Tachibana and Nomizu-Smyth type theorems on k-positively curved manifolds

Based on joint works with M. Mariani, F. Mastropietro, M. Rigoli

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Motivation: the Bochner technique

M connected oriented Riemannian manifold of dimension $n \geq 2$

For $k \geq 0$ we denote by $\Omega^k(M)$ the space of C^∞ differential k-forms on M

Definition

$\omega \in \Omega^k(M)$ is **harmonic** if it is both **closed** ($\mathrm{d}\omega = 0$) and **coclosed** ($\delta\omega = 0$).

$$\mathcal{H}^k(M) = \{\omega \in \Omega^k(M) : \omega \text{ is harmonic}\}$$

Theorem (Hodge-de Rham)

If M is closed, $\varphi : \mathcal{H}^k(M) \to H^k_{dR}(M) : \omega \mapsto [\omega]$ is a linear isomorphism.

In particular, the k-th Betti number $b_k(M) = \dim \mathcal{H}^k(M)$.

Bochner technique

The (negative definite) Bochner Laplacian Δ of a tensor field T is

$$\Delta T = \operatorname{tr} \nabla^2 T$$

We define the (negative) Hodge Laplacian Δ_H on differential k-forms as

$$-\Delta_H = \mathrm{d}\delta + \delta\mathrm{d}$$

If $\omega \in \mathcal{H}^k(M)$ is harmonic, then $\Delta_H \omega = 0$. The converse is true if M is closed:

$$\Delta_H \omega = 0 \quad \Rightarrow \quad 0 = \int_M \langle \omega, \mathrm{d}\delta\omega \rangle + \int_M \langle \omega, \delta\mathrm{d}\omega \rangle = \int_M |\mathrm{d}\omega|^2 + \int_M |\delta\omega|^2$$

Theorem

For every $\omega \in \Omega^1(M)$

$$\Delta\omega = \Delta_H \omega + \operatorname{Ric}(\omega^\sharp, \cdot\,)$$

Bochner technique

Theorem (Bochner '46)

If M^n is closed with $\mathrm{Ric} \geq 0$, then every $\omega \in \mathcal{H}^1(M)$ is parallel and $b_1(M) \leq n$.

Moreover, if $\mathrm{Ric} > 0$ at some point on M then $\mathcal{H}^1(M) = \{0\}$ and $b_1(M) = 0$.

Proof For all $\omega \in \mathcal{H}^1(M)$ we have

$$\frac{1}{2}\Delta|\omega|^2 = |\nabla\omega|^2 + \langle\omega, \Delta\omega\rangle = |\nabla\omega|^2 + \underbrace{\langle\omega, \Delta_H\omega\rangle}_{=0} + \underbrace{\mathrm{Ric}(\omega^\sharp, \omega^\sharp)}_{\geq 0} \geq |\nabla\omega|^2$$

and if $\mathrm{Ric} > 0$ at some $x \in M$ then the inequality is strict unless $\omega_x = 0$ at x.

By the maximum principle (or the divergence theorem) the function $|\omega|^2$ is constant and $\nabla\omega = 0$. In particular, $\omega \equiv 0$ on M if $\mathrm{Ric} > 0$. Since ω is parallel, it is completely determined by its value at any point $p \in M$, so

$$b_1(M) = \dim \mathcal{H}^1(M) = \dim\{\omega_p : \omega \in \mathcal{H}^1(M)\} \leq \dim T_p^* M = n . \quad \square$$

Bochner technique

Theorem (Bochner '46)

If M^n is closed with $\mathrm{Ric} \geq 0$, then every $\omega \in \mathcal{H}^1(M)$ is parallel and $b_1(M) \leq n$.

Moreover, if $\mathrm{Ric} > 0$ at some point on M then $\mathcal{H}^1(M) = \{0\}$ and $b_1(M) = 0$.

By Poincaré-Hodge duality, we also have

Theorem

If M^n is closed with $\mathrm{Ric} \geq 0$, then every $\omega \in \mathcal{H}^{n-1}(M)$ is parallel & $b_{n-1}(M) \leq n$.

Moreover, if $\mathrm{Ric} > 0$ at some point on M then $\mathcal{H}^{n-1}(M) = \{0\}$ and $b_{n-1}(M) = 0$.

In particular, so far one can prove the following

Corollary

A closed M^3 with $\mathrm{Ric} \geq 0$ has the rational homology of $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{S}^1$ or $\mathbb{T}^3$.

Moreover, if $\mathrm{Ric} > 0$ at some point then M has the rational homology of $\mathbb{S}^3$.

