A new perspective on generalizing harmonic maps

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Article

- V. Branding, S. N., C. Oniciuc, *On conformal biharmonic maps and hypersurfaces*, https://arxiv.org/abs/2311.04493.

Outline

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How could we generalize the well-known harmonic maps?

Let (M^m, g) and (N^n, h) be two Riemannian manifolds.

- Energy functional

$$E : C^\infty(M,N) \to \mathbb{R}, \qquad E(\phi) = \frac{1}{2}\int_M |d\phi|^2 \, v_g.$$

- Euler-Lagrange equation (harmonic equation)

$$\begin{aligned} \tau(\phi) &= \operatorname{trace} \bar{\nabla} d\phi \\ &= 0, \end{aligned}$$

  where $\bar{\nabla}$ represents the connection on the pull-back bundle $\phi^{-1}TN$.
- Critical points of E are called harmonic maps.

Biharmonic maps

Let (M^m, g) and (N^n, h) be two Riemannian manifolds.

- Bienergy functional

$$E_2 : C^\infty(M,N) \to \mathbb{R}, \qquad E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 v_g$$

- Euler-Lagrange equation (biharmonic equation)

$$\begin{aligned} \tau_2(\phi) &= -\bar{\Delta}\tau(\phi) - \operatorname{trace} R^N(d\phi(\cdot), \tau(\phi))d\phi(\cdot) \\ &= 0. \end{aligned}$$

- Critical points of E_2 are called biharmonic maps.

The biharmonic equation (G.Y. Jiang - 1986)

$$\tau_2(\phi) = -\bar{\Delta}\tau(\phi) - \operatorname{trace} R^N(d\phi(\cdot), \tau(\phi))d\phi(\cdot) = 0,$$

where

$$\bar{\Delta}\tau(\phi) = -\operatorname{trace}\left(\bar{\nabla}\bar{\nabla} - \bar{\nabla}_{\nabla}\right)$$

is the rough Laplacian on sections of $\phi^{-1}TN^n$ and

$$R^N(X,Y)Z = \nabla^N_X\nabla^N_Y Z - \nabla^N_Y\nabla^N_X Z - \nabla^N_{[X,Y]}Z.$$

- is a fourth-order non-linear elliptic equation;
- any harmonic map is a stable biharmonic map;
- a non-harmonic biharmonic map is called proper-biharmonic;

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Does biharmonicity serve as a 'good' generalization of harmonicity?

- Biharmonic isometric immersions enjoy many intriguing properties; there are many examples in the Euclidean spheres (Balmuş, Montaldo, Oniciuc – 2008); in the Euclidean space there exists a challenging conjecture of B.-Y. Chen claiming that any biharmonic submanifold is minimal.

From the point of view of submanifolds theory, the answer is YES!

- Both harmonic and biharmonic equations are invariant under homotetic transformations of the domain;
- The harmonic equation is invariant under conformal transformations of the domain in dimension two;
- The biharmonic equation does not enjoy conformal invariance in any dimension.

From the point of view of conformal geometry, the answer is NO!

Introducing the c-biharmonicity

Theorem 3.1 (Berard – 2008)

Let (M^m, g) and (N^n, h) be two Riemannian manifolds, with m even. Then, there exists a functional $E^c_{m/2}$ acting on $C^\infty(M,N)$ which is conformally invariant with respect to the domain metric g. The corresponding Euler-Lagrange equation is a m-th-order non-liniar elliptic equation with the leading term $(\bar{\Delta})^{m/2-1}\tau(\phi)$.

Definition

The critical points of $E^c_{m/2}$ are called conformal-harmonic maps, or c-harmonic maps.

Remark

When $m = 2$, we have $E^c_1 = E$.

Introducing the c-biharmonicity

The equation $\tau^c_{m/2}(\phi) = 0$ does not have an explicit expression. However, since we want to work with this equation, we propose

- to keep the degree of the Euler-Lagrange equation to be four;
- to extend the expression of E^c_2 to any manifold M^m, not only M^4.

This functional will be conformally invariant only when $m = 4$.

Definition

- *The (new) functional E^c_2 is called conformal-bienergy functional or c-bienergy functional.*
- *The critical points of E^c_2 are called conformal-biharmonic maps, or c-biharmonic maps.*

Introducing the c-biharmonicity

Let (M^m, g) and (N^n, h) be two Riemannian manifolds.

