Critical compatible metrics on contact 3-manifolds

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Outline

1. introducing the main actors and some conjectures
2. brute force approach
3. subtle approach

Main references:

[**1**] Y. Mitsumatsu, D. Peralta-Salas & R. Slobodeanu: *On the existence of critical compatible metrics on contact 3-manifolds.* arXiv 2311.15833
[**2**] S.S. Chern, R.S. Hamilton, *On Riemannian metrics adapted to three-dimensional contact manifolds. With an appendix by Alan Weinstein*, in Lecture Notes in Math. 1111, Springer 1985.
[**3**] S. Tanno, *Variational problems on contact Riemannian manifolds*, Trans. Amer. Math. Soc. (1989).
[**4**] S. Hozoori, *Dynamics and topology of conformally Anosov contact 3-manifolds*, Diff. Geom. Appl (2020).

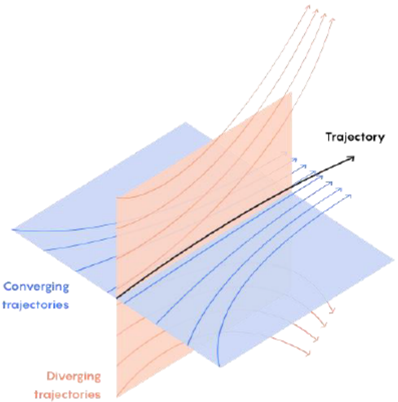
Guest star

- A flow φ_t (associated to a v.f. $X = \dot{\varphi}_t$) on a closed 3-manifold M is **Anosov** if $\exists$ Riemannian metric on M, $A, \Lambda > 0$ and line subbundles E^s and E^u such that $TM = E^s \oplus E^u \oplus \text{span}\{X\}$, $d\varphi_t\left(E^{s,u}\right) = E^{s,u}$ and, $\forall t > 0$,

$$\|d\varphi_t(Y)\| \leqslant Ae^{-\Lambda t}\|Y\|, Y \in E^s,\ \|d\varphi_t(Y)\| \geqslant Ae^{\Lambda t}\|Y\|, Y \in E^u.$$

So geometrically Anosov flows are distinguished by the contracting and expanding behaviour of two invariant directions

- $E^s \oplus \text{span}\{X\}$ and $E^u \oplus \text{span}\{X\}$ integrable plane fields
- typical example: geodesic flow on $T^1\Sigma$, $\Sigma =$ hyperbolic surf
- underlying manifold needs to have a fundamental group with exponential growth $\Rightarrow$ no Anosov flows on $\mathbb{T}^3$ or $\mathbb{S}^3$
- Anosov flows on compact 3-manifolds do not have invariant closed surfaces.
- **conformally Anosov** flows, introduced by Mitsumatsu (as projectively Anosov flows) and Eliashberg-Thurston, are generalizations of Anosov flows



At any point on an Anosov flow, trajectories converge in one direction (blue) and diverge in the other (orange).

Picture credits: Merrill Sherman/Quanta Magazine; source: Thomas Barthelmé

contact structures & compatible metrics

- $M =$ oriented and closed smooth 3-manifold.
- $\alpha =$ contact form on M, i.e. 1-form s.t. $\alpha \wedge \mathrm{d}\alpha \neq 0$ on M.
- $\zeta \subset TM$: (coorientable) contact structure, i.e. a 2-plane field for which there is a contact form α s.t. $\zeta = \ker \alpha$. Such ζ is a maximally non-integrable distribution on M.
- $R =$ Reeb field associated to α: the unique vector field determined by

$$\alpha(R) = 1, \qquad i_R \mathrm{d}\alpha = 0.$$

- any Reeb field preserves the volume form $\alpha \wedge \mathrm{d}\alpha$ on M.

bi-contact structures and Anosovity of their intersection

- A **bi-contact structure** on a 3-manifold M is defined as a pair of transverse contact plane fields (ζ_1, ζ_2) defined by 1-forms η_1 and η_2 such that $\eta_1 \wedge d\eta_1$ and $\eta_2 \wedge d\eta_2$ are volume forms on M of opposite orientations.
- A vector field X is **supported by the bi-contact structure** (η_1, η_2) if $X \in \ker \eta_1 \cap \ker \eta_2$.

