(λ, λ)-Eigenfunctions on Compact Manifolds

Thomas Jack Munn

Department of Mathematics
Faculty of Science
Lund University

These lecture notes are available at:

`www.matematik.lu.se/matematiklu/personal/munn/slides/slides.html`

Brest - 4/9/2024

Outline

Outline

Outline

1. Eigenfunctions and Eigenfamilies
   - The Operators τ and κ
   - Definitions and Examples
   - Applications and Properties
2. (λ, λ)-Eigenfunctions
3. Generalised Eigenfamilies

Let (M^m, g) be a Riemannian manifold, $T^{\mathbb{C}}M$ be the complexification of its tangent bundle TM and extend g to a complex-bilinear form on $T^{\mathbb{C}}M$.

Let (M^m, g) be a Riemannian manifold, $T^{\mathbb{C}}M$ be the complexification of its tangent bundle TM and extend g to a complex-bilinear form on $T^{\mathbb{C}}M$.

Then the **gradient** of a **complex-valued** function $\phi = u + i\,v : (M, g) \to \mathbb{C}$ is the section of $T^{\mathbb{C}}M$ satisfying $\nabla\phi = \nabla u + i\,\nabla v$.

Let (M^m, g) be a Riemannian manifold, $T^{\mathbb{C}}M$ be the complexification of its tangent bundle TM and extend g to a complex-bilinear form on $T^{\mathbb{C}}M$.

Then the **gradient** of a **complex-valued** function $\phi = u + i\,v : (M, g) \to \mathbb{C}$ is the section of $T^{\mathbb{C}}M$ satisfying $\nabla\phi = \nabla u + i\,\nabla v$.

The complex-linear **Laplace-Beltrami operator** τ on (M, g) acts locally on ϕ as

$$\tau(\phi) = \operatorname{div}(\nabla\phi) = \sum_{X \in \mathrm{ONF}} X^2(\phi) - (\nabla_X X)(\phi).$$

Let (M^m, g) be a Riemannian manifold, $T^{\mathbb{C}}M$ be the complexification of its tangent bundle TM and extend g to a complex-bilinear form on $T^{\mathbb{C}}M$.

Then the **gradient** of a **complex-valued** function $\phi = u + i\,v : (M, g) \to \mathbb{C}$ is the section of $T^{\mathbb{C}}M$ satisfying $\nabla\phi = \nabla u + i\,\nabla v$.

The complex-linear **Laplace-Beltrami operator** τ on (M, g) acts locally on ϕ as

$$\tau(\phi) = \mathrm{div}(\nabla\phi) = \sum_{X \in \mathrm{ONF}} X^2(\phi) - (\nabla_X X)(\phi).$$

For two functions $\phi, \psi : (M, g) \to \mathbb{C}$ we have

$$\tau(\phi \cdot \psi) = \tau(\phi) \cdot \psi + 2 \cdot \kappa(\phi, \psi) + \phi \cdot \tau(\psi),$$

where the **conformality operator** κ satisfies

$$\kappa(\phi, \psi) = g(\nabla\phi, \nabla\psi).$$

Definition 1.1 (Gudmundsson & Sakovich (2008) [2])

Let (M, g) be a Riemannian manifold, $\lambda, \mu \in \mathbb{C}$. Then a complex-valued function $\phi : M \to \mathbb{C}$ is said to be a **(λ, μ)-eigenfunction** if it is eigen with respect to both the Laplace-Beltrami operator τ and the conformality operator κ with respective eigenvalues λ, μ, i.e.

$$\tau(\phi) = \lambda \cdot \phi \quad \text{and} \quad \kappa(\phi, \phi) = \mu \cdot \phi^2.$$

Definition 1.1 (Gudmundsson & Sakovich (2008) [2])

Let (M, g) be a Riemannian manifold, $\lambda, \mu \in \mathbb{C}$. Then a complex-valued function $\phi : M \to \mathbb{C}$ is said to be a (λ, μ)**-eigenfunction** if it is eigen with respect to both the Laplace-Beltrami operator τ and the conformality operator κ with respective eigenvalues λ, μ, i.e.