- Conformal-bienergy (c-bienergy) functional

$$E_2^c(\phi) = E_2(\phi) + \int_M \left(\frac{1}{3} \operatorname{Scal} |d\phi|^2 - \operatorname{trace}\langle d\phi\left(\operatorname{Ric}(\cdot)\right), d\phi(\cdot)\rangle \right) v_g$$

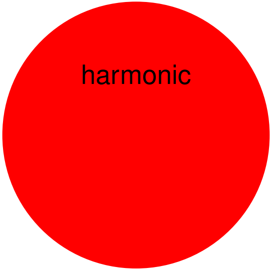
- Euler-Lagrange equation (c-biharmonic equation)

$$\begin{aligned} \tau_2^c(\phi) &= \tau_2(\phi) + 2\operatorname{trace}(\bar{\nabla} d\phi)(\operatorname{Ric}(\cdot), \cdot) - \frac{2}{3}\operatorname{Scal}\tau(\phi) + \frac{1}{3} d\phi(\nabla \operatorname{Scal}) \\ &= 0. \end{aligned}$$

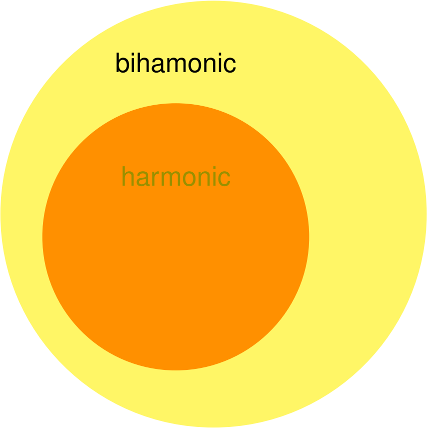
Remark

In the particular case when $\phi : M^4 \to \mathbb{R}$, then τ_2^c is the Paneitz operator.

Harmonic vs. biharmonic vs. c-biharmonic maps



Harmonic vs. biharmonic vs. c-biharmonic maps



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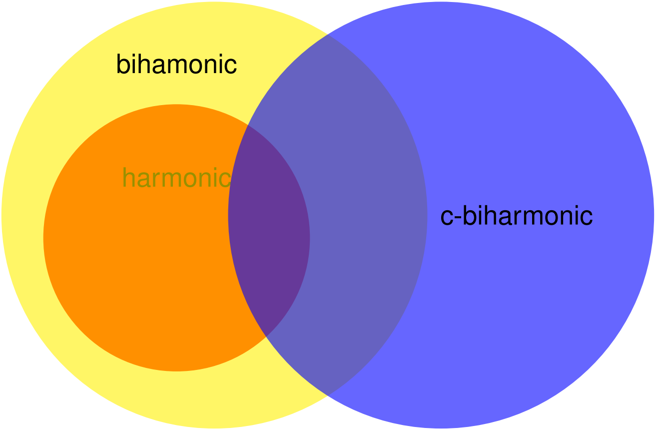

Outline

Bienergy functional vs. c-bienergy functional

Proposition 4.1

We have $E_2^c = E_2$ in each of the following cases:

i) *if the dimension of the domain manifold is $m = 1$;*
ii) *if the domain manifold is a 3-dimensional Einstein manifold;*
iii) *if the domain manifold M^m is Ricci-flat.*

Harmonic maps vs. c-biharmonic maps

Proposition 4.2

The identity map $\mathrm{Id}:(M^m,g)\to(M^m,g)$ *is c-biharmonic if and only if* Scal *is constant.*

Harmonic maps vs. c-biharmonic maps

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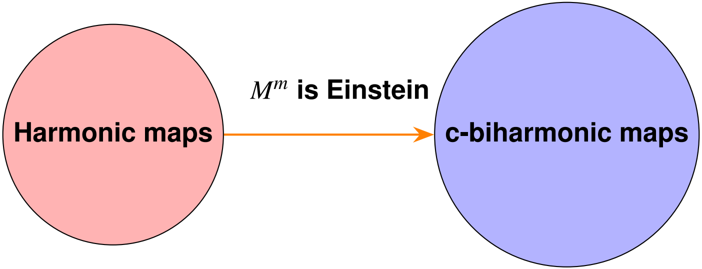