Lichnerowicz operator Γ

A. Lichnerowicz defined for each $k \geq 1$ a self-adjoint endomorphism

$$\Gamma : T^0_k M \to T^0_k M$$

given, with respect to any local orthonormal frame $\{\theta^i\}_{1 \leq i \leq n}$ for T^*M, by

$$(\Gamma T)_{i_1 \dots i_k} = \sum_{\ell=1}^{k} R_{i_\ell t} T_{i_1 \dots t \dots i_k} - 2 \sum_{1 \leq \ell < h \leq k} R_{i_\ell t i_h s} T_{i_1 \dots t \dots s \dots i_k}$$

where $\mathrm{Riem} = R = R_{ijtl}\, \theta^i \otimes \theta^j \otimes \theta^t \otimes \theta^l$ and $\mathrm{Ric} = R_{ij}\, \theta^i \otimes \theta^j$.

Theorem (Weitzenböck identity)

For every $\omega \in \Omega^k(M)$, $k \geq 1$,

$$\Delta \omega = \Delta_H \omega + \Gamma \omega$$

In particular, for every $\omega \in \mathcal{H}^k(M)$

$$\frac{1}{2} \Delta |\omega|^2 = |\nabla \omega|^2 + \langle \Gamma \omega, \omega \rangle$$

Curvature operator $\mathfrak{R}$

The curvature operator of M is the self-adjoint endomorphism

$$\mathfrak{R} : \Lambda^2 M \to \Lambda^2 M$$

given w. r. to any local o.n. frame $\{\theta^i\}$ for T^*M by

$$(\mathfrak{R}\omega)_{ij} = R_{ijts}\omega_{ts}$$

- If M has constant sectional curvature $\kappa \in \mathbb{R}$ then $\mathfrak{R}\omega = 2\kappa\omega$ for all $\omega \in \Lambda^2 M$
- Given $c \in \mathbb{R}$, we write $\mathfrak{R} \geq c$ if $\langle \mathfrak{R}\omega, \omega \rangle \geq c|\omega|^2$ for all $\omega \in \Lambda^2 M$
- $\mathfrak{R}$ is *positive* ($\mathfrak{R} > 0$) if $\langle \mathfrak{R}\omega, \omega \rangle > 0$ whenever $\omega \neq 0$
- For any tangent 2-plane $v \wedge w \leq TM$

$$\mathrm{Sect}(v \wedge w) = \frac{1}{2} \frac{\langle \mathfrak{R}(v^\flat \wedge w^\flat), v^\flat \wedge w^\flat \rangle}{|v^\flat \wedge w^\flat|^2}$$

In particular, $\mathfrak{R} \geq c \;\Rightarrow\; \mathrm{Sect} \geq c/2$

The converse is generally false: $\mathbb{CP}^2$ has $1/4 \leq \mathrm{Sect} \leq 1$ but $\mathfrak{R} \geq 0$, $\ker \mathfrak{R} \neq \{0\}$

Curvature operator on 3-manifolds

For $n \geq 3$ recall the Ricci decomposition of the Riemann tensor R

$$R = W + \frac{1}{n-2} Z \owedge g + \frac{S}{2(n-1)(n-2)} g \owedge g$$

where S is the scalar curvature, $Z = \mathring{\mathrm{Ric}} = \mathrm{Ric} - \frac{S}{n} g$ and W is the Weyl tensor.

If $n = 3$ then W is always zero and $\mathfrak{R}$ is diagonalized by decomposable 2-forms:

- let v_1, v_2, v_3 be a basis of eigenvectors of Ric with eigenvalues λ_1, λ_2, λ_3
- the 2-forms $\omega_1 = v_2^\flat \wedge v_3^\flat$, $\omega_2 = v_3^\flat \wedge v_1^\flat$, $\omega_3 = v_1^\flat \wedge v_2^\flat$ satisfy

$$\mathfrak{R}\omega_1 = 2K_{23}\omega_1 \,, \quad \mathfrak{R}\omega_2 = 2K_{31}\omega_2 \,, \quad \mathfrak{R}\omega_3 = 2K_{12}\omega_3$$

where $K_{ij} = \mathrm{Sect}(v_i \wedge v_j) \equiv \frac{1}{2} S - \varepsilon_{ijk} \lambda_k$

Therefore, *only for* $n = 3$, for any $c \in \mathbb{R}$

$$\mathfrak{R} \geq c \quad \Leftrightarrow \quad \mathrm{Sect} \geq \frac{c}{2} \quad \Leftrightarrow \quad \frac{S-c}{2} \geq \mathrm{Ric} \quad (\Rightarrow \; \mathrm{Ric} \geq c)$$

Remark $\mathfrak{R} \geq 0 \Leftrightarrow \mathrm{Sect} \geq 0$ is still stronger than $\mathrm{Ric} \geq 0$.