$$\tau(\phi) = \lambda \cdot \phi \quad \text{and} \quad \kappa(\phi, \phi) = \mu \cdot \phi^2.$$

A set $\mathcal{E}$ of complex-valued functions on M is said to be a (λ, μ)**-eigenfamily** on M if for all $\phi, \psi \in \mathcal{E}$ we have

$$\tau(\phi) = \lambda \cdot \phi \quad \text{and} \quad \kappa(\phi, \psi) = \mu \cdot \phi \psi.$$

Theorem 1.2 (Fuglede (1978) + Ishihara (1979), ($\lambda = 0$ and $\mu = 0$))

A complex-valued function $\phi : (M, g) \to \mathbb{C}$ on a Riemannian manifold is a **harmonic morphism** *if and only if it is an* **eigenfunction** *with $\lambda = 0$ and $\mu = 0$.*

Example 1.3 (Coordiate projections)

Consider the functions $\phi_i : \mathbb{C}^n \simeq \mathbb{R}^{2n} \to \mathbb{C}$ defined by the projections

$$\phi_i : z \mapsto z_i.$$

Then

$$\tau(\phi_i) = 0, \quad \kappa(\phi_i, \phi_j) = 0$$

so $\mathcal{F} = \{\phi_i \mid i = 1, \dots n\}$ is a $(0,0)$-eigenfamily.

Example 1.3 (Coordiate projections)

Consider the functions $\phi_i : \mathbb{C}^n \simeq \mathbb{R}^{2n} \to \mathbb{C}$ defined by the projections

$$\phi_i : z \mapsto z_i.$$

Then

$$\tau(\phi_i) = 0, \quad \kappa(\phi_i, \phi_j) = 0$$

so $\mathcal{F} = \{\phi_i \mid i = 1, \dots n\}$ is a $(0, 0)$-eigenfamily.

Example 1.4 (Restriction to odd-dimensional spheres)

Consider $S^{2n-1} \subset \mathbb{R}^{2n} \simeq \mathbb{C}^n$. Then we can define $\tilde{\phi}_i = \phi_i|_{S^{2n-1}} : S^{2n-1} \to \mathbb{C}$. We have that

$$\tau(\tilde{\phi}_i) = (-2n + 1) \cdot \phi_i, \quad \kappa(\tilde{\phi}_i, \tilde{\phi}_j) = -1 \cdot \phi_i \phi_j$$

so $\tilde{\mathcal{F}} = \{\tilde{\phi}_i \mid i = 1, \dots n\}$ is a $(-2n + 1, -1)$-eigenfamily

Eigenfamilies can be used to produce complex-valued harmonic morphisms:

Theorem 1.5 (Gudmundsson & Sakovich (2008) [2])

Let (M, g) be a semi-Riemannian manifold and

$$\mathcal{F} = \{\phi_1, \dots \phi_n\}$$

be a finite eigenfamily of complex valued functions on M. If $P, Q : \mathbb{C}^n \to \mathbb{C}$ are linearly independent homogeneous polynomials of the same positive degree.

Eigenfamilies can be used to produce complex-valued harmonic morphisms:

Theorem 1.5 (Gudmundsson & Sakovich (2008) [2])

Let (M, g) be a semi-Riemannian manifold and

$$\mathcal{F} = \{\phi_1, \ldots \phi_n\}$$

be a finite eigenfamily of complex valued functions on M. If $P, Q : \mathbb{C}^n \to \mathbb{C}$ are linearly independent homogeneous polynomials of the same positive degree. Then the quotient

$$\frac{P(\phi_1, \ldots, \phi_n)}{Q(\phi_1, \ldots, \phi_n)}$$

is a non-constant harmonic morphism on the open and dense subset

$$\{p \in M | Q(\phi_1(p), \ldots, \phi_n(p)) \neq 0\}.$$

Eigenfunctions can be used to produce minimal submanifolds:

Theorem 1.6 (Baird & Eells (1981), ($\lambda = 0$ and $\mu = 0$))

Let $\phi : (M, g) \to \mathbb{C}$ *be a* **horizontally conformal** $[\kappa(\phi, \phi) = 0]$ *function from a Riemannian manifold. Then* ϕ *is* **harmonic** $[\tau(\phi) = 0]$ *if and only if its fibres are* **minimal** *at regular points of* ϕ $[\nabla\phi \neq 0]$.