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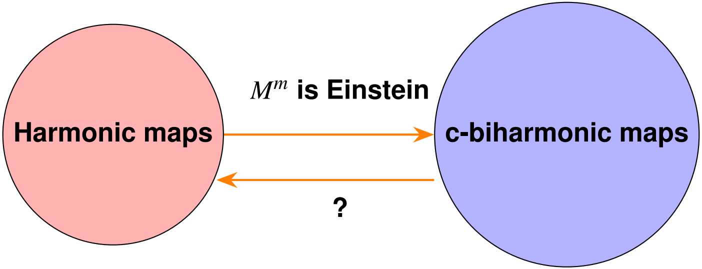

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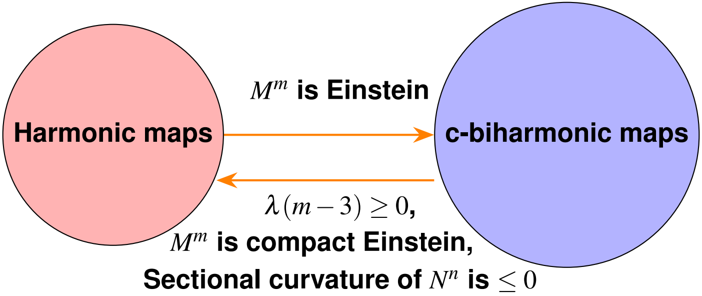

$$\mathrm{Ric} = \lambda Id$$

Harmonic maps vs. c-biharmonic maps

Proposition 4.3

Let $\phi : M^m \to N^n$ be a harmonic map, where M is an Einstein manifold with $\mathrm{Ric} = \lambda\,\mathrm{Id}$ and λ is a real constant. Then ϕ is c-biharmonic.

Proposition 4.4

Let $\phi : M^m \to N^n$ be a smooth map and assume that M is a compact Einstein manifold with $\mathrm{Ric} = \lambda\,\mathrm{Id}$ and λ is a real constant, and N has non-positive sectional curvature. If $\lambda\,(m-3) \geq 0$, then ϕ is c-biharmonic if and only if it is harmonic.

When $M = \mathbb{S}^4(r)$ we can recover a result from (Lamm - 2005), that is

Corollary 4.5

Let $\phi : \mathbb{S}^4(r) \to N^n$ be a smooth map and assume that N has non-positive sectional curvature. Then ϕ is c-biharmonic if and only if it is harmonic.

Outline

c-biharmonic hypersurfaces in space forms

Let $\phi : M^m \to N^{m+1}(c)$ be a hypersurface. Then

$$\frac{1}{m}\tau_2^c(\phi) = \left(-\Delta f + \frac{f}{3}\left(5|A|^2 - 2m^2 f^2 - c\left(2m^2 - 11m + 6\right)\right) - \frac{2}{m}\operatorname{trace} A^3\right)\eta$$
$$- mf\nabla f - 2A(\nabla f) + \frac{1}{3m}\nabla \operatorname{Scal}. \quad (1)$$

We will consider the simplest cases:

- totally geodesic hypersurfaces in space forms;
- minimal hypersurfaces in space forms;
- CMC hypersurfaces in space forms;
- hypersurfaces with $\nabla A = 0$ in space forms.

Totally geodesic and minimal hypersurfaces in space forms

Proposition 5.1

Any totally geodesic hypersurface in $N^{m+1}(c)$ is c-biharmonic.

Proposition 5.2

Let $\phi : M^m \to N^{m+1}(c)$ be a minimal hypersurface. Then M is c-biharmonic if and only if $\operatorname{trace} A^3 = 0$ and Scal is constant.

Corollary 5.3

Any minimal Einstein hypersurface in a space form is c-biharmonic.

Corollary 5.4

A minimal hypersurface with $\nabla A = 0$ in a space form is c-biharmonic if and only if $\operatorname{trace} A^3 = 0$.

Biharmonic hypersurfaces vs. c-biharmonic hypersurfaces

Using a result of (Lusala, Scherfner, Sousa – 2005), we obtain

Remark

- *There is an example of c-biharmonic hypersurfaces with* 4 *constant distinct principal curvatures:* $1+\sqrt{2}$, $1-\sqrt{2}$, $-1+\sqrt{2}$ *and* $-1-\sqrt{2}$;
- *All known examples of proper biharmonic hypersurfaces have only* 1 *or* 2 *constant distinct principal curvatures.*

CMC hypersurfaces in space forms

From (1) we obtain

Proposition 5.5

Let $\phi : M^m \to N^{m+1}(c)$ *be a CMC hypersurface. Then* M *is c-biharmonic if and only if*

$$\begin{cases} mf\left(5|A|^2 - 2m^2f^2 - c\left(2m^2 - 11m + 6\right)\right) - 6\,\mathrm{trace}\,A^3 = 0, \\ \nabla\,\mathrm{Scal} = 0. \end{cases}$$

CMC Einstein hypersurfaces in space forms

We have seen in Corollary 5.3 that a minimal Einstein hypersurface in a space form is c-biharmonic. When the hypersurface is non-minimal, the situation is more rigid.

