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Reduction for minimal submanifolds from harmonic morphisms

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Example scenario: $\mathfrak{M}$ a space of maps $M^m \to N^n$, $L : \mathfrak{M} \to \mathbb{R}$ a nice functional.

Usually the equation $\delta L = 0$ is a PDE and it gets more difficult when $\dim(M)$ gets larger:

- If $\dim(M) = 0$, we want to extremise a function on N.
- If $\dim(M) = 1$, we (locally) want to solve $\dim(N)$ coupled ODEs.
- If $\dim(M) \geqslant 2$, we want to solve $\dim(N)$ coupled PDEs in $\dim(M)$ variables.

Reduction of dimension

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If there is a nice group action G on M and N that leaves L invariant, then one could look for G-equivariant solutions. Often this leads to another PDE for maps of the form

$$M/G \to N,$$

which has less "effective" variables.

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Theorem ([PT 86])

Let $\pi : (M, g) \to (N, h)$ *be a Riemannian submersion so that:*

(i) $\pi^{-1}(\{y\})$ *is compact for all* $y \in N$*,*

(ii) the mean curvature of the fibres $H_{\pi^{-1}(\{y\})}$ *is a basic field of the submersion.*

Then for all submanifolds $B \subset N$ *one has that* $\varphi^{-1}(B)$ *is minimal in* (M, g) *iff* B *is minimal in* $(N, \mathrm{vol}(\pi^{-1}(\{y\}))^{2/\dim(B)} h)$.

Remark

- Condition (ii) means that there is a vectorfield $V \in \Gamma(TN)$ so that $d\pi(H_{\varphi^{-1}(y)})_y = V_y$.
- The condition of an π-invariant submanifold being minimal is reduced to a PDE on N.

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Reduction of dimension – Example Application

Definition

Let $f : S^{m-1} \to \mathbb{R}$ be an isoparametric function. Define

$$F : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}^2, x \mapsto \left(\|x\|^2, f(\frac{x}{\|x\|}) \right)$$

The level sets of F are all rescalings of the level sets of f.

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Example applications:

Q.M. Wang **[WS 94]**: Classifies all complete immersed minimal submanifolds of $\mathbb{R}^m$ of the form $F^{-1}(\gamma)$ for a curve γ by applying a similar theorem as above and reducing to an ODE in $\mathbb{R}^2$.

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[Ri 24]: Use the same symmetry to find new embedded self-shrinkers in $\mathbb{R}^m \setminus \{0\}$, by applying the above theorem and reducing to an ODE in $\mathbb{R}^2$.

Goal

The goal of today:

- Present reduction techniques for minimal submanifolds by using harmonic morphisms.
- Investigate examples (polynomials!).
- Find new polynomial harmonic morphisms $\mathbb{R}^m \to \mathbb{R}^n$.

This is work in progress.

Definition

A map $\varphi : (M, g) \to (N, h)$ is a harmonic morphism if

$$\Delta^N f = 0 \implies \Delta^M (f \circ \varphi) = 0$$

for any locally defined function germ $f : N \to \mathbb{R}$.

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Definition

Suppose $\varphi : (M, g) \to (N, h)$ is smooth

(i) φ is called harmonic if it extremises $E(\varphi) = \frac{1}{2}\int_M \|d\varphi\|^2$, i.e. iff

$$\tau(\varphi) = \mathrm{Tr}(\nabla d\varphi) = 0$$

in coordinates x_i on M, y_α on N:

$$\sum_{ij} g^{ij}\partial_i\partial_j\varphi^\alpha - \sum_{ij} g^{ij}\Gamma(g)_{ij}^k\partial_k\varphi^\alpha + \sum_{ij} g^{ij}\sum_{\beta\gamma}\Gamma(h)_{\beta\gamma}^\alpha\partial_i\varphi^\beta\partial_j\varphi^\gamma = 0$$

(ii) φ is called weakly horizontally conformal (WHC) if there exists a $\lambda : M \to \mathbb{R}$ so that:

$$\lambda^2 g(d\varphi\, v, d\varphi\, w) = h(v, w)$$

for all $v, w \perp \ker(d\varphi)$. We call λ the *conformality factor* of φ.

Harmonic morphisms

Theorem ([Fu 78], [Is 79])

$\varphi : (M, g) \to (N, h)$ *is a harmonic morphism iff it is WHC and harmonic.*

Examples

- Harmonic Riemannian submersions.
- Holomorphic functions from a Kähler manifold to a Riemann surface.
- The projection $(M \times N, g_M + f^2 h_N) \to (N, h_N)$ to the warped factor in a warped product.