Characterization of conformally Anosov [Mitsumatsu, 1995]

supported by a bi-contact structure $\Leftrightarrow$ conformally Anosov

- Reeb flow R of a contact manifold (M, α) is supported by a **calibrated bi-contact structure** if $\exists$ contact forms η_1, η_2 such that $R \in \ker \eta_1 \cap \ker \eta_2$ and, for some constant $\varkappa \neq 0$,

$$\begin{aligned} \eta_1 \wedge d\eta_1 = -\eta_2 \wedge d\eta_2 = \varkappa\Omega\,, \\ \eta_1 \wedge d\eta_2 = \eta_2 \wedge d\eta_1 = 0\,, \\ \alpha \wedge \eta_1 \wedge \eta_2 = \Omega\,, \end{aligned} \tag{1}$$

main actors: contact structures & compatible metrics

- Riemannian **metric** g on M is called **compatible with $\boldsymbol{\alpha}$** if $|\alpha|_g = 1$ and there exists a constant $\theta > 0$ such that

$$*\mathrm{d}\alpha = \theta\alpha\,,$$

  where $*$ is the Hodge star operator associated with g.
- a contact structure ζ and a metric g are compatible if there is a defining contact form α for ζ that is compatible with g.
- the volume element defined by g satisfies $\mathrm{vol}_g = \frac{1}{\theta}\alpha \wedge \mathrm{d}\alpha$.
- on (M, α) consider the space of compatible metrics $\mathcal{M}_\theta(\alpha)$

contact structures & compatible metrics

- a method to **construct a compatible metric**: start with a $(1,1)$-tensor ϕ that satisfies $\phi^2 = -I + \alpha \otimes R$ (so actually a complex structure on the contact planes, extended along the Reeb field direction by $\phi R = 0$) and

  $$d\alpha(\phi X, \phi Y) = d\alpha(X, Y), \quad d\alpha(\phi X, X) > 0\,, \quad X, Y \in \zeta = \ker \alpha,$$

  then define

  $$g(X, Y) := \frac{1}{2}\mathrm{d}\alpha(\phi X, Y) + \alpha(X)\alpha(Y), \quad X, Y \in \Gamma(TM)$$

- define (1,1)-tensor $h := \frac{1}{2}\mathcal{L}_R\phi$, related to the **torsion** tensor $\tau = \mathcal{L}_R g$ via: $\tau(\cdot, \cdot) = 2g(h\phi\cdot, \cdot)$
- h is symmetric, $h\phi + \phi h = 0$, so that if X is an eigenvector of h with eigenvalue λ then ϕX is an eigenvector with eigenvalue $-\lambda$. Moreover $R \in \ker(h)$

The variational problem studied

- **Chern-Hamilton energy** $E : \mathcal{M}_\theta(\alpha) \to [0, \infty)$,

$$E(g) = \int_M |\tau|^2 \mathrm{vol}_g$$

- Sasakian metrics are defined by R being Killing: $\tau = 0$. They are absolute minima of E ("vacuum fields").

Euler-Lagrange equations, Tanno 1989

A compatible metric is a critical point of the Chern-Hamilton energy functional if and only if it satisfies the equation:

$$\nabla_R h = 2h\phi\,, \qquad (2)$$

which is equivalent to $(\nabla_R \mathcal{L}_R g)(\cdot, \cdot) = 2\mathcal{L}_R g(\phi\cdot, \cdot)$.

Deng (1991) computes also the second variation. If they exist, critical compatible metrics are always (local) minima of E.

Important properties

1) First integral property

If g is a critical compatible metric, then $\lambda^2 \in C^\infty(M)$ is a first integral of R, i.e., $R(\lambda^2) = 0$.

2) Curvature eq [Tanno]. Notation $\eta = \alpha$, $\xi = R$ (Reeb)

g is critical iff $\eta_s R^s_{irj} \xi^r = 2g_{ij} - 2\eta_i \eta_j - \nabla_r \eta_i \nabla^r \eta_j$.

3) Conformal Anosovity [Perrone, 2005]

On a compact contact metric 3-manifold with nowhere vanishing torsion τ, if the compatible metric g is critical for the Chern-Hamilton functional, then R is conformally Anosov.