Theorem 5.6

Let $\phi : M^m \to N^{m+1}(c)$ be a non-minimal CMC Einstein hypersurface with $\mathrm{Ric} = \lambda\,\mathrm{Id}$, where λ is a real constant. Then, M is c-biharmonic if and only if M is umbilical,

$$\lambda = \frac{6mc(m-1)}{2m^2-5m+6}, \qquad f^2 = \frac{c\left(-2m^2+11m-6\right)}{2m^2-5m+6}, \qquad c \neq 0.$$

In this case, one of the following holds

i) $c > 0$ *and* $m \leq 4$;

ii) $c < 0$ *and* $m \geq 5$.

Hypersurfaces with $\nabla A = 0$ in space forms

We have seen in Corollary 5.4 that a minimal hypersurface with $\nabla A = 0$ in a space form is c-biharmonic iff. $\operatorname{trace} A^3 = 0$. When the hypersurface with $\nabla A = 0$ is non-minimal, we obtain

Proposition 5.7

Let $\phi : M^m \to N^{m+1}(c)$ be a non-minimal hypersurface with $\nabla A = 0$. Then, M is c-biharmonic if and only

$$|A|^2\left(6|A|^2 - 5m^2f^2 - 6cm\right) + m^2f^2\left(2m^2f^2 + c\left(2m^2 - 11m + 12\right)\right) = 0. \qquad (2)$$

c-biharmonic hypersurfaces with $\nabla A = 0$

$\mathbb{R}^{m+1}$

$\mathbb{S}^{m+1}$

$\mathbb{H}^{m+1}$

c-biharmonic hypersurfaces with $\nabla A = 0$

$\mathbb{R}^{m+1}$

$\mathbb{S}^{m+1}$

$\mathbb{H}^{m+1}$

c-biharmonic hypersurfaces in $\mathbb{R}^{m+1}$

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c-biharmonic hypersurfaces with $\nabla A = 0$ in $\mathbb{R}^{m+1}$

- **The hyperplanes of $\mathbb{R}^{m+1}$ are totally geodesic and c-biharmonic;**
- **No m-dimensional hypersphere or cylinder is c-biharmonic in $\mathbb{R}^{m+1}$.**

c-biharmonic hypersurfaces with $\nabla A = 0$

$\mathbb{R}^{m+1}$

$\mathbb{S}^{m+1}$

$\mathbb{H}^{m+1}$

c-biharmonic hypersurfaces with $\nabla A = 0$

$\mathbb{R}^{m+1}$

$\mathbb{S}^{m+1}$

$\mathbb{H}^{m+1}$

c-biharmonic hypersurfaces with $\nabla A = 0$ in $\mathbb{S}^{m+1}$

c-biharmonic hypersurfaces with $\nabla A = 0$ in $\mathbb{S}^{m+1}$

c-biharmonic hypersurfaces with $\nabla A = 0$ in $\mathbb{S}^{m+1}$

c-biharmonic hypersurfaces in $\mathbb{S}^{m+1}$ with 1 distinct principal curvature

Theorem 5.8

The hypersphere $\mathbb{S}^m(r)$ is c-biharmonic in $\mathbb{S}^{m+1}$ if and only if

i) $r = 1$, *i.e., $\mathbb{S}^m(r)$ is totally geodesic in $\mathbb{S}^{m+1}$ and for any $m \geq 1$;*

ii) $m = 1$ *or* $m = 3$ *and* $r = 1/\sqrt{2}$;

iii) $m = 2$ *and* $r = 1/\sqrt{3}$;

iv) $m = 4$ *and* $r = \sqrt{3}/2$.

c-biharmonic hypersurfaces in $\mathbb{S}^{m+1}$ with 2 distinct principal curvatures

