
&+a^2f^4f'\left(-12a^4c_2c_3^2H_0f'y' + 4\left(a^4c_3^2 - 12a^2\left(c_3^2-1\right)H_0^2 - 48H_0^4\right)f'^2y'^2\right)\\
&\qquad +2a^4b^4f^8f'y'^4 + a^8c_3^4f^4f' + 2c_4^2f'^5 = 0\\
&a^4c_3^2f^3\left(f'y'' - f''y'\right) - a^2b^4f^6y'^3 + a^2b^2f^4y'\left(a^2c_3^2 - 6c_2H_0f'y'\right)\\
&+f^2f'\left(2a^4c_2c_3^2H_0 + \left(a^4c_3^2 + 12a^2\left(c_3^2-1\right)H_0^2 + 48H_0^4\right)f'y'\right) - 2c_2c_4H_0f'^3 = 0
\end{aligned} \tag{6}$$

for some non-zero constants a, c_2, c_3 satisfying $b^2 = a^2 - 4H_0^2 > 0$, where we put $c_4 = a^2c_3^2 + 4H_0^2$. Then, the Robertson-Walker space-time $L^5_1(f, 0)$ admits a space-like, biconservative PMCV surface M with the mean curvature H_0 parametrized by

$$\phi(u, v) = \left(u, \frac{\sin(av)}{af(u)}, \frac{\cos(av)}{af(u)}, y(u), \frac{2H_0 - c_2a^2f(u)y(u)}{c_3a^2f(u)}\right). \tag{7}$$

PMCV Surfaces in $L_1^5(f, 0)$

Theorem

Conversely, if a Robertson-Walker space-time $L_1^5(f, 0)$ admits a space-like, biconservative PMCV surface, then f must be a solution of (6) and the surface must be locally congruent to the surface given by (7).

PMCV Surfaces in $L^5_1(f,0)$

Theorem

Conversely, if a Robertson-Walker space-time $L^5_1(f,0)$ admits a space-like, biconservative PMCV surface, then f must be a solution of (6) and the surface must be locally congruent to the surface given by (7).

Sketch Proof:
Consider a space-like surface $M^2 \subset L^5_1(f,c)$.

- Choose an orthonormal frame $\{e_1, e_2; e_3, e_4, e_5\}$ such that

$$\eta = \cosh\theta e_3, \qquad H = H_0 e_4$$

- In this case, we have

$$\nabla_{e_1} e_1 = \nabla_{e_1} e_2 = \nabla_{e_2} e_1 = \nabla_{e_2} e_2 = 0,$$
$$A_{e_4} = \begin{pmatrix} 0 & 0 \\ 0 & 2H_0 \end{pmatrix}, \qquad A_{e_3} = \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix},$$
$$A_{e_5} = \begin{pmatrix} \tau & 0 \\ 0 & -\tau \end{pmatrix}.$$

- We used techniques similar to the case $M^2 \subset L^4_1(f,0)$.

Section 3.4:

Surfaces in a Lorentzian Product

Surfaces in $\mathbb{E}^1_1 \times \mathbb{Q}^4_\varepsilon$

Lemma

Let M be a space-like surface in $L^5_1(f,0)$ with mean curvature H_0 and $\{e_1, e_2; e_3, e_4, e_5\}$ be the orthonormal frame field defined by $\eta = \cosh\theta e_3$ and $H = H_0 e_4$. If M is PMCV and biconservative, then

$$
\begin{aligned}
\hat{\nabla}_{e_1} e_1 &= \tau_0 e_5 - c\cosh^2\theta_0 e_6, & \hat{\nabla}_{e_2} e_1 &= 0,\\
\hat{\nabla}_{e_1} e_2 &= 0, & \hat{\nabla}_{e_2} e_2 &= 2H_0 e_4 - \tau_0 e_5 - c e_6,\\
\hat{\nabla}_{e_1} e_3 &= -\tanh\theta_0\tau_0 e_5 + c\frac{\sinh 2\theta_0}{2} e_6, & \hat{\nabla}_{e_2} e_3 &= 0,\\
\hat{\nabla}_{e_1} e_4 &= 0, & \hat{\nabla}_{e_2} e_4 &= -2H_0 e_2,\\
\hat{\nabla}_{e_1} e_5 &= -\tau_0 e_1 - \tanh\theta_0\tau_0 e_3, & \hat{\nabla}_{e_2} e_5 &= \tau_0 e_2,\\
\hat{\nabla}_{e_1} e_6 &= \cosh^2\theta_0 e_1 + \frac{\sinh 2\theta_0}{2} e_3, & \hat{\nabla}_{e_2} e_6 &= e_2
\end{aligned}
$$

