V-minimal submanifolds

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Outline:

- define the notion of V-minimality, for V a smooth vector field on a Riemannian manifold (a natural extension of the classical notion of minimality).
- find examples for locally conformal Kaehler (l.c.K) manifolds and Pseudo Horizontally Homothetic (PHH) submersion.

Question:
In the *presence of a vector field* V on a manifold M, one can naturally ask whether the definition of harmonicity or minimality (for example) can be perturbed to underline *the influence of* V.

V - Harmonic Maps and Morphisms

The notions of

- *V-harmonic maps* was introduced by: *Chen-Jost-Wang (J. Geom. Anal., 2015)* and *Chen-Jost-Qiu (Ann. Global Anal. Geom., 2012)* (see also *Chen-Qiu* (Adv. Math., 2016), *Qiu* (Proc. Amer. Math. Soc., 2017))
- *V-harmonic morphisms* was introduced by: *Zhao (Proc.Amer.Math.Soc. Vol.148, No.3,2020)*.

Let (M, g) and (N, h) be Riemannian manifolds, V a smooth vector field on M and $\varphi : M \to N$ a smooth map.

Definition (Chen-Jost-Wang)

The map φ is called V*-harmonic* if it satisfies:

$$\tau_V(\varphi) := \tau(\varphi) + d\varphi(V) = 0, \tag{1}$$

where $\tau(\varphi)$ is the tension field of the map φ. Since the differential $d\varphi$ of φ can be viewed as a section of the bundle $T^*M \otimes \varphi^{-1}TN$, $d\varphi(V)$ is a section of the bundle $\varphi^{-1}TN$. The tension $\tau_V(\varphi)$ is called the V*-tension field of* φ.

- V-harmonicity is not defined via a variational problem, but rather by imposing the vanishing of the V-tension field of φ.
- If $V = 0$, the two notions coincide, more generally, the same is true if V is vertical.
- A smooth function $f : M \to \mathbf{R}$ is said to be V*-harmonic* if:

$$\Delta_V(f) := \Delta(f) + \langle V, \nabla f \rangle = 0.$$

In local coordinates $(x_i)_{i=\overline{1,m}}$ on M and $(y_\alpha)_{\alpha=\overline{1,n}}$ on N, respectively, the tension fields have the following expressions (see *Baird-Wood* (Book-Oxford Univ.Press, 2003), *Eells-Lemaire* (Bull. London Math. Soc., 1978)): $\tau(\varphi) = \sum\limits_{\alpha=1}^{n} \tau(\varphi)^\alpha \frac{\partial}{\partial y_\alpha}$, where, denoting by ${}^M\Gamma_{ij}^k$ and ${}^N L_{\beta\gamma}^\alpha$ the Christoffel symbols of M and N, and $\varphi^\alpha = \varphi \circ y_\alpha$,

$$\tau(\varphi)^\alpha = \sum_{i,j=1}^{m} g^{ij} \left(\frac{\partial^2 \varphi^\alpha}{\partial x_i \partial x_j} - \sum_{k=1}^{m} {}^M\Gamma_{ij}^k \frac{\partial \varphi^\alpha}{\partial x_k} + \sum_{\beta,\gamma=1}^{n} {}^N L_{\beta\gamma}^\alpha \frac{\partial \varphi^\beta}{\partial x_i} \frac{\partial \varphi^\gamma}{\partial x_j} \right). \quad (2)$$

For V a smooth vector field on M, given in local coordinates on M by $V = \sum\limits_{i=1}^{m} V_i \frac{\partial}{\partial x_i}$, the V**-tension field of the map** φ **has the following expression in local coordinates:**

$$\tau_V(\varphi) = \sum_{\alpha=1}^{n} \tau(\varphi)^\alpha \frac{\partial}{\partial y_\alpha} + \sum_{\alpha=1}^{n} \sum_{i=1}^{m} V_i \frac{\partial \varphi^\alpha}{\partial x_i} \frac{\partial}{\partial y_\alpha}. \quad (3)$$

The V-tension field of the composition of two maps (see Zhao) $\varphi : M \to N$ and $\psi : N \to P$ is given by:

$$\tau_V(\psi \circ \varphi) = d\psi(\tau_V(\varphi)) + \mathrm{trace}\nabla d\psi(d\varphi, d\varphi). \tag{4}$$

As in the case of harmonic maps, Zhao defined V*-harmonic morphisms* as:

maps $\varphi : M \to N$, between Riemannian manifolds, which pulls back local harmonic functions on N to local V-harmonic functions on M.