Relation between Γ and $\mathfrak{R}$

To each $T \in T^0_k M$ we associate a $T^0_k M$-valued 2-form $\hat{T} \in T^0_k M \otimes \Lambda^2 M$:

Let $\{\theta^i\}$ be an o.n. frame on T^*M. For each $\alpha = (i_1, \ldots, i_k) \in \{1, \ldots, n\}^k$ set

$$T^\alpha_{ts} = T^{(i_1,\ldots,i_k)}_{ts} = \frac{1}{2}\sum_{\ell=1}^{k} T_{i_1\ldots t\ldots i_k}\delta_{i_\ell s} - \frac{1}{2}\sum_{\ell=1}^{k} T_{i_1\ldots s\ldots i_k}\delta_{i_\ell t}$$

then define

$$\hat{T} = \sum_{i_1,\ldots,i_k,t,s} T^{(i_1,\ldots,i_k)}_{ts}\theta^{i_1}\otimes\cdots\otimes\theta^{i_k}\otimes\theta^t\otimes\theta^s$$

Theorem (Berger '61, Meyer '71, Tachibana '74, Petersen)

For every $T, S \in T^0_k M$

$$\langle \Gamma T, S\rangle = \sum_\alpha \langle \mathfrak{R} T^\alpha, S^\alpha\rangle = \sum_\alpha R_{ijts} T^\alpha_{ij} S^\alpha_{ts}$$

In particular, if $\mathfrak{R} \geq c$ then

$$\langle \Gamma T, T\rangle \geq c\sum_\alpha |T^\alpha|^2 = c|\hat{T}|^2$$

Bochner technique for k-forms

Theorem (Berger '61, Gallot-Meyer '72, '75)

If M^n is closed with $\mathfrak{R} \geq 0$, then every $\omega \in \mathcal{H}^k(M)$ is parallel and $b_k(M) \leq \binom{n}{k}$.

If further $\mathfrak{R} > 0$ at some point, then $\mathcal{H}^k(M) = \{0\}$ and $b_k(M) = 0$ for $0 < k < n$.

Proof Direct computation shows that for all $\omega \in \Omega^k(M)$

$$|\hat{\omega}|^2 = \min\{k, n-k\}|\omega|^2 .$$

Hence, if $\mathfrak{R} \geq 0$ then for all $\omega \in \mathcal{H}^k(M)$

$$\frac{1}{2}\Delta|\omega|^2 = |\nabla\omega|^2 + \langle \Gamma\omega, \omega \rangle \geq |\nabla\omega|^2$$

and if $0 < k < n$ and $\mathfrak{R} > 0$ at some $x \in M$ then the inequality is strict unless $\omega = 0$ at x. Then the proof goes on as in case $k = 1$. $\square$

Remark $\mathfrak{R} > 0$ cannot be relaxed to $\mathrm{Sect} > 0$: $b_2(\mathbb{CP}^2) = 1$.

$\langle \Gamma A, A \rangle$ for symmetric bilinear tensors A

Let $\{\theta^i\}_{1 \le i \le n}$ be a local o. n. frame on T^*M. If $A \in T^0_2 M$ is symmetric,

$$\langle \Gamma A, A \rangle = 2R_{ij}A_{jt}A_{ti} - 2R_{itjs}A_{ij}A_{ts}$$

If $\theta^i = v_i^\flat$ where v_i are eigenvectors of A with eigenvalues λ_i, then

$$\langle \Gamma A, A \rangle = 2\sum_{i=1}^{n} R_{ii}\lambda_i^2 - 2\sum_{i,j=1}^{n} R_{ijij}\lambda_i\lambda_j = \sum_{i,j=1}^{n} R_{ijij}(\lambda_i - \lambda_j)^2$$

A direct computation shows that

$$\frac{1}{2}\sum_{i,j=1}^{n}(\lambda_i - \lambda_j)^2 = n|\mathring{A}|^2 = |\hat{A}|^2 \qquad \text{where } \mathring{A} = A - \frac{\operatorname{tr} A}{n} g$$

Theorem (Berger-Ebin '69)

If $\operatorname{Sect} \ge c/2$ *then for every* $A \in S^0_2 M$

$$\langle \Gamma A, A \rangle \ge nc|\mathring{A}|^2 = c|\hat{A}|^2$$

Harmonic curvature

Definition

The Riemann curvature tensor R of M^n is **harmonic** if $\operatorname{div} R = 0$.

- R is harmonic iff $\nabla \mathrm{Ric}$ is totally symmetric ($\Leftrightarrow$ Ric is a *Codazzi tensor*)

- Einstein manifolds ($\mathrm{Ric} = \lambda g$, $\lambda \in \mathbb{R}$) have harmonic curvature
- Locally symmetric manifolds ($\nabla R = 0$) have harmonic curvature

The converse is false: ex. $(\mathbb{R}^n_+, x_n^{4/(n-2)} g_{\mathrm{can}})$ for $n \geq 3$ [Gray '78]

- If $n = 2$, $\operatorname{div} R = 0 \quad \Leftrightarrow \quad M$ has constant curvature
- If $n = 3$, $\operatorname{div} R = 0 \quad \Leftrightarrow \quad M$ is locally conformally flat and S is constant
- If $n \geq 4$, $\operatorname{div} R = 0 \quad \Leftrightarrow \quad \operatorname{div} W = 0$ and S is constant

Berger theorem

Theorem (Berger)

Let $M^{n \geq 3}$ be closed with harmonic curvature.