Eigenfunctions can be used to produce minimal submanifolds:

Theorem 1.6 (Baird & Eells (1981), ($\lambda = 0$ and $\mu = 0$))

Let $\phi : (M, g) \to \mathbb{C}$ be a **horizontally conformal** $[\kappa(\phi, \phi) = 0]$ *function from a Riemannian manifold. Then ϕ is* **harmonic** $[\tau(\phi) = 0]$ *if and only if its fibres are* **minimal** *at regular points of ϕ* $[\nabla\phi \neq 0]$.

Theorem 1.7 (Gudmundsson & TM (2024))

Let $\phi : (M, g) \to \mathbb{C}$ be a complex-valued **eigenfunction** *on a Riemannian manifold, such that $0 \in \phi(M)$ is a regular value for ϕ. Then the* **fibre over zero** $\mathcal{F}_0 = \phi^{-1}(\{0\})$ *is a* **minimal submanifold** *of M.*

Eigenfunctions can be used to produce p-harmonic functions:

Theorem 1.8 (Gudmundsson & Sobak (2020))

Let $\phi : (M, g) \to \mathbb{C}$ be a (λ, μ)-eigenfunction from a Riemannian manifold. Then for any positive integer p the non-vanishing function

$$\Phi_p : W = \{x \in M \mid \phi(x) \notin (-\infty, 0]\} \to \mathbb{C}$$

with

$$\Phi_p(x) = \begin{cases} c_1 \cdot \log(\phi(x))^{p-1} & \text{if } \mu = 0, \lambda \neq 0 \\ c_1 \cdot \log(\phi(x))^{2p-1} + c_1 \cdot \log(\phi(x))^{2p-2} & \text{if } \mu \neq 0, \lambda = 0 \\ c_1 \cdot \phi(x)^{1-\frac{\lambda}{\mu}} \log(\phi(x))^{p-1} + c_2 \cdot \log(\phi(x))^{p-1} & \text{if } \mu \neq 0, \lambda \neq 0 \end{cases}$$

is proper p-harmonic on the domain W.

When (locally) describing an eigenfunction in polar form, some additional relations are obtained.

Lemma 1.9 (Riedler & Siffert (2024) [7])

Let (U, g) be a Riemannian manifold, not necessarily compact or complete, and let $\phi : U \to \mathbb{C}$ be a (λ, μ)-eigenfunction with λ, μ both real and $\phi(x) \neq 0$ for all $x \in U$. Suppose $\phi(x) = e^{ih(x)}|\phi(x)|$ for some smooth function $h : U \to \mathbb{R}$. Then:

1. $\tau(h) = 0$;
2. $\tau(\ln|\phi|) = \lambda - \mu$;
3. $\kappa(h, |\phi|) = 0$;
4. $\kappa(\ln|\phi|, \ln|\phi|) = \kappa(h, h) + \mu$.

In the context of (λ, λ)-eigenfunctions we have the additional result.

Proposition 2.1 (Riedler & Siffert (2024) [7])

Let (M, g) be a compact connected Riemannian manifold and $\phi : M \to \mathbb{C}$ a non-constant (λ, μ)-eigenfunction.

In the context of (λ, λ)-eigenfunctions we have the additional result.

Proposition 2.1 (Riedler & Siffert (2024) [7])

Let (M, g) be a compact connected Riemannian manifold and $\phi : M \to \mathbb{C}$ a non-constant (λ, μ)-eigenfunction. The following are equivalent:

1. $\lambda = \mu$.