Theorem 5.9

The generalized Clifford torus $\mathbb{S}^{m_1}(r_1) \times \mathbb{S}^{m_2}(r_2)$*, where* $m_1 + m_2 = m$ *and* $r_1^2 + r_2^2 = 1$*, is c-biharmonic in* $\mathbb{S}^{m+1}$ *if and only if one of the following cases holds*

i) $m_1 = m_2 = 2$ *and either*

$$r_1^2 = r_2^2 = \frac{1}{2} \qquad \text{or} \qquad r_1^2 = \frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}\right),\ r_2^2 = \frac{1}{2}\left(1 + \frac{1}{\sqrt{3}}\right).$$

c-biharmonic hypersurfaces in $\mathbb{S}^{m+1}$ with 2 distinct principal curvatures

Theorem (continuation)

ii) *$m_1 \neq 2$ or $m_2 \neq 2$ and*

$$r_1^2 = \frac{T_*}{1+T_*},\ r_2^2 = \frac{1}{1+T_*},$$

where T_* is the unique positive solution of the polynomial equation

$$a_3T^3 + a_2T^2 + a_1T + a_0 = 0, \tag{3}$$

with the coefficients given by

$$\begin{cases} a_0 = m_1\left(2m_1^2 - 5m_1 + 6\right) \\ a_1 = m_1\left((m_1-m_2)^2 + \left(m_1 - \frac{11}{2}\right)^2 + (m_2-3)^2 - \frac{133}{4}\right) \\ a_2 = -m_2\left((m_1-m_2)^2 + \left(m_2 - \frac{11}{2}\right)^2 + (m_1-3)^2 - \frac{133}{4}\right) \\ a_3 = -m_2\left(2m_2^2 - 5m_2 + 6\right) \end{cases}.$$

c-biharmonic hypersurfaces with $\nabla A = 0$

$\mathbb{R}^{m+1}$

$\mathbb{S}^{m+1}$

$\mathbb{H}^{m+1}$

c-biharmonic hypersurfaces with $\nabla A = 0$

$\mathbb{R}^{m+1}$

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c-biharmonic hypersurfaces with $\nabla A = 0$ in $\mathbb{H}^{m+1}$

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c-biharmonic hypersurfaces with $\nabla A = 0$ in $\mathbb{H}^{m+1}$

We consider the one sheet hyperboloid model lying in the Minkowski space $\mathbb{R}^{n,1}$. More precisely, in $\mathbb{R}^{n+1}$, $n \geq 2$, we define the inner product

$$\langle X, Y \rangle := \sum_{i=1}^{n} X^i Y^i - X^{n+1} Y^{n+1},$$

where $X = \left(X^1, X^2, \ldots, X^{n+1}\right)$ and $Y = \left(Y^1, Y^2, \ldots, Y^{n+1}\right)$ are vectors in $\mathbb{R}^{n+1}$. The hyperbolic space is defined by

$$\mathbb{H}^n = \left\{ \bar{x} \in \mathbb{R}^{n+1} \mid \langle \bar{x}, \bar{x} \rangle = -1 \quad \text{and} \quad x^{n+1} > 0 \right\}.$$

Hypersurfaces with $\nabla A = 0$ in $\mathbb{H}^{m+1}$ (Ryan – 1971; Liu, Su – 2002)

I. $M^m = \left\{\bar{x} \in \mathbb{H}^{m+1} \mid x^1 = r \geq 0\right\}$

II. $M^m = \left\{\bar{x} \in \mathbb{H}^{m+1} \mid x^{m+2} = x^{m+1} + a,\ a > 0\right\}$

III. $M^m = \left\{\bar{x} \in \mathbb{H}^{m+1} \mid \sum_{i=1}^{m+1} \left(x^i\right)^2 = r^2,\ r > 0\right\}$

IV. $$M_k^m = \left\{(\bar{x}_1, \bar{x}_2) \in \mathbb{H}^{m+1} \mid |\bar{x}_1|^2 = \sum_{i=1}^{k+1} \left(x^i\right)^2 = r^2, \right.$$
$$\left. |\bar{x}_2|^2 = \sum_{i=k+2}^{m+1} \left(x^i\right)^2 - \left(x^{m+2}\right)^2 = -\left(1 + r^2\right), r > 0, 1 \leq k \leq m-1\right\}$$

Case I.