If the codomain has dimension 2, harmonic morphisms fibre the domain by minimal submanifolds:

Theorem ([BE 81])

$\varphi : (M, g) \to (N, h)$ *WHC and* $\dim N = 2$*, then* φ *is harmonic iff* $\varphi^{-1}(\{y\})$ *is minimal at its regular points for all* $y \in M$.

In higher dimensions there are different extensions of this, we highlight:

Theorem ([BG 92])

$\varphi : (M, g) \to (N, h)$ *WHC with conformailtiy factor* λ*, then*

$$\tau_{\dim N}(\varphi) := \lambda^{\dim N - 2}\left(\tau(\varphi) + (\dim N - 2) d\varphi(\nabla \ln \lambda)\right) = 0$$

iff $\varphi^{-1}(\{y\})$ *is minimal at its regular points for all* $y \in N$.

Remark

- τ_p is called the p-tension field.
- For non WHC maps τ_p is defined as the variation of the functional $\frac{1}{p}\int_M \|d\varphi\|^p$.
- In general the equation $\tau_p(\varphi) = 0$ is much less studied than $\tau_2(\varphi) = 0$.
- WHC and $\tau_p = 0 \iff p$-harmonic morphism.

Reduction via harmonic morphisms

Reduction via harmonic morphisms

Initial observation: The previous two results fit into the following statement:

Theorem

$\varphi : (M^m, \to (N^n, h)$ *a WHC submersion,* $p \in \{1, ..., n\}$. *The following are equivalent:*

(i) $\tau_p(\varphi) = 0$

(ii) $\forall B \subset N$ *minimal codimension* p *submanifolds* $\varphi^{-1}(B)$ *is minimal.*

(iii) $\forall B \subset N$ *codimension* p *submanifolds*

$$\lambda^2 H_B = d\varphi(H_{\varphi^{-1}(B)})$$

where H_B *and* $H_{\varphi^{-1}(B)}$ *denote the mean curvatures of* B *and* $\varphi^{-1}(B)$.

Remark

(Submersive) harmonic morphisms are then precisely the WHC maps pull back minimal codim 2 submanifolds to minimal codim 2 submanifolds.

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Proof sketch

The Theorem follows from two calculations. Let $\varphi : (M, g) \to (N, h)$ be a submersive WHC:

Lemma (1), (**[BG 92]**?)

X, Y, Z *vectorfields in* $\Gamma(TN)$, *let* $\widehat{X}, \widehat{Y}, \widehat{Z}$ *denote their horizontal lifts to* M. *Then:*

$$h(\nabla^N_X Y, Z) = \lambda^2 g(\nabla^M_{\widehat{X}} \widehat{Y}, \widehat{Z}) + \widehat{X}(\ln\lambda)h(Y, Z) + \widehat{Y}(\ln\lambda)h(X, Z) - \widehat{Z}(\ln\lambda)h(X, Y)$$

One uses this to see:

Lemma (2)

Let $B \subset N$ *codimension* p *submanifold,* $Z \in \Gamma(TN)$ ***orthogonal to*** B*:*

$$h(\frac{\tau_p}{\lambda^{p-2}}, Z) = h(d\varphi(H_{\varphi^{-1}(B)}) - \lambda^2 H_B, Z)$$

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Proof sketch

Then

(i) $\iff$ (iii) $\implies$ (ii) are clear. (ii) $\implies$ (i) follows from:

Lemma

$y \in N$, *for all vector subspaces* $V \subset T_y N$ *there is a* $B \subset N$ *minimal with* $y \in B$ *and* $T_y B = V$.

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Remark

Lemmas (1) and (2) do not require full WHC, an infinitesimal version suffices.

Corollary (1)

$\varphi : (M, g) \to (N, h)$, $B \subset N$ *minimal and* φ *submersive and WHC to 1st order along* $\varphi^{-1}(B)$. *TFAE:*

(i) $\varphi^{-1}(B)$ *is miminal.*

(ii) $\tau_{\dim N - \dim B}(\varphi) \in T_{\varphi(x)}B$ *for all* $x \in \varphi^{-1}(B)$.

Remark

For B a point this is already contained in Baird-Gudmundsson 1994.

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The following statement is the one I want to apply in the next few slides:

Corollary (2)

$\varphi : (M, g) \to (N, h)$ *submersive harmonic morphism with conformality factor* λ. $B \subset N$ *minimal with* $\dim B \neq \dim N - 2$. *TFAE:*

(i) $\varphi^{-1}(B)$ *is minimal.*

(ii) $d\varphi(\nabla \ln \lambda) \in T_{\varphi(x)}B$ *for all* $x \in \varphi^{-1}(B)$.

Proof.