In the context of (λ, λ)-eigenfunctions we have the additional result.

Proposition 2.1 (Riedler & Siffert (2024) [7])

Let (M, g) be a compact connected Riemannian manifold and $\phi : M \to \mathbb{C}$ a non-constant (λ, μ)-eigenfunction. The following are equivalent:

1. $\lambda = \mu$.
2. $|\phi|^2$ *is constant.*

In the context of (λ, λ)-eigenfunctions we have the additional result.

Proposition 2.1 (Riedler & Siffert (2024) [7])

Let (M, g) be a compact connected Riemannian manifold and $\phi : M \to \mathbb{C}$ a non-constant (λ, μ)-eigenfunction. The following are equivalent:

1. $\lambda = \mu$.
2. $|\phi|^2$ *is constant.*
3. $\phi(x) \neq 0$ *for all* $x \in M$.

In the context of (λ, λ)-eigenfunctions we have the additional result.

Proposition 2.1 (Riedler & Siffert (2024) [7])

Let (M, g) be a compact connected Riemannian manifold and $\phi : M \to \mathbb{C}$ a non-constant (λ, μ)-eigenfunction. The following are equivalent:

1. $\lambda = \mu$.
2. $|\phi|^2$ *is constant.*
3. $\phi(x) \neq 0$ *for all* $x \in M$.

In particular, item (4) on the previous slide simplifies to

$$\kappa(h, h) = -\lambda.$$

Main Results

Theorem 2.2 (TM & Riedler)

Let (M, g) be a compact and connected Riemannian manifold, $\phi : M \to \mathbb{C}$ a non-constant smooth map, $\lambda < 0$.

Main Results

Theorem 2.2 (TM & Riedler)

Let (M, g) be a compact and connected Riemannian manifold, $\phi : M \to \mathbb{C}$ a non-constant smooth map, $\lambda < 0$. The following are equivalent:

1. *ϕ is a (λ, λ)-eigenfunction.*
2. *For all $x_0 \in M$ the map $\pi : (M, g) \to (S^1, \frac{1}{|\lambda|} dt^2)$, $x \mapsto \frac{\phi(x)}{|\phi(x_0)|}$ is a well-defined and harmonic Riemannian submersion.*

Proof

$(1 \Rightarrow 2)$. From Proposition 2.1 the map $\pi : M \to S^1, x \mapsto \frac{\phi(x)}{|\phi(x_0)|}$ is well defined. For any point $x_1 \in M$ there is a neighbourhood U of x_1 and a function $h : U \to \mathbb{R}$ such that

$$\phi(x) = |\phi(x_0)| e^{ih(x)}.$$

Proof

$(1 \Rightarrow 2)$. From Proposition 2.1 the map $\pi : M \to S^1, x \mapsto \frac{\phi(x)}{|\phi(x_0)|}$ is well defined. For any point $x_1 \in M$ there is a neighbourhood U of x_1 and a function $h : U \to \mathbb{R}$ such that

$$\phi(x) = |\phi(x_0)| e^{ih(x)}.$$

Then Lemma 1.9 tells us that

$$\kappa(h,h) = \|dh\|^2 = -\lambda = |\lambda| \text{ and } \tau(h) = 0.$$

Proof

$(1 \Rightarrow 2)$. From Proposition 2.1 the map $\pi : M \to S^1, x \mapsto \frac{\phi(x)}{|\phi(x_0)|}$ is well defined. For any point $x_1 \in M$ there is a neighbourhood U of x_1 and a function $h : U \to \mathbb{R}$ such that

$$\phi(x) = |\phi(x_0)| e^{ih(x)}.$$

Then Lemma 1.9 tells us that

$$\kappa(h,h) = \|dh\|^2 = -\lambda = |\lambda| \text{ and } \tau(h) = 0.$$

Since the map

$$(\mathbb{R}, \cdot) \to (S^1, dt^2), \quad t \mapsto e^{it}$$

is a local isometry, the first equation implies that $\pi : (M, g) \to (S^1, \frac{1}{|\lambda|} dt^2)$ is a Riemannian submersion and the second implies that it is harmonic.