Theorem 5.10

The hypersurface $M^m = \left\{\bar{x} \in \mathbb{H}^{m+1} \mid x^1 = r \geq 0\right\}$ is c-biharmonic in $\mathbb{H}^{m+1}$ if and only if

i) $r = 0$, *i.e.,* M^m *is totally geodesic in* $\mathbb{H}^{m+1}$;

ii) $m \geq 5$ *and*

$$r^2 = \frac{2m^2 - 11m + 6}{6m}.$$

Cases II and III.

Proposition 5.11

The hypersurface $M^m = \left\{\bar{x} \in \mathbb{H}^{m+1} \mid x^{m+2} = x^{m+1} + a,\ a > 0\right\}$ *cannot be c-biharmonic in* $\mathbb{H}^{m+1}$.

Proposition 5.12

The hypersurface $M^m = \left\{\bar{x} \in \mathbb{H}^{m+1} \mid \sum_{i=1}^{m+1} \left(x^i\right)^2 = r^2,\ r > 0\right\}$ *cannot be c-biharmonic in* $\mathbb{H}^{m+1}$.

Case IV.

$$M_k^m = \left\{ (\bar{x}_1, \bar{x}_2) \in \mathbb{H}^{m+1} \mid |\bar{x}_1|^2 = \sum_{i=1}^{k+1} \left(x^i\right)^2 = r^2, \right.$$

$$\left. |\bar{x}_2|^2 = \sum_{i=k+2}^{m+1} \left(x^i\right)^2 - \left(x^{m+2}\right)^2 = -\left(1+r^2\right), r > 0, 1 \leq k \leq m-1 \right\}$$

Proposition 5.13

The hypersurface M_k^m is c-biharmonic if and only if $r^2 = T_*$ is a positive solution of the polynomial equation

$$\zeta(T) = a_3 T^3 + a_2 T^2 + a_1 T + a_0 = 0, \tag{4}$$

with the coefficients given by

$$\begin{cases} a_0 = 2k^3 - 5k^2 + 6k \\ a_1 = 2k\left(k(3m-5) - m^2 + 3m + 6\right) \\ a_2 = -2m^3 + 11m^2 - 6m + 4k\left(m^2 - m + 3\right) \\ a_3 = 6m^2 \end{cases}.$$

Case IV. – non-existence results

Proposition 5.14

The hypersurface M_k^m *cannot be c-biharmonic in* $\mathbb{H}^{m+1}$ *if*

i) $m \leq 7$ *and for any* $1 \leq k \leq m-1$;

ii) $m \in \{8, 9\}$ *and*

$$\frac{m^2 - 3m - 6}{3m - 5} \leq k;$$

iii) $m \geq 10$ *and*

$$\frac{m\left(m^2 - 11m + 6\right)}{4\left(m^2 - m + 3\right)} \leq k.$$

Case IV. -existence results

Theorem 5.15

If $k = 1$, then the hypersurface M_1^m is c-biharmonic in $\mathbb{H}^{m+1}$ if and only if $m \geq 8$. In this case, r^2 is one of the two positive solutions of the polynomial equation (4).

Proposition 5.16

If $m \geq 8$ and $1 \leq k < k_m$, then M_k^m is c-biharmonic in $\mathbb{H}^{m+1}$ if and only if r^2 is one of the two positive solutions of the polynomial equation (4), *where*

$$k_m = \frac{2m^3 - 6m^2 - 3m + m\sqrt{4m^4 - 24m^3 - 48m^2 + 84m - 63}}{4\left(3m^2 - 2m + 3\right)}.$$

Remark

For m large enough,

$$k_m \in \left(\frac{m}{4}, \frac{m}{3}\right).$$

Outline

Jacobi operator

Let $\phi : \mathbb{S}^m(r) \to \mathbb{S}^n$ be a smooth map.

$$J_2^c(V) = J_2(V) + \frac{2}{3}\frac{(m-1)(m-3)}{r^2}J(V).$$

Remark

For $m = 1$ or $m = 3$, we note that $J_2^c = J_2$.

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$$J_2^c(V) = J_2(V) + \frac{2}{3}\frac{(m-1)(m-3)}{r^2}J(V).$$

Remark

For $m = 1$ or $m = 3$, we note that $J_2^c = J_2$.