- *If* $\mathrm{Sect} \geq 0$ *then* Ric *is parallel.*
- *If* $\mathrm{Sect} > 0$ *then* M *is Einstein.*

Proof The Ricci tensor of M satisfies

$$\frac{1}{2}\Delta|\mathrm{Ric}|^2 = |\nabla \mathrm{Ric}|^2 + \frac{1}{2}\langle \Gamma \mathrm{Ric}, \mathrm{Ric}\rangle$$

and if $\mathrm{Sect} \geq c \geq 0$ then $\langle \Gamma \mathrm{Ric}, \mathrm{Ric}\rangle \geq 2c|\hat{\mathrm{Ric}}|^2 = 2nc|Z|^2 \geq 0$.

So $\nabla \mathrm{Ric} = 0$ by maximum principle, and therefore $\nabla Z = 0$ and $\nabla S = 0$.

If $\mathrm{Sect} > 0$ at some $x \in M$ then it must be $Z = 0$ at x, hence $Z \equiv 0$ on M. $\square$

Berger theorem

Theorem (Berger)

Let $M^{n\geq 3}$ be closed with harmonic curvature.

- *If* $\mathrm{Sect} \geq 0$ *then* Ric *is parallel.*
- *If* $\mathrm{Sect} > 0$ *then M is Einstein.*

If $n = 3$ then $\nabla\mathrm{Ric} = 0 \Leftrightarrow \nabla R = 0$ and $\mathrm{Ric} = \lambda g \Leftrightarrow R = (\lambda/12) g \owedge g$

Corollary

Let M^3 be closed with harmonic curvature.

- *If* $\mathrm{Sect} \geq 0$ *then M is isometric to a quotient of* $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{R}$ *or* $\mathbb{R}^3$.
- *If* $\mathrm{Sect} > 0$ *then M is isometric to a quotient of* $\mathbb{S}^3$.

Remark In dimension $n = 4$, $\mathbb{CP}^2$ is Einstein but not of constant curvature.

Tachibana theorem

Theorem (Tachibana '74)

Let $M^{n\geq 4}$ be closed with harmonic curvature.

- *If $\mathfrak{R} \geq 0$ then M is locally symmetric.*
- *If $\mathfrak{R} > 0$ then M is isometric to a quotient of $\mathbb{S}^n$.*

Proof The Riemann curvature tensor R of M satisfies

$$\frac{1}{2}\Delta|R|^2 = |\nabla R|^2 + \frac{1}{2}\langle \Gamma R, R\rangle$$

and if $\mathfrak{R} \geq c \geq 0$ then $\langle \Gamma R, R\rangle \geq c|\hat{R}|^2 \geq 0$. So $\nabla R = 0$ by maximum principle.

$\nabla R = 0$ also implies $\nabla W = 0$, $\nabla \mathrm{Ric} = \nabla Z = 0$ and $\nabla S = 0$.

If $\mathfrak{R} > 0$ at some $x \in M$ then it must be $\hat{R} = 0$ at x. By direct computation

$$|\hat{R}|^2 = 2(n-1)|W|^2 + \frac{4n}{n-2}|Z|^2$$

so W and Z vanish at x, hence everywhere on M by parallelism. $\square$

Tachibana theorem for $n = 3$

Theorem (proved in C.-Mariani-Rigoli '24)

Let M^3 be closed with harmonic curvature.

- *If $\mathrm{Ric} \geq 0$ then M is isometric to a quotient of $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{R}^3$.*
- *If $\mathrm{Ric} > 0$ then M is isometric to a quotient of $\mathbb{S}^3$.*

Proof $\langle \Gamma R, R \rangle = P(\lambda, \mu, \nu)$, where $\lambda \leq \mu \leq \nu$ are the eigenvalues of Ric and

$$P(\lambda, \mu, \nu) = 8[\lambda(\lambda - \mu)(\lambda - \nu) + \mu(\mu - \lambda)(\mu - \nu) + \nu(\nu - \lambda)(\nu - \mu)] .$$

It is shown that if $\lambda, \mu, \nu \geq 0$ then $P(\lambda, \mu, \nu) \geq 0$, with equality holding iff $\lambda = \mu = \nu$ or $0 = \lambda < \mu = \nu$. Then the proof goes on as in case $n \geq 4$, by also using the following result of M. H. Noronha. $\square$

Theorem (Noronha '93)

Let M^n be closed with $\mathrm{Ric} \geq 0$ and locally conformally flat. Then M is either

- *globally conformally equivalent to a quotient of $\mathbb{S}^n$, or*
- *isometric to a quotient of $\mathbb{S}^{n-1} \times \mathbb{R}$ or $\mathbb{R}^n$*

If M is locally symmetric then M is isometric to a quotient of $\mathbb{S}^n$, $\mathbb{S}^{n-1} \times \mathbb{R}$ or $\mathbb{R}^n$.