One has:

$$\tau_{\dim N - \dim B}(\varphi) = \lambda^{\dim N - \dim B - 2}\left(\tau_2(\varphi) + (\dim N - \dim B - 2)d\varphi(\nabla \ln \lambda)\right)$$

where $\tau_2(\varphi) = 0$ since φ is a harmonic morphism. □

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Application to polynomial harmonic morphisms

Polynomials to $\mathbb{C}$

Let $P : \mathbb{R}^m \to \mathbb{C}$ be a homogeneous polynomial harmonic morphism. The only minimal submanifolds of codimension $\notin \{0, 2\}$ are straight lines.
Its usually easier to calculate $dP(\nabla \frac{1}{\lambda^2})$ rather than $dP(\nabla \ln \lambda)$. We calculate this for some examples:

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1. $P : \mathbb{C}^m \to \mathbb{C}$, $(z_1, \ldots, z_m) \mapsto \sum_i z_i^2$. $\quad dP(\nabla \frac{1}{\lambda^2})(z_1, \ldots, z_m) = 32\, P(z_1, \ldots, z_m)$.
2. $P : \mathbb{C}^2 \to \mathbb{C}$, $(z_1, z_2) \mapsto z_1^2 + 2z_2^2$. $\quad dP(\nabla \frac{1}{\lambda^2})(z_1, z_2) = 32z_1^2 + 256z_2^2$.
3. $P : \mathbb{C}^3 \times \mathbb{R} \to \mathbb{C}$, $((z, u, w), t) \mapsto z^2 w + u^2 \overline{w} + 2izut$.

$$dP(\nabla \frac{1}{\lambda^2})(z, u, w, t) = 32(|z|^2 + |u|^2 + |w|^2 + t^2)P(z, u, w, t).$$

4. $P : \mathbb{C}^3 \times \mathbb{R} \to \mathbb{C}$, $((z, u, w), t) \mapsto z^2 w + \gamma^2 u^2 \overline{w} + 2\gamma izut$, here $\gamma \in \mathbb{C}$. Then $dP(\nabla \frac{1}{\lambda^2})$ is very complicated and not proportional to P unless $\gamma \in \{-1, 0, 1\}$.
5. $P : \mathbb{C}^4 \to \mathbb{C}$, $(z, u, v, w) \mapsto z^2 wv - u^2 \overline{wv} + zu(|w|^2 - |v|^2)$. Then

$$dP(\nabla \frac{1}{\lambda^2})(z, u, v, w) = 8\left[(|z|^2 + |u|^2 + |v|^2 + |w|^2)^2 + 2(|z|^2 + |u|^2)(|v|^2 + |w|^2)\right] \cdot P(z, u, v, w)$$

Polynomials to $\mathbb{C}$

We see a pattern:

- The first guess tends to have $dP(\nabla \ln \lambda)$ being proportional to P. ($\longrightarrow$ lines through zero get pulled back to minimal hypersurfaces)
- The second guess tends to have $\bigcup_{x \in P^{-1}(\{y\})} \{dP(\nabla \ln \lambda)\}$ spanning $\mathbb{C}$. ($\longrightarrow$ no line pulls back to a minimal hypersurface)

Remark

Important: The general discussion for $\mathbb{R}^m \to \mathbb{C}$ is not new! It is discussed **[BG 94]**, and recently Kislitsyn **[Ki 24]** finds another way to write the condition $dP(\nabla \ln \lambda) \propto P$.

Polynomials $\mathbb{R}^m \to \mathbb{R}^n$

Motivated by the examples to $\mathbb{C}$ we define:

Definition

A homogeneous polynomial harmonic morphisms $P : \mathbb{R}^m \to \mathbb{R}^n$ is said to be *umbilical* if there is a function $f : \mathbb{R}^m \to \mathbb{R}$ so that

$$dP(\nabla \ln \lambda)(x) = f(x)\, P(x)$$

for all $x \in \mathbb{R}^m$.

Proposition

Let $C \subset \mathbb{R}^n$ be a minimal cone and $P : \mathbb{R}^m \to \mathbb{R}^n$ an umbilical harmonic morphism. Then $P^{-1}(C)$ is a minimal cone at its regular points.

In particular: Umbilical harmonic morphisms pull back minimal submanifolds of S^{n-1} to minimal submanifolds of S^{m-1} (albeit not necessarily preserving closedness!)

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In searching for examples we have the problem that not many harmonic morphisms $\mathbb{R}^m \to \mathbb{R}^n$ with $n > 2$ are known. To my knowledge we only have:

0. Orthogonal projections and homotheties.
1. Degree 2 polynomials (classified by Ou **[Ou 97]**, and Ou & Wood **[OW 96]**).
2. Multiplications $\mathbb{K}^m \to \mathbb{K}$ where $\mathbb{K}$ is the quaternions or octonions.