Recalling that (cf. *Baird-Wood* (Book-Oxford Univ.Press, 2003)) a smooth map $\varphi : M^m \to N^n$, $x \in M$ is called *horizontally weakly conformal* (HWC) at x if either:

a) $\mathrm{d}\varphi_x = 0$, or

b) $\mathrm{d}\varphi_x$ maps the horizontal space $\mathcal{H}_x = \{\ker(\mathrm{d}\varphi_x)\}^\perp$ conformally onto $T_{\varphi(x)}N$,
(i.e. $\mathrm{d}\varphi_x$ is surjective and there exists a number $\Lambda(x) \neq 0$ such that $h(\mathrm{d}\varphi_x(X), \mathrm{d}\varphi_x(Y)) = \Lambda(x)g(X, Y)$, for any X, Y horizontal vector fields),

the following results characterize the V-harmonic morphisms:

Theorem (see Zhao)

Let $\varphi : (M^m, g) \to (N^n, h)$ *be a smooth map between Riemannian manifolds. Then the following conditions are equivalent:*

1) φ *is a* V*-harmonic morphism;*
2) φ *is a horizontally weakly conformal* V*-harmonic map;*
3) $\forall \psi : W \to P$ *a smooth map from an open subset* $W \subset N$ *with* $\varphi^{-1}(W) \neq \emptyset$, *to a Riemann manifold* P, *we have:* $\tau_V(\psi \circ \varphi) = \lambda^2 \tau(\psi)$, *for some smooth function* $\lambda^2 : M \to [0, \infty)$;
4) $\forall \psi : W \to P$ *a harmonic map from an open subset* $W \subset N$ *with* $\varphi^{-1}(W) \neq \emptyset$, *to a Riemann manifold* P, *the map* $\psi \circ \varphi$ *is a* V*-harmonic map;*
5) $\exists \lambda^2 : M \to [0, \infty)$ *a smooth function such that:* $\Delta_V(f \circ \varphi) = \lambda^2 \Delta f$, *for any function* f *defined on an open subset* W *of* N *with* $\varphi^{-1}(W) \neq \emptyset$.

Corollary (Zhao)

For $\varphi : M \to N$ a V-harmonic morphism with dilation λ and $\psi : N \to P$ a harmonic morphism with dilation θ, the composition $\psi \circ \varphi$ is a V-harmonic morphism with dilation $\lambda(\theta \circ \varphi)$.

Theorem (Zhao)

For a horizontally weakly conformal map $\varphi : M \to N$ with dilation λ, any two of the following conditions imply the third:

1) *φ is a V-harmonic map (and so a V-harmonic morphism);*
2) *$V + \nabla \log(\lambda^{2-n})$ is vertical at regular points;*
3) *the fibres of φ are minimal at regular points.*

V - minimal submanifolds

Let (M, g) be a Riemannian manifold, V a smooth vector field on M, and K a submanifold of M.
With respect to g_x, we have the orthogonal decomposition $T_xM = T_xK \oplus T_xK^{\perp}, \forall x \in K$ and, according to this decomposition,

$$\nabla^M_X Y = \nabla^K_X Y + A(X, Y), \text{ for all } X, Y \in \Gamma(TK).$$

The symmetric bilinear map $A : TK \times TK \to TK^{\perp}$ is *the second fundamental form of the submanifold* K.

Definition

The submanifold K of M is called V*-minimal* if

$$\mathrm{trace}(A) - V \in \Gamma(TK).$$

Remark

The V-minimality can be rephrased as:

- $\mathrm{trace}(A) - V^{\perp} = 0$, *since the trace of the second fundamental form has the image in the normal bundle,*
  or
- *If $\varphi : M \to N$ is a differentiable map and K is a fibre of φ. Then the V-minimality condition translates to:*

$$\mathrm{trace}(A) - V \text{ is a vertical vector.}$$

Proposition

Let K be a closed submanifold of a Riemann manifold (M, g). Then there exists V a smooth vector field on M, such that K is V-minimal.