Any volume-preserving conformally Anosov flow is in fact Anosov. Above we can conclude R Anosov.

Proof. global orthonormal frame of eigenvectors of $h \to R$ stays at the intersection of 2 contact structures $\to$ (conf.) Anosovity

Examples of critical metrics

- the standard metric on the tangent sphere bundle of a compact Riem. manifold of const. curvature ± 1 [Blair]
- (related ex.!) For any $\lambda > 0$, in the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ of $SL(2,\mathbb{R})$, consider the basis:

$$R = \frac{\lambda}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ e_1 = \sqrt{\frac{\lambda}{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ e_2 = -\sqrt{\frac{\lambda}{2}}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix};$$

which satisfies the commutation relations

$$[R, e_1] = \lambda e_2\,, \quad [e_1, e_2] = -2R\,, \quad [e_2, R] = -\lambda e_1\,.$$

By left translation $\to$ global frame on $SL(2,\mathbb{R})$.
The dual co-frame $\{\alpha, \eta_1, \eta_2\}$ satisfy:

$$d\alpha = 2\eta_1 \wedge \eta_2\,, \quad d\eta_1 = -\lambda\alpha\wedge\eta_2\,, \quad d\eta_2 = -\lambda\alpha\wedge\eta_1$$

g = (left invariant) metric for which this frame is orthonormal $\to$ critical, compatible with α [Perrone 2005].
- in all examples the energy density $|\tau|^2_g = 8\lambda^2 \equiv$ constant.
- Our main result: these are essentially all possible critical compatible metrics (besides Sasakian)

old Chern-Hamilton conjecture

on a closed contact 3-manifold (M, α) whose corresponding Reeb vector field induces a Seifert foliation, there always exists a critical compatible metric.

Solved by:

- D. Blair [J. Austral. Math. Soc. 37 (1984)]: *for regular contact compact manifolds, a contact metric is critical if and only if it is a Sasakian metric*
- Ph. Rukimbira [Houston J. Math. 21 (1995)]: *same is true for almost regular*

generalized conjecture

Generalized Chern-Hamilton conjecture [Hozoori, 2020]

For any closed contact 3-manifold (M, ζ), there exists a compatible metric that realizes the minimum (among compatible metrics) of the Chern-Hamilton energy functional.

Brute force approach on the torus

- on $\mathbb{T}^3$ we have a family of (tight) contact structures $\eta_m = \sin(mx_3)\mathrm{d}x_1 + \cos(mx_3)\mathrm{d}x_2$, $m \in \mathbb{Z}$ that satisfy $*\mathrm{d}\eta_m = m\eta_m$. There is no contact diffeomorphism between $(\mathbb{T}^3, \zeta_n)$ and $(\mathbb{T}^3, \zeta_m)$ if $n \neq m$). Moreover, any tight contact structure on $\mathbb{T}^3$ is contactomorphic to one of η_m's.
- they are good candidates to test the new conjecture: not regular, $\mathbb{T}^3$ cannot be Sasakian [Itoh, 1997]
- $\alpha := \frac{m}{2}\eta_m$ admits the flat metric $g_0 = \frac{m^2}{4}(dx_1^2 + dx_2^2 + dx_3^2)$ as compatible metric. This metric is not critical.
- try to construct a critical compatible metric for α. Start with the global frame (orthonormal w.r.t. g_0):

$$
\begin{aligned}
&R = \tfrac{2}{m}\left(\sin(mx_3)\partial_1 + \cos(mx_3)\partial_2\right),\\
&X_1 = \tfrac{2}{m}\partial_3,\ X_2 = \tfrac{2}{m}\left(\cos(mx_3)\partial_1 - \sin(mx_3)\partial_2\right)
\end{aligned}
$$

  such that R is the Reeb field associated to α and $\{X_1, X_2\}$ span the contact distribution $\ker\alpha$.

CONTN'D

- ϕ must be given by:

$$\phi X_1 = -aX_1 - \frac{a^2+1}{b} X_2, \qquad \phi X_2 = bX_1 + aX_2, \qquad \phi R = 0$$

where a, b are smooth functions on $\mathbb{T}^3$, $b > 0$.