And direct sums and compositions of the above.

We first review the degree 2 case.

Definition

1. A *Clifford system* of $\mathbb{R}^n$ on $\mathbb{R}^m$ is a system $A_1, \ldots, A_n$ of symmetric $m \times m$ matrices so that:

$$A_i A_j + A_j A_i = 2\delta_{ij} \mathbb{1}_{m\times m}$$

2. If $A_1, \ldots, A_n$ is a Clifford system on $\mathbb{R}^m$ then we associate to it

$$P : \mathbb{R}^m \to \mathbb{R}^n, \qquad P(x)_i = \langle x, A_i x \rangle$$

Remark

- A Clifford system is the same as an orthogonal representation of the Clifford algebra $\rho : \mathcal{C}(\mathbb{R}^n, -\langle,\rangle) \to \mathrm{End}(\mathbb{R}^m)$.
- The map P is the same as $\mathbb{R}^m \to (\mathbb{R}^n)^*$, $x \mapsto [v \mapsto \langle x, \rho(v)x \rangle]$.
- Every degree 2 harmonic morphism is a weighted direct sum of such maps.

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Quadratic polynomials

Proposition

Let $P = \bigoplus_i \lambda_i P_{\rho_i}$ *a quadratic harmonic morphism. Then* P *is umbilical if and only if the weights* $|\lambda_i|$ *are all equal. In this case* 0 *is the only critical point of* P.

Polynomials of this type are already called umbilical in the literature (hence the name).

Corollary

If $P : \mathbb{R}^m \to \mathbb{R}^n$ *is a quadratic umbilical polynomial harmonic morphism then* P *pulls back closed minimal submanifolds of* S^{n-1} *to closed minimal submanifolds of* S^{m-1}.

New polynomials

We now describe some new examples of polynomial harmonic morphisms $\mathbb{R}^m \to \mathbb{R}^n$. The simplest example is as follows:

Definition

Let $\rho_1 : \mathcal{C}(\mathbb{R}^n) \to \mathrm{End}(\mathbb{R}^{m_1})$, $\rho_2 : \mathcal{C}(\mathbb{R}^{m_1}) \to \mathrm{End}(\mathbb{R}^{m_2})$ be Clifford systems. We define:

$$P : \mathbb{R}^{m_1} \oplus \mathbb{R}^{m_2} \to (\mathbb{R}^n)^*, \qquad (x, y) \mapsto [v \mapsto \langle y, \rho_2(\rho_1(v)x)y\rangle]$$

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$$P : \mathbb{R}^{m_1} \oplus \mathbb{R}^{m_2} \to (\mathbb{R}^n)^*, \qquad (x, y) \mapsto [\nu \mapsto \langle y, \rho_2(\rho_1(\nu)x)y\rangle]$$

Definition

Let $\rho_k : \mathcal{C}(\mathbb{R}^{m_{k-1}}) \to \mathrm{End}(\mathbb{R}^{m_k})$ be Clifford systems. For $\nu \in \mathbb{R}^{m_0}$ and $(x_\ell, \dots, x_1) \in \bigoplus_{k=1}^{\ell} \mathbb{R}^{m_k}$ let $A_0(\nu) = \nu$ and

$$A_k(\nu) = \rho_k(A_{k-1}(\nu))x_k.$$

Then define

$$P : \bigoplus_{k=1}^{\ell} \mathbb{R}^{m_k} \to (\mathbb{R}^{m_0})^*, \qquad (x_\ell, \dots, x_1) \mapsto [\nu \mapsto \langle x_\ell, A_\ell(\nu)x_\ell\rangle].$$

New polynomials

Proposition

The polynomials defined on the previous slides are umbilical harmonic morphisms.

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Open questions

The following questions motivated this investigation:

- The minimal cones generated by $\mathbb{C}^m \to \mathbb{C}$, $(z_1, \dots, z_m) \mapsto \sum_i z_i^2$ are isometric to Simon's cone, which is *area-minimising* for $m \geqslant 4$.

  Can one adapt the proof to figure out when general umbilical polynomial harmonic morphisms $\mathbb{R}^m \to \mathbb{C}$ generate area-minimising cones?
- For a holomorphic map $\mathbb{C}^m \to \mathbb{C}$, all of its level sets are Kähler varieties, hence area-minisming.

  When are the level sets of a polynomial harmonic morphism $\mathbb{R}^m \to \mathbb{C}$ area-minimisers? Stable?
- For an umbilical polynomial harmonic morphism $P : \mathbb{R}^m \to \mathbb{R}^n$ the level set $P^{-1}(\{0\})$ is minimal near its regular points.

  Is it stationary / stable / area-minimising as a singular variety?

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Thank you for your attention!!

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