Manifolds with $\mathfrak{R} \geq 0$

Theorem (Hamilton '82)

Let M^3 be closed with $\mathrm{Ric} \geq 0$. Then M is diffeomorphic to a quotient of $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{R}^3$. If $\mathrm{Ric} > 0$ somewhere, then M is diffemorphic to a quotient of $\mathbb{S}^3$.

Theorem (Hamilton '86, Chow-Lu-Ni '06, Böhm-Wilking '08, Ni-Wu '07)

Let $M^{n \geq 4}$ be closed with $\mathfrak{R} \geq 0$. Then M is diffeomorphic to a quotient of a product of finitely many factors of the following types:

i) *standard spheres*
ii) *Euclidean spaces*
iii) *closed symmetric spaces*
iv) *closed Kähler manifolds with positive curvature operator on real $(1,1)$-forms*

If $\mathfrak{R} > 0$ somewhere on M, then M is diffeomorphic to a quotient of $\mathbb{S}^n$.

The above statements remain true if condition $\mathfrak{R} \geq 0$ (resp., $\mathfrak{R} > 0$) is replaced by $\lambda_1 + \lambda_2 \geq 0$ (resp., $\lambda_1 + \lambda_2 > 0$) where λ_1, λ_2 are the first two eigenvalues of $\mathfrak{R}$

Petersen-Wink's improved conditions for the Bochner technique

Definition

For each $x \in M$ let $\lambda_1(x) \leq \lambda_2(x) \leq \cdots \leq \lambda_{\binom{n}{2}}(x)$ be the eigenvalues of $\mathfrak{R}_x$.

For each integer $1 \leq N \leq \binom{n}{2}$ we define $\mathfrak{R}^{(N)} : M \to \mathbb{R}$ by

$$\mathfrak{R}^{(N)} = \frac{1}{N} \sum_{i=1}^{N} \lambda_i$$

Main goal

Obtain bounds of the form $\langle \Gamma T, T \rangle \geq \mathfrak{R}^{(N)} |\hat{T}|^2$ for suitable classes of tensors T.

Petersen-Wink's improved conditions for the Bochner technique

Let $\{\theta^i\}_{1\leq i\leq n}$ be a local o.n. frame on T^*M. For $T\in T^0_kM$ and $\omega\in\Lambda^2M$ set

$$\langle\omega,\,\hat{T}\rangle=\sum_{i_1,\dots,i_k}\omega_{ts}\,T^{(i_1,\dots,i_k)}_{ts}\,\theta^{i_1}\otimes\cdots\otimes\theta^{i_k}\in T^0_kM\,.$$

If $\{\omega_i\}$ is an o.n. basis for Λ^2M of eigenvectors of $\mathfrak{R}$ with eigenvalues λ_i, then

$$\langle\Gamma T,\,T\rangle=\sum_i\lambda_i|\langle\omega_i,\,\hat{T}\rangle|^2\,,\quad|\hat{T}|^2=\sum_i|\langle\omega_i,\,\hat{T}\rangle|^2\,.$$

Theorem (Petersen-Wink '21)

Let $T\in T^0_kM$ and suppose that there exists an integer $1\leq N<\binom{n}{2}$ such that

$$|\langle\omega,\,\hat{T}\rangle|^2\leq\frac{1}{N}|\omega|^2|\hat{T}|^2$$

for all $\omega\in\Lambda^2M$. Then

$$\langle\Gamma T,\,T\rangle\geq\mathfrak{R}^{(N)}|\hat{T}|^2\,.$$

The proof is an application of an elementary lemma.

Petersen-Wink's improved conditions for the Bochner technique

Elementary lemma

Let $a_1, \dots, a_n$ be $n \geq 2$ real numbers such that, for some integer $1 \leq N < n$,

$$a_i^2 \leq \frac{1}{N} \sum_{j=1}^{n} a_j^2 \qquad \forall\, 1 \leq i \leq n\,.$$

Then for any non-decreasing sequence of real numbers $\lambda_1 \leq \cdots \leq \lambda_n$

$$\sum_{i=1}^{n} \lambda_i a_i^2 \geq \frac{1}{N} \sum_{i=1}^{N} \lambda_i \sum_{j=1}^{n} a_j^2$$