- the compatible metric in the standard frame $\{\partial_1, \partial_2, \partial_3\}$:

$$\begin{pmatrix} \frac{m^2}{4}\left(b\cos^2(mx_3)+\sin^2(mx_3)\right) & -\frac{m^2}{8}(b-1)\sin(2mx_3) & \dots \\ -\frac{m^2}{8}(b-1)\sin(2mx_3) & \frac{m^2}{4}\left(b\sin^2(mx_3)+\cos^2(mx_3)\right) & \dots. \\ -\frac{am^2}{8}\cos(mx_3) & \frac{am^2}{8}\sin(mx_3) & \dots \end{pmatrix}$$

- $\frac{1}{2}\alpha \wedge \mathrm{d}\alpha = \frac{m^3}{8}\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 = \mathrm{vol}_g \Rightarrow \sqrt{\det g} = \frac{m^3}{8} \Rightarrow$ $a \equiv 0$.
- $\lambda^2 = \frac{\left(\partial_1 b\ \sin(mx_3) + \partial_2 b\ \cos(mx_3)\right)^2 + m^2\, b^4}{m^2\, b^2}$ is nowhere vanishing as $b = g(X_2, X_2) > 0$. Therefore, from property 3) we deduce that the $(\mathbb{T}^3, \alpha)$ is (conformally) Anosov, absurd

There exists no critical metric compatible with the contact forms η_m.

"subtle approach"

Idea(s): prove that in any case R is Anosov. Perrone's property 3) holds due to Mitsumatsu characterization. For this he needs global eigenframe for h that was assumed nonvanishing. If h is vanishing, maybe we still apply a "local version" of Mitsumatsu characterization? Happily the answer was yes:

Theorem (Mitsumatsu, Peralta-Salas, R.S.)

A closed contact 3-manifold (M, α) admits a critical compatible metric g if and only if:

1. It supports a Sasakian metric, or
2. Its associated Reeb field is an Anosov flow which is supported by a calibrated bi-contact structure. This is equivalent to the Anosov flow being C^∞-conjugate to one of the algebraic Anosov flows modeled on $\widetilde{SL}(2, \mathbb{R})$, and M diffeomorphic to a compact quotient of $\widetilde{SL}(2, \mathbb{R})$.

In case 1, $\mathcal{L}_R g = 0$ and in case 2, $|\mathcal{L}_R g| \equiv$ constant on M. Any critical compatible metric g is a global minimizer of the energy.

ideas from the proof

critical metric $\Rightarrow$ Reeb is Anosov, supported by calibrated bi-contact struct

- Similar to Perrone prove

Lemma

Let (N,α) be a compact contact 3-manifold, possibly with boundary. Assume that g is a critical compatible metric such that the function $\lambda^2 = |h|^2/2$ is nowhere vanishing. Then the associated Reeb field R is supported by a C^∞ bi-contact structure (η_1, η_2) that satisfies:

$$\begin{aligned} \eta_1 \wedge d\eta_1 &= -\eta_2 \wedge d\eta_2 = \lambda\Omega\,, \\ \eta_1 \wedge d\eta_2 &= \eta_2 \wedge d\eta_1 = 0\,, \\ \alpha \wedge \eta_1 \wedge \eta_2 &= \Omega. \end{aligned} \tag{3}$$

Here $\Omega := \frac{1}{2}\alpha \wedge d\alpha$. Moreover $\mathcal{L}_R\eta_1 = -\lambda\eta_2$, $\mathcal{L}_R\eta_2 = -\lambda\eta_1$ and

$$g = \alpha \otimes \alpha + \eta_1 \otimes \eta_1 + \eta_2 \otimes \eta_2\,. \tag{4}$$