Proof (continued)

$(2 \Rightarrow 1)$. Let $\iota : (S^1, \frac{1}{|\lambda|} dt^2) \to (\mathbb{C}, \langle \cdot, \cdot \rangle)$ denote the standard inclusion of the unit circle. We then have that

$$\phi(x) = (\iota \circ \pi)(x)$$

for all x.

Proof (continued)

$(2 \Rightarrow 1)$. Let $\iota : (S^1, \frac{1}{|\lambda|} dt^2) \to (\mathbb{C}, \langle \cdot, \cdot \rangle)$ denote the standard inclusion of the unit circle. We then have that

$$\phi(x) = (\iota \circ \pi)(x)$$

for all x. Since π is a harmonic Riemannian submersion, a calculation shows that

$$\begin{aligned} \kappa(\iota \circ \pi, \iota \circ \pi) &= \kappa(\iota, \iota) \circ \pi, \\ \tau(\iota \circ \pi) &= \tau(\iota) \circ \pi. \end{aligned}$$

Since $\kappa(\iota, \iota) = \lambda \cdot \iota^2$ and $\tau(\iota) = \lambda \cdot \iota$, we immediately verify that ϕ is a (λ, λ)-eigenfunction.

Theorem 2.3 (TM & Riedler)

Let (M, g) be compact and connected, $\lambda < 0$, and $\pi : (M, g) \to (S^1, h = \frac{1}{|\lambda|} dt^2)$ a smooth map. The following are equivalent:

1. *The map π is a harmonic Riemannian submersion.*

Theorem 2.3 (TM & Riedler)

Let (M, g) be compact and connected, $\lambda < 0$, and $\pi : (M, g) \to (S^1, h = \frac{1}{|\lambda|} dt^2)$ a smooth map. The following are equivalent:

1. *The map π is a harmonic Riemannian submersion.*
2. *M is a mapping torus*

$$M_0 \times_\eta [0, 2\pi] = \frac{M_0 \times [0, 2\pi]}{(x, 0) \sim (\eta(x), 2\pi)}$$

with metric

$$g = g(t) + \frac{1}{|\lambda|} dt^2$$

and monodromy map $\eta : M_0 \to M_0$ with $\eta^* g(2\pi) = g(0)$, M_0 is compact, the volume density of $g(t)$ is constant in t, and $\pi([x, t]) = e^{it}$.

Outline of Proof

$(1 \Rightarrow 2)$

- M is compact $\Rightarrow \pi : M \to S^1$ is a fibre bundle.

Outline of Proof

$(1 \Rightarrow 2)$

- M is compact $\Rightarrow \pi : M \to S^1$ is a fibre bundle.
- Let $X = \nabla \hat{\pi}$, where $\hat{\pi}$ is the local lift of π to $\mathbb{R}$.

Outline of Proof

$(1 \Rightarrow 2)$

- M is compact $\Rightarrow \pi : M \to S^1$ is a fibre bundle.
- Let $X = \nabla\hat{\pi}$, where $\hat{\pi}$ is the local lift of π to $\mathbb{R}$. Since $\hat{\pi}$ is a Riemannian submersion:

$$d\hat{\pi}(X) = g(X, X) = \frac{d\hat{\pi}(X)^2}{|\lambda|},$$

so $d\hat{\pi}(X) = \|X\|^2 = -\lambda$.

Outline of Proof

$(1 \Rightarrow 2)$

- M is compact $\Rightarrow \pi : M \to S^1$ is a fibre bundle.
- Let $X = \nabla\hat{\pi}$, where $\hat{\pi}$ is the local lift of π to $\mathbb{R}$. Since $\hat{\pi}$ is a Riemannian submersion:

$$d\hat{\pi}(X) = g(X, X) = \frac{d\hat{\pi}(X)^2}{|\lambda|},$$

  so $d\hat{\pi}(X) = \|X\|^2 = -\lambda$.
- Now, let η_t denote the flow of X it follows that

$$\partial_t(\hat{\pi}\eta_t(x)) = d\hat{\pi}(X) = -\lambda,$$

  and so $\hat{\pi}(\eta_t(x)) = -\lambda \cdot t + \hat{\pi}(x)$, i.e. $\pi(\eta_t(x)) = e^{it}\pi(x)$.