Proof

$$\sum_{i=1}^{n} \lambda_i a_i^2 \geq \sum_{i=1}^{N} \lambda_i a_i^2 + \lambda_{N+1} \sum_{i=N+1}^{n} a_i^2 = \sum_{i=1}^{N} (\lambda_i - \lambda_{N+1}) a_i^2 + \lambda_{N+1} \sum_{i=1}^{n} a_i^2$$

$$\geq \frac{1}{N} \sum_{i=1}^{n} (\lambda_i - \lambda_{N+1}) \sum_{j=1}^{n} a_j^2 + \lambda_{N+1} \sum_{i=1}^{n} a_i^2 = \frac{1}{N} \sum_{i=1}^{N} \lambda_i \sum_{j=1}^{n} a_j^2\,.$$

Petersen-Wink's approach to k-forms

Theorem (Petersen-Wink '21)

Let M^n be a Riemannian manifold and $0 < k < n$. For all $T \in \Lambda^k M$ and $\omega \in \Lambda^2 M$

$$|\langle \omega, \hat{T} \rangle|^2 \leq \frac{1}{\min\{k, n-k\}} |\omega|^2 |\hat{T}|^2$$

In particular, for all $T \in \Lambda^k M$

$$\langle \Gamma T, T \rangle \geq \mathfrak{R}^{(\min\{k,n-k\})} |\hat{T}|^2$$

Theorem (Petersen-Wink '21)

If M^n is closed with $\mathfrak{R}^{(p)} \geq 0$ for some $1 \leq p \leq n/2$, then every $\omega \in \mathcal{H}^k(M)$ is parallel and $b_k(M) \leq \binom{n}{k}$, provided $0 \leq k \leq p$ or $n - p \leq k \leq n$.

Moreover, if $\mathfrak{R}^{(p)} > 0$ at some point then $\mathcal{H}^k(M) = \{0\}$ and $b_k(M) = 0$, provided $1 \leq k \leq p$ or $n - p \leq k \leq n - 1$.

Algebraic curvature tensors

Definition

$T \in T_4^0 M$ is an **algebraic curvature tensor** if it shares the algebraic symmetries of the Riemann tensor, that is, if for all vectors v_1, v_2, v_3, v_4

$$T(v_1, v_2, v_3, v_4) = -T(v_2, v_1, v_3, v_4) = T(v_3, v_4, v_1, v_2)$$

$$T(v_1, v_2, v_3, v_4) + T(v_2, v_3, v_1, v_4) + T(v_3, v_1, v_2, v_4) = 0$$

For any algebraic curvature tensor T we can define a Ricci contraction Ric_T

$$\mathrm{Ric}_T(v_1, v_2) = \mathrm{tr}\left[(w_1, w_2) \mapsto T(v_1, w_1, v_2, w_2)\right],$$

a total trace $S_T = \mathrm{tr}\,\mathrm{Ric}_T$ and, if $n \geq 3$, an associated Weyl-type tensor W_T s.t.

$$T = W_T + \frac{1}{n-2} Z_T \owedge g + \frac{S_T}{2(n-1)(n-2)} g \owedge g$$

where $Z_T = \mathring{\mathrm{Ric}}_T = \mathrm{Ric}_T - \frac{S_T}{n} g$. We have

$$|\hat{T}|^2 = 2(n-1)|W_T|^2 + \frac{4n}{n-2}|Z_T|^2 = |\hat{W}_T|^2 + \frac{4}{n-2}|\hat{Z}_T|^2$$

Harmonic curvature tensors

Definition

An algebraic curvature tensor field $T \in C^\infty(T^0_4 M)$ is **harmonic** if $\operatorname{div} T = 0$ and the second Bianchi identity holds, i.e. if for all vector fields v_1, v_2, v_3, w, z

$$\nabla_{v_1} T(v_2, v_3, w, z) + \nabla_{v_2} T(v_3, v_1, w, z) + \nabla_{v_3} T(v_1, v_2, w, z) = 0$$

For any harmonic curvature tensor T on a Riemannian manifold

$$\frac{1}{2}\Delta |T|^2 = |\nabla T|^2 + \frac{1}{2}\langle \Gamma T, T\rangle$$

Petersen-Wink's approach to algebraic curvature tensors

Theorem (Petersen-Wink '21)

Let $M^{n\geq 3}$ be a Riemannian manifold. For each algebraic curvature tensor T with $Z_T = 0$ and for each $\omega \in \Lambda^2 M$

$$|\langle \omega, \hat{T} \rangle|^2 \leq \frac{2}{n-1} |\omega|^2 |\hat{T}|^2$$

In particular, for any such T

$$\langle \Gamma T, T \rangle \geq \mathfrak{R}^{(\lfloor \frac{n-1}{2} \rfloor)} |\hat{T}|^2$$

Theorem (Petersen-Wink '21)

Let $M^{n\geq 4}$ be a closed Einstein manifold.