- if $\{p \in M : \lambda^2(p) = 0\} = \emptyset$, essentially as in the result of Perrone, R is Anosov and λ is constant (Anosov cannot have first integrals), i.e. the bicontact structure is calibrated.
- prove that R is C^∞-conjugate to an algebraic Anosov flow. Define $e_s := \frac{1}{\sqrt{2}}(e_1 + e_2)$ and $e_u := \frac{1}{\sqrt{2}}(e_1 - e_2)$. We have $[R, e_s] = \lambda e_s$, $[R, e_u] = -\lambda e_u$ from the Lemma.
  As $d\alpha(e_u, e_s) = 2$, $[e_s, e_u] = 2R + f_s e_s + f_u e_u$, for some smooth functions f_s and f_u. Take the time t flow $\phi_t = \exp(tR)$ of the Reeb vector field R:

$$[e_s, e_u] = [e^{-t\lambda} e_s, e^{t\lambda} e_u] = \phi_{t*}[e_s, e_u] = 2R + e^{-t\lambda} f_s \circ \phi_{-t}\, e_s + e^{t\lambda} f_u \circ \phi_{-t}\, e_u$$

and thus we have $f_s = e^{-t\lambda} f_s \circ \phi_{-t}$ and $f_u = e^{t\lambda} f_u \circ \phi_{-t}$ for all $t \in \mathbb{R}$. This immediately implies $f_s = f_u \equiv 0$, and therefore we obtain the relations

$$[R, e_s] = \lambda e_s\,, \quad [R, e_u] = -\lambda e_u\,, \quad [e_s, e_u] = 2R.$$

- if $\{p \in M : \lambda^2(p) = 0\} \neq \emptyset$, take U =connected component of $M \setminus \{p \in M : \lambda^2(p) = 0\}$. Prove that there is a compact set $N \subset U$ that is diffeomorphic to $\mathbb{T}^2 \times [c-\delta, c+\delta]$ and is fibred by the level sets of the function $\psi = \lambda^2|_U$.
- obtain a contradiction using:

Extend Mitsumatsu characterisation

Let N be a compact 3-manifold with smooth boundary. If we have a bi-contact structure on N, and the vector field at the intersection of the two contact bundles is tangent to ∂N, then it is conformally Anosov. However, the flow is not Anosov and, in particular, it does not preserve a volume.

ideas from the proof

if R is supported by a C^∞ calibrated bi-contact structure (η_1, η_2), then there exists a critical metric compatible with α

By hypothesis we have $\alpha \wedge \eta_1 \wedge \eta_2 = \Omega := \frac{1}{2}\alpha \wedge d\alpha$ and

$$\begin{aligned} \eta_1 \wedge d\eta_1 = \lambda\Omega \,, \qquad & \eta_2 \wedge d\eta_2 = -\lambda\Omega \,, \\ \eta_1 \wedge d\eta_2 = 0 \,, \qquad & \eta_2 \wedge d\eta_1 = 0 \,. \end{aligned} \tag{5}$$

Consider the Riemannian metric $g := \alpha \otimes \alpha + \eta_1 \otimes \eta_1 + \eta_2 \otimes \eta_2$, whose volume element is $\mathrm{vol}_g = \alpha \wedge \eta_1 \wedge \eta_2 = \Omega$, by assumption. Since in addition $|\alpha|_g = 1$ we deduce that

$$*_g d\alpha = 2\alpha \,,$$

so g is a metric compatible with the contact form α.

We can prove that g is critical (Tanno eqs). Start by evaluating (5) on the (positive) orthonormal frame $\{R, e_1, e_2\}$, g-dual to $\{\alpha, \eta_1, \eta_2\}$. Define the (1,1)-tensor ϕ by $\phi R = 0$, $\phi e_1 = -e_2$, $\phi e_2 = e_1$. ETC

Global minimization

Theorem

Let g be a critical compatible metric on (M, α). Then it is a global minimizer of the Chern-Hamilton energy functional.

Proof. Let (η_1, η_2) the (calibrated) bi-contact structure, whose dual frame is (e_1, e_2). On the contact distribution $\ker \alpha$ with the frame given by $e_s := \frac{1}{\sqrt{2}}(e_1 + e_2)$ and $e_u := \frac{1}{\sqrt{2}}(e_1 - e_2)$. Consider the dual co-frame $\{\eta_u, \eta_s\}$, so that

$$g = \alpha \otimes \alpha + \eta_u \otimes \eta_u + \eta_s \otimes \eta_s \,.$$

Chern-Hamilton energy is $E(g) = 8\lambda^2 \operatorname{Vol}(M)$, $\lambda = c > 0$ const. A general Riemannian metric compatible with α is:

$$\widetilde{g} = \alpha \otimes \alpha + p\eta_u \otimes \eta_u + r(\eta_u \otimes \eta_s + \eta_s \otimes \eta_u) + q\eta_s \otimes \eta_s \,,$$

where p, q, r are C^∞ functions s.t. $p > 0$, $q > 0$, $pq - r^2 = 1$. One can now elementary prove that $E(\widetilde{g}) - E(g) \geqslant 0$.