Outline of Proof

$(1 \Rightarrow 2)$

- M is compact $\Rightarrow \pi : M \to S^1$ is a fibre bundle.
- Let $X = \nabla\hat{\pi}$, where $\hat{\pi}$ is the local lift of π to $\mathbb{R}$. Since $\hat{\pi}$ is a Riemannian submersion:
$$d\hat{\pi}(X) = g(X,X) = \frac{d\hat{\pi}(X)^2}{|\lambda|},$$
so $d\hat{\pi}(X) = \|X\|^2 = -\lambda$.
- Now, let η_t denote the flow of X it follows that
$$\partial_t(\hat{\pi}\eta_t(x)) = d\hat{\pi}(X) = -\lambda,$$
and so $\hat{\pi}(\eta_t(x)) = -\lambda \cdot t + \hat{\pi}(x)$, i.e. $\pi(\eta_t(x)) = e^{it}\pi(x)$.
- It follows that M is a mapping torus
$$M \cong \frac{M_0 \times [0,2\pi]}{(x,0) \sim (\eta(x),2\pi)}$$
with monodromy map $\eta = \eta_{2\pi}$, and $\pi([x,t]) = e^{it}$.

Outline of Proof (continued)

- It follows that M is a mapping torus

$$M \cong \frac{M_0 \times [0, 2\pi]}{(x, 0) \sim (\eta(x), 2\pi)}$$

with monodromy map $\eta = \eta_{2\pi}$, and $\pi([x, t]) = e^{it}$.

- Under the above diffeomorphism $X \equiv \partial_t$. X is horizontal, so the metric on M has the form

$$g = g(t) + \tfrac{1}{|\lambda|} dt^2$$

where $\eta^* g(2\pi) = g(0)$ and $g(t)$ is a family of metrics on the fibre M_0.

Outline of Proof (continued)

- It follows that M is a mapping torus

$$M \cong \frac{M_0 \times [0, 2\pi]}{(x,0) \sim (\eta(x), 2\pi)}$$

  with monodromy map $\eta = \eta_{2\pi}$, and $\pi([x,t]) = e^{it}$.

- Under the above diffeomorphism $X \equiv \partial_t$. X is horizontal, so the metric on M has the form

$$g = g(t) + \tfrac{1}{|\lambda|} dt^2$$

  where $\eta^* g(2\pi) = g(0)$ and $g(t)$ is a family of metrics on the fibre M_0.

- Since X is divergence free, its flow preserves the volume form, so the volume density induced by $g(t)$ is constant with respect to t.

Outline of Proof (continued)

$(2 \Rightarrow 1)$. Note that the map

$$\widehat{\pi} : (\mathbb{R} \times M_0, dt^2 + g(t)) \to (\mathbb{R}, \cdot), \qquad (t, x) \mapsto t$$

is clearly a Riemannian submersion. It follows from the fact that the volume density is constant over time that $\widehat{\pi}$ is harmonic.

Outline of Proof (continued)

$(2 \Rightarrow 1)$. Note that the map

$$\widehat{\pi} : (\mathbb{R} \times M_0, dt^2 + g(t)) \to (\mathbb{R}, \cdot), \qquad (t, x) \mapsto t$$

is clearly a Riemannian submersion. It follows from the fact that the volume density is constant over time that $\widehat{\pi}$ is harmonic.
Now the diagram

$$\begin{array}{ccc} (M_0 \times \mathbb{R}, dt^2 + g(t)) & \xrightarrow{\widehat{\pi}} & (\mathbb{R}, \cdot) \\ \big\downarrow /\mathbb{Z} & & \big\downarrow /\mathbb{Z} \\ (M_0 \times_\eta I, dt^2 + g(t)) & \xrightarrow{\pi} & (S^1, dt^2) \end{array}$$

commutes by by construction, here the vertical arrows are the natural covering maps. Since the vertical maps are also local isometries it follows that π is a harmonic Riemannian submersion □