- *If $\mathfrak{R}^{(\lfloor \frac{n-1}{2} \rfloor)} \geq 0$ then M is locally symmetric.*
- *If $\mathfrak{R}^{(\lfloor \frac{n-1}{2} \rfloor)} > 0$ then M is isometric to a quotient of $\mathbb{S}^n$.*

In case $n = 4$ the above statements remain true when $\mathfrak{R}^{(\lfloor \frac{n-1}{2} \rfloor)}$ is replaced by $\mathfrak{R}^{(2)}$.

Generalized Tachibana theorem

Theorem (C.-Mariani-Rigoli '24)

Let T be an algebraic curvature tensor on $M^{n\geq 4}$. Then

$$\langle \Gamma T, T\rangle \geq \mathfrak{R}^{(\lfloor \frac{n-1}{2} \rfloor)} |\hat{T}|^2$$

Theorem (C.-Mariani-Rigoli '24)

Let $M^{n\geq 4}$ be closed with harmonic curvature.

- *If $\mathfrak{R}^{(\lfloor \frac{n-1}{2} \rfloor)} \geq 0$ then M is locally symmetric.*
- *If $\mathfrak{R}^{(\lfloor \frac{n-1}{2} \rfloor)} > 0$ then M is isometric to a quotient of $\mathbb{S}^n$.*

Remark If $n = 4$ the statements remain true with $\mathfrak{R}^{(\lfloor \frac{n-1}{2} \rfloor)}$ replaced by $\mathfrak{R}^{(2)}$. In particular, the corresponding Tachibana type theorem has been proved in the preprint [Bettiol-Jackson Goodman '22]

Lower bound on $\langle \Gamma T, T \rangle$

The proof of the lower bound $\langle \Gamma T, T \rangle \geq \mathfrak{R}^{(\lfloor \frac{n-1}{2} \rfloor)} |\hat{T}|^2$ is based on two lemmas.

Lemma 1 (C.-Mariani-Rigoli '24)

Let T, T' be algebraic curvature tensors on $M^{n \geq 3}$. Then

$$\langle \Gamma T, T' \rangle = \langle \Gamma W_T, W_{T'} \rangle + \frac{4}{n-2} \langle \Gamma Z_T, Z_{T'} \rangle$$

Definition

For each integer $1 \leq N \leq \binom{n}{2}$ we define $\mathrm{Sect}^{(N)} : M \to \mathbb{R}$ by

$$\mathrm{Sect}^{(N)}(x) = \inf \left\{ \frac{1}{N} \sum_{i=1}^{N} \mathrm{Sect}(\pi_i) : \pi_1, \ldots, \pi_N \text{ orthogonal 2-planes in } T_x M \right\}$$

Lemma 2 (C.-Mariani-Rigoli '24)

For all $A \in S_2^0 M$

$$\langle \Gamma A, A \rangle \geq 2\, \mathrm{Sect}^{(\lfloor n/2 \rfloor)} |\hat{A}|^2$$

Lower bound on $\langle \Gamma T, T\rangle$

Proof For any collection $\{\pi_i\}$ of $\lfloor n/2\rfloor$ mutually orthogonal 2-planes we have

$$\frac{2}{\lfloor n/2\rfloor}\sum_{i=1}^{\lfloor n/2\rfloor}\mathrm{Sect}(\pi_i)=\frac{1}{\lfloor n/2\rfloor}\sum_{i=1}^{\lfloor n/2\rfloor}\frac{\langle \mathfrak{R}\omega_i,\omega_i\rangle}{|\omega_i|^2}\geq \mathfrak{R}^{(\lfloor \frac{n}{2}\rfloor)}\geq \mathfrak{R}^{(\lfloor \frac{n-1}{2}\rfloor)}$$

where each ω_i is a 2-form metrically equivalent to a bivector spanning π_i. Then

$$\begin{aligned}\langle \Gamma T, T\rangle &= \langle \Gamma W_T, W_T\rangle+\frac{4}{n-2}\langle \Gamma Z_T, Z_T\rangle\\ &\geq \mathfrak{R}^{(\lfloor \frac{n-1}{2}\rfloor)}|\hat{W}_T|^2+\frac{4}{n-2}\mathfrak{R}^{(\lfloor \frac{n}{2}\rfloor)}|\hat{Z}_T|^2\\ &\geq \mathfrak{R}^{(\lfloor \frac{n-1}{2}\rfloor)}\left(|\hat{W}_T|^2+\frac{4}{n-2}|\hat{Z}_T|^2\right)=\mathfrak{R}^{(\lfloor \frac{n-1}{2}\rfloor)}|\hat{T}|^2. \quad\square\end{aligned}$$

Remark If $n=4$ then $\langle \Gamma Z_T, Z_T\rangle \geq \mathfrak{R}^{(2)}|\hat{Z}_T|^2$ holds by the previous steps, while $\langle \Gamma W_T, W_T\rangle \geq \mathfrak{R}^{(2)}|\hat{W}_T|^2$ follows from Petersen-Wink.