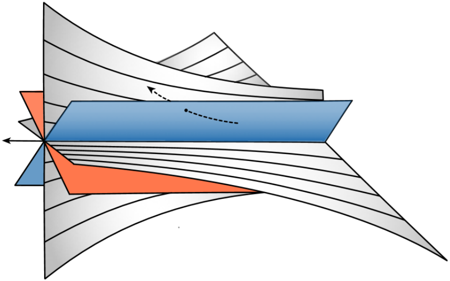
Final remarks

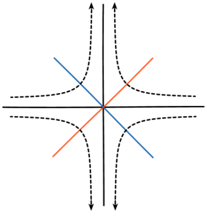
- The manifold in our theorem carries one of the 8 geometries in the sense of Thurston. In the Sasakian case, according to Geiges' classification, the manifold is Seifert fibred and admits an $\mathbb{S}^3$-geometry, a Nil^3-geometry or an $\widetilde{SL}(2,\mathbb{R})$-geometry, and the structures are left invariant. In the Anosov case, the manifold admits an $\widetilde{SL}(2,\mathbb{R})$-geometry.
- a closed contact 3-manifold that is overtwisted it does not admit a critical compatible metric. (using [Hozoori] that proved: a conformally Anosov contact compact 3-manifold is universally tight)

OPEN PROBLEM

Find a good energy functional for selecting the "best compatible metric".

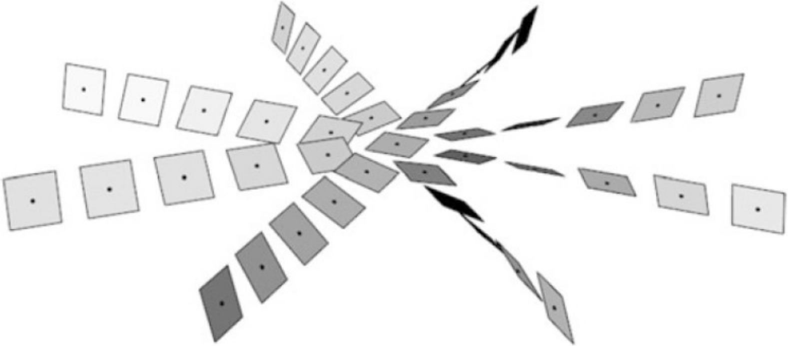
Mulţumesc pentru atenţie!





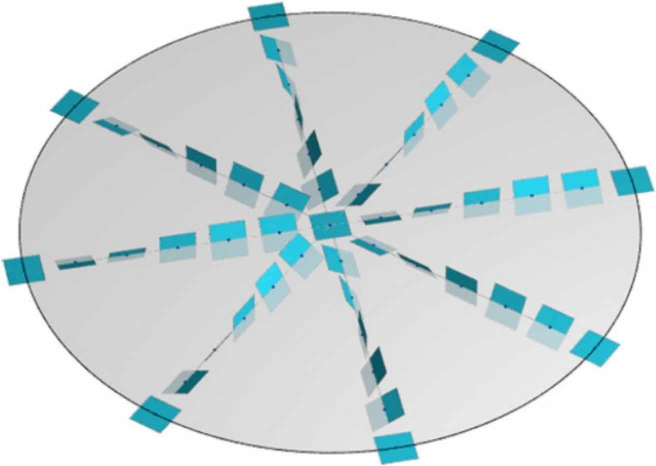
Picture credits: Federico Salmoiraghi, *Surgery on Anosov flows using bi-contact geometry*

Tight contact structure



Picture credits: Patrick Massot, *Topological Methods in 3-Dimensional Contact Geometry - An Illustrated Introduction to Giroux's Convex Surfaces Theory*

Overtwisted contact structure



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