Proposition 2.4

Let (M, g) be a compact and connected Riemannian manifold, $\lambda \in \mathbb{C}$ and $\mathcal{F}$ a (λ, λ)-eigenfamily on M. Then $\dim(span_{\mathbb{C}}(\mathcal{F})) \in \{0, 1\}$.

Proposition 2.4

Let (M, g) be a compact and connected Riemannian manifold, $\lambda \in \mathbb{C}$ and $\mathcal{F}$ a (λ, λ)-eigenfamily on M. Then $\dim(span_{\mathbb{C}}(\mathcal{F})) \in \{0, 1\}$.

Proof.

Suppose that ϕ, ψ are non-constant and in the same (λ, λ)-eigenfamily $\mathcal{F}$. Then it follows from Theorem 1.5 that the quotient

$$\frac{\phi}{\psi} : M \setminus \psi^{-1}(\{0\}) \to \mathbb{C}$$

is a harmonic morphism, in particular a harmonic map.

Proposition 2.4

Let (M, g) be a compact and connected Riemannian manifold, $\lambda \in \mathbb{C}$ and $\mathcal{F}$ a (λ, λ)-eigenfamily on M. Then $\dim(\mathit{span}_{\mathbb{C}}(\mathcal{F})) \in \{0, 1\}$.

Proof.

Suppose that ϕ, ψ are non-constant and in the same (λ, λ)-eigenfamily $\mathcal{F}$. Then it follows from Theorem 1.5 that the quotient

$$\frac{\phi}{\psi} : M \setminus \psi^{-1}(\{0\}) \to \mathbb{C}$$

is a harmonic morphism, in particular a harmonic map. Since $\psi(x) \neq 0$ for all $x \in M$, the domain of $\frac{\phi}{\psi}$ is all of M. By compactness of M $\frac{\phi}{\psi}$ must then be constant, i.e. ψ and ϕ are linearly dependent. □

Definition 3.1

Let $\mathcal{F} = \{\phi_1, ..., \phi_k\}$ be a finite family of functions $M \to \mathbb{C}$ and $\lambda_i \in \mathbb{C}$, $1 \leq i \leq k$ a vector in $\mathbb{C}^k$ and A_{ij}, $1 \leq i, j \leq k$ a symmetric complex $k \times k$ matrix.

1. We call $\mathcal{F}$ is an (λ_i, A_{ij})-**eigenfamily** if for all $\phi_i, \phi_j \in \mathcal{F}$:

$$\kappa(\phi_i, \phi_j) = A_{ij}\phi_i\phi_j, \qquad \tau(\phi_i) = \lambda_i\phi_i.$$

Definition 3.1

Let $\mathcal{F} = \{\phi_1, \ldots, \phi_k\}$ be a finite family of functions $M \to \mathbb{C}$ and $\lambda_i \in \mathbb{C}$, $1 \le i \le k$ a vector in $\mathbb{C}^k$ and A_{ij}, $1 \le i, j \le k$ a symmetric complex $k \times k$ matrix.

1. We call $\mathcal{F}$ is an (λ_i, A_{ij})**-eigenfamily** if for all $\phi_i, \phi_j \in \mathcal{F}$:

$$\kappa(\phi_i, \phi_j) = A_{ij}\phi_i\phi_j, \qquad \tau(\phi_i) = \lambda_i\phi_i.$$

2. We say the family is λ**-diagonal** if additionally $\lambda_i = A_{ii}$ for all i.

Definition 3.1

Let $\mathcal{F} = \{\phi_1, ..., \phi_k\}$ be a finite family of functions $M \to \mathbb{C}$ and $\lambda_i \in \mathbb{C}$, $1 \leq i \leq k$ a vector in $\mathbb{C}^k$ and A_{ij}, $1 \leq i, j \leq k$ a symmetric complex $k \times k$ matrix.