Generalized Tachibana theorem for complete manifolds

Theorem (C.-Mariani-Rigoli '24)

Let M^3 be a complete 3-manifold with harmonic curvature.

- *If $\mathrm{Ric} \geq 0$ then M is isometric to a quotient of $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{R}^3$.*
- *If $\mathrm{Ric} > 0$ then M is isometric to a quotient of $\mathbb{S}^3$.*

Proof We recall the following

Theorem (Zhu '94, Carron-Herzlich '06)

If M^n is complete with $\mathrm{Ric} \geq 0$ and locally conformally flat, then M is either

i) *isometric to a quotient of $\mathbb{S}^{n-1} \times \mathbb{R}$ or $\mathbb{R}^n$*

ii) *conformally equivalent to a quotient of $\mathbb{S}^n$*

iii) *non-flat and conformally equivalent to a quotient of $\mathbb{R}^n$*

ii) $\Rightarrow$ M is compact $\Rightarrow$ we apply Tachibana theorem for closed manifolds.

iii) cannot occur: otherwise, the universal cover of M would be $\mathbb{R}^n$ with complete metric of constant $S > 0$ and conformally equivalent to the Euclidean metric, which is impossible by [Caffarelli-Gidas-Spruck '89]. □

Generalized Tachibana theorem for complete manifolds

Theorem (C.-Mariani-Rigoli '24)

Let $M^{n\geq 4}$ be a complete manifold with harmonic curvature and such that

$$\int_1^{+\infty} \frac{R\mathrm{d}R}{\mathrm{vol}_g(B_R(p))} = +\infty \qquad \text{for some } p \in M$$

- *If $\mathfrak{R}^{(\lfloor \frac{n-1}{2} \rfloor)} \geq 0$ then M is locally symmetric.*
- *If $\mathfrak{R}^{(\lfloor \frac{n-1}{2} \rfloor)} > 0$ then M is isometric to a quotient of $\mathbb{S}^n$.*

Corollary

Let M^4 be a complete 4-manifold with harmonic curvature.

- *If $\mathfrak{R} \geq 0$ then M is locally symmetric.*
- *If $\mathfrak{R} > 0$ then M is isometric to a quotient of $\mathbb{S}^4$.*

Nomizu-Smyth theorem

Let $\psi : M^n \to \overline{M}^{n+1}$ be a two-sided isometrically immersed hypersurface in a space of constant curvature. If the mean curvature H of the immersion ψ is constant, then the second fundamental form II satisfies

$$\frac{1}{2}\Delta|\mathrm{II}|^2 = |\nabla \mathrm{II}|^2 + \langle \Gamma \mathrm{II}, \mathrm{II}\rangle$$

Theorem (Nomizu-Smyth '69)

Let M^n be a closed manifold with $\mathrm{Sect} \geq 0$ and let $\psi : M \to \overline{M}$ be as above with $\overline{M} = \mathbb{R}^{n+1}$, $\mathbb{H}^{n+1}$ or $\mathbb{S}^{n+1}$.

- *If $\overline{M} = \mathbb{R}^{n+1}$ or $\mathbb{H}^{n+1}$ then $\psi(M)$ is a totally umbilic sphere.*
- *If $\overline{M} = \mathbb{S}^{n+1}$ then $\psi(M)$ is an umbilic sphere or a Clifford torus $\mathbb{S}^k \times \mathbb{S}^{n-k}$.*

Theorem (C.-Mastropietro-Rigoli)

Let $\psi : M^n \to \mathbb{S}^{n+1}$ be a two-sided isometric immersion with constant H. If M is closed and has $\mathrm{Sect}^{(\lfloor n/2 \rfloor)} \geq 0$ then $\psi(M)$ is an umbilic sphere or a Clifford torus.

Nomizu-Smyth type theorem

Proposition (C.-Mastropietro-Rigoli)

If $\psi : M^n \to \mathbb{S}^{n+1}$ be a two-sided, 2-convex isometric immersion. Then

$$\langle \Gamma \mathrm{II}, \mathrm{II} \rangle \geq \left(n - \frac{n^3 H^2}{(n-2)^2} \right) |\Phi|^2 \quad \text{and} \quad \langle \Gamma \mathrm{II}, \mathrm{II} \rangle \geq (n - |\mathrm{II}|^2) |\Phi|^2$$

where $\Phi = \mathrm{II} - Hg$ is the traceless part of the second fundamental form.

Theorem (C.-Mastropietro-Rigoli)

Let $\psi : M^n \to \mathbb{S}^{n+1}$ be a two-sided, 2-convex isometric immersion with constant H. If M is closed and either

$$|H| \leq \frac{n-2}{n} \qquad \text{or} \qquad |\mathrm{II}|^2 \leq n$$

then $\psi(M)$ is a totally umbilic sphere or a Clifford torus.

Thank you for your time!