Surfaces in $\mathbb{E}^1_1 \times \mathbb{Q}^4_\varepsilon$

Lemma

Let M be a space-like surface in $L^5_1(f,0)$ with mean curvature H_0 and $\{e_1, e_2; e_3, e_4, e_5\}$ be the orthonormal frame field defined by $\eta = \cosh\theta e_3$ and $H = H_0 e_4$. If M is PMCV and biconservative, then

$$
\begin{aligned}
\hat{\nabla}_{e_1} e_1 &= \tau_0 e_5 - c\cosh^2\theta_0 e_6, & \hat{\nabla}_{e_2} e_1 &= 0,\\
\hat{\nabla}_{e_1} e_2 &= 0, & \hat{\nabla}_{e_2} e_2 &= 2H_0 e_4 - \tau_0 e_5 - c e_6,\\
\hat{\nabla}_{e_1} e_3 &= -\tanh\theta_0 \tau_0 e_5 + c\frac{\sinh 2\theta_0}{2} e_6, & \hat{\nabla}_{e_2} e_3 &= 0,\\
\hat{\nabla}_{e_1} e_4 &= 0, & \hat{\nabla}_{e_2} e_4 &= -2H_0 e_2,\\
\hat{\nabla}_{e_1} e_5 &= -\tau_0 e_1 - \tanh\theta_0 \tau_0 e_3, & \hat{\nabla}_{e_2} e_5 &= \tau_0 e_2,\\
\hat{\nabla}_{e_1} e_6 &= \cosh^2\theta_0 e_1 + \frac{\sinh 2\theta_0}{2} e_3, & \hat{\nabla}_{e_2} e_6 &= e_2
\end{aligned}
$$

Proposition

There are no space-like biconservative PMCV surface in $\mathbb{E}^1_1 \times \mathbb{H}^4$.

PMCV Surfaces in $\mathbb{E}^1_1 \times \mathbb{S}^4$

Theorem

Let M be a space-like surface in $\mathbb{E}^1_1 \times S^4$. Then M is biconservative and PMCV if and only if it is congruent to the surface locally parametrized by

$$\phi(u,v) = \left(-b_1 u, \frac{\sqrt{b_1^2+1}\cos\left(\sqrt{b_1^2+2}u\right)}{\sqrt{b_1^2+2}}, \frac{\sqrt{b_1^2+1}\sin\left(\sqrt{b_1^2+2}u\right)}{\sqrt{b_1^2+2}}, b_2, b_3 \sin\frac{v}{b_3}, b_3 \cos\frac{v}{b_3} \right) \tag{8}$$

for some non-zero constants b_1, b_2, b_3 satisfying $b_2^2 + b_3^2 = \frac{1}{b_1^2+2}$.

Section 4:

Concluding Remark

Biconservative surfaces in $L_1^n(f,c)$

First, we obtain

Lemma

Let M be an oriented space-like biconservative PMCV surface in $L_1^n(f,c), n \geq 6$. Then, we have two cases:

Case 1. $\dim N_1M = 2$ at every point of M, $\eta \in N_1M$ and $\nabla^\perp(N_1M) \subset N_1M$,

Case 2. $\dim N_2M = 3$ at every point of M, $\eta \in N_2M - N_1M$ and $\nabla^\perp(N_2M) \subset N_2M$.

We have the following reduction of codimension:

Theorem

Let M be an oriented space-like biconservative PMCV surface in $L_1^n(f,c), n \geq 6$. Then, there exists a totally geodesic submanifold N of $L_1^n(f,c)$ such that $M \subset N$ and $\dim N$ is either 4 or 5.

Sketch Proof

- If M^r is space-like in $L_1^n(f,c)$, then $\Pi^2(M)$ is r-dimensional submanifold in $\mathbb{Q}_c^{n-1}$.
- We proved that $\Pi^2(M)$ lies on $\bar{N}$ by [9]
- We have $M \subset N = I \times_f \bar{N}$
- N is totally geodesic in the RW space.

[9] [Erbacher, J. Differ. Geometry, 1971]

THANK YOU