1. We call $\mathcal{F}$ is an (λ_i, A_{ij})**-eigenfamily** if for all $\phi_i, \phi_j \in \mathcal{F}$:

$$\kappa(\phi_i, \phi_j) = A_{ij}\phi_i\phi_j, \qquad \tau(\phi_i) = \lambda_i\phi_i.$$

2. We say the family is λ**-diagonal** if additionally $\lambda_i = A_{ii}$ for all i.
3. $\mathcal{F}$ is said to be **reduced** if A_{ij} is a non-degenerate matrix.

Definition 3.2

For A a positive definite $k \times k$ matrix, let (T^k, A^{-1}) denote the flat torus $T^k = (S^1)^k$ equipped with metric A^{-1}.

Definition 3.2

For A a positive definite $k \times k$ matrix, let (T^k, A^{-1}) denote the flat torus $T^k = (S^1)^k$ equipped with metric A^{-1}.

Theorem 3.3 (TM & Riedler)

Let (M, g) be a compact Riemannian manifold, A a symmetric real $k \times k$ matrix, and $\mathcal{F} = \{\phi_1, ..., \phi_k\}$ a family of functions $M \to \mathbb{C}$. The following are equivalent:

1. *$\mathcal{F}$ is a reduced $(-A_{ii}, -A_{ij})$-eigenfamily.*

Definition 3.2

For A a positive definite $k \times k$ matrix, let (T^k, A^{-1}) denote the flat torus $T^k = (S^1)^k$ equipped with metric A^{-1}.

Theorem 3.3 (TM & Riedler)

Let (M, g) be a compact Riemannian manifold, A a symmetric real $k \times k$ matrix, and $\mathcal{F} = \{\phi_1, \dots, \phi_k\}$ a family of functions $M \to \mathbb{C}$. The following are equivalent:

1. *$\mathcal{F}$ is a reduced $(-A_{ii}, -A_{ij})$-eigenfamily.*
2. *A is positive definite and for all $x_0 \in M$ the map $\pi : (M, g) \to (T^k, A^{-1})$, $x \mapsto \left(\frac{\phi_1(x)}{|\phi_1(x_0)|}, \dots, \frac{\phi_k(x)}{|\phi_k(x_0)|}\right)$ is a well-defined and harmonic Riemannian submersion.*

Thank you for watching!

[1] S. Gudmundsson, T. J. Munn, *Minimal submanifolds via complex-valued eigenfunctions*, J. Geom. Anal. **34** (2024), 190.

[2] S. Gudmundsson, A. Sakovich, *Harmonic morphisms from the classical compact semisimple Lie groups*, Ann. Global Anal. Geom. **33** (2008), 343-356.

[3] S. Gudmundsson, A. Sakovich, *Harmonic morphisms from the classical non-compact semisimple Lie groups*, Differential Geom. Appl. **27** (2009), 47-63.

[4] S. Gudmundsson, A. Siffert, M. Sobak, *Explicit proper p-harmonic functions on the Riemannian symmetric spaces* $\mathbf{SU}(n)/\mathbf{SO}(n)$, $\mathbf{Sp}(n)/\mathbf{U}(n)$, $\mathbf{SO}(2n)/\mathbf{U}(n)$, $\mathbf{SU}(2n)/\mathbf{Sp}(n)$, J. Geom. Anal. **32**, 147 (2022).

[5] S. Gudmundsson, M. Sobak, *Proper r-harmonic functions from Riemannian manifolds*, Ann. Global Anal. Geom. **57** (2020), 217-223.

[6] O. Riedler *Polynomial harmonic morphisms and eigenfamilies on spheres*, arXiv preprint arXiv:2310.19565 (2023).

[7] O. Riedler and A. Siffert *Global Eigenfamilies on Compact Manifolds*, arXiv preprint arXiv:2401.17750 (2024).