- $r = 1$ and $n = m+1$;
- $r = 1$ and $n = m$;
- $r \in (0,1)$ and $n = m+1$;

The stability of the equator

Theorem 6.1 (Smith – 1975)

We consider $\imath : \mathbb{S}^m \to \mathbb{S}^{m+1}$ the canonical inclusion of the totally geodesic hypersphere $\mathbb{S}^m$. Then, with respect to the energy functional E, we have

i) *if $m = 2$, then the index and the nullity of $\mathbb{S}^m$ are $m-1$ and $3(m+1)$, respectively;*

ii) *if $m \geq 3$, then the index and the nullity of $\mathbb{S}^m$ are $m+2$ and $(m+1)(m+2)/2$, respectively.*

Theorem 6.2

We consider $\imath : \mathbb{S}^m \to \mathbb{S}^{m+1}$ the canonical inclusion of the totally geodesic hypersphere $\mathbb{S}^m$. Then, with respect to the c-bienergy functional E_2^c, we have

i) *if $m = 2$ or $m = 4$, then the index and the nullity of $\mathbb{S}^m$ are 0 and $(m+1)(m+4)/2$, respectively;*

ii) *if $m = 3$, then the index and the nullity of $\mathbb{S}^m$ are 0 and $(m+1)(m+2)/2$, respectively;*

iii) *if $m \geq 5$, then the index and the nullity of $\mathbb{S}^m$ are $m+2$ and $(m+1)(m+2)/2$, respectively.*

The stability of the identity map

Theorem 6.3 (Smith – 1975)

We consider $\mathrm{Id} : \mathbb{S}^m \to \mathbb{S}^m$ *the identity map of* $\mathbb{S}^m$*. Then, with respect to the energy functional* E*, we have*

i) *if* $m = 2$*, then the index and the nullity of* $\mathbb{S}^m$ *are* 0 *and* $(m+1)(m+2)/2$*, respectively;*

ii) *if* $m \geq 3$*, then the index and the nullity of* $\mathbb{S}^m$ *are* $m+1$ *and* $m(m+1)/2$*, respectively.*

Theorem 6.4

We consider $\mathrm{Id} : \mathbb{S}^m \to \mathbb{S}^m$ *the identity map of* $\mathbb{S}^m$*. Then, with respect to the c-bienergy functional* E_2^c*, we have*

i) *if* $m = 2$ *or* $m = 4$*, then the index and the nullity of* $\mathbb{S}^m$ *are* 0 *and* $(m+1)(m+2)/2$*, respectively;*

ii) *if* $m = 3$*, then the index and the nullity of* $\mathbb{S}^m$ *are* 0 *and* $m(m+1)/2$*, respectively;*

iii) *if* $m \geq 5$*, then the index and the nullity of* $\mathbb{S}^m$ *are* $m+1$ *and* $m(m+1)/2$*, respectively.*

The stability of the small hyperspheres of $\mathbb{S}^{m+1}$

Theorem 6.5 (Loubeau, Oniciuc – 2005)

We consider $\imath : \mathbb{S}^m(1/\sqrt{2}) \to \mathbb{S}^{m+1}$ the canonical inclusion of the small hypersphere. Then, with respect to the bienergy functional E_2, we have the index and the nullity of $\mathbb{S}^m(1/\sqrt{2})$ are 1 and $(m+1)(m+2)/2$, respectively.

Theorem 6.6

We consider $\imath : \mathbb{S}^m(r) \to \mathbb{S}^{m+1}$ the canonical inclusion of the small hypersphere of radius $r \in (0,1)$. Then, with respect to the c-bienergy functional E_2^c, we have

i) *if $m = 2$ and $r = 1/\sqrt{3}$, then the index and the nullity of $\mathbb{S}^m(r)$ are 1 and 6, respectively;*

ii) *$m = 3$ and $r = 1/\sqrt{2}$, then the index and the nullity of $\mathbb{S}^m(r)$ are 1 and $(m+1)(m+2)/2$, respectively;*

iii) *if $m = 4$ and $r = \sqrt{3}/2$, then the index and the nullity of $\mathbb{S}^m(r)$ are 1 and 20, respectively.*

Stability of c-biharmonic maps

Theorem 6.7 (Xin – 1980)

Let $\phi : \mathbb{S}^m \to N^n$ be a non-constant harmonic map. If $m \geq 3$, then, with respect to the energy functional E, ϕ is unstable.

Theorem 6.8

Let $\phi : \mathbb{S}^m \to N^n$ be a non-constant harmonic map. If $m \geq 5$, then, with respect to the c-bienergy functional E_2^c, ϕ is unstable.

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Thank you for your attention!