Sketch of proof:

- choose the vector field trace (A), on K
- extend trace (A) to V on M

Question: In the case of a Riemannian submersion, can we adapt the result such that it holds true for a universal vector field V across all fibers?

Recall that (see *Barrett O'Neill* (Michigan Math. J., 1966)):

- for a Riemannian submersion $\varphi : M \to N$, can be defined two tensors, one of which is the second fundamental form of all the fibers.
  If $\mathcal{H}$ and $\mathcal{V}$ denote the horizontal and the vertical distribution on M, then the second fundamental form of all fibers $\varphi^{-1}(y)$, $y \in N$, gives rise to a (1,2)-tensor field on M, defined by:

$$T_X Y = \mathcal{H}\nabla^M_{\mathcal{V}X}(\mathcal{V}Y) + \mathcal{V}\nabla^M_{\mathcal{V}X}(\mathcal{H}Y),$$

  for all arbitrary vector field X, Y in M and ∇^M the covariant derivative on M.

In this context, we prove the following.

Proposition

Let $\varphi : M \to N$ be a Riemannian submersion. Then there exists a smooth vector field V on M such that any fiber of φ is V-minimal.

Sketch of proof:

- consider the tensor $T_X Y$
- on M, choose $V =$ trace $(\mathcal{H}\nabla^M_{\mathcal{V}X}(\mathcal{V}Y))$

V–minimality and locally conformal Kähler manifolds

Locally conformal Kähler manifolds are natural generalization of the class of Kähler manifolds, and they have been much studied since the work of I. Vaisman in the '70s (Israel J. Math., 1976).

A manifold (M^{2n}, J, g) is called *locally conformally Kähler* (l.c.K) if g can be rescaled locally, in a neighborhood of any point in M, so as to be Kähler.

Definition (see Vaisman (Israel J. of Math., 1976) or Dragomir-Ornea (Book-Birkhäuser, 1998))

A complex n-dimensional Hermitian manifold (M^{2n}, J, g), where J denotes its complex structure and g its Hermitian metric is called a *locally conformal Kähler (l.c.K.) manifold* if there is an open cover $\{U_i\}_{i\in I}$ of M and a family of $\mathcal{C}^\infty$ functions $f_i : U_i \to \mathbf{R}$, $i \in I$, such that each local metric:

$$g_i = \exp(-f_i) g_{|U_i} \tag{5}$$

is Kählerian, where $g_{|U_i} = \iota_i^\star$, and $\iota_i : U_i \to M$ is the inclusion.
The manifold is called *globally conformal Kähler (g.c.K) manifold* if there exist a $\mathcal{C}^\infty$ function $f : M \to \mathbf{R}$, such that the metric $exp(-f)g$ is Kähler.

There are many fundamental examples of l.c.K manifolds which are not Kahler and they play a major rolle in the classification of compact complex manifolds.

For example:

Let $\lambda \in \mathbf{C}, |\lambda| \neq 1,$ and Δ_λ be the cyclic group generated by the transformations $z \to \lambda z$ of $\mathbf{C}^n - \{0\}.$

The quotient space $\mathbf{C}^n - \{0\}/\Delta_\lambda$ has the structure of a complex manifold, called the *complex Hopf manifold*.

- it is compact
- it admits no global Kähler metrics

An equivalent characterization of a l.c.K manifold can be given as follows (see *Vaisman* or *Dragomir-Ornea* or *Marrero-Rocha* (Geometriae Dedicata, 1994)):

- M^{2n}-an almost Hermitian manifold, with metric g, Riemann connection ∇ and almost complex structure J.
- the Nijenhuis tensor of M: $N_J(X,Y) = [JX,JY] - J[JX,Y] - J[X,JY] - [X,Y], \forall X,Y \in \mathcal{X}(M)$,
- the Kähler 2-form Ω: $\Omega(X,Y) = g(X,JY)$,
- the Lee 1-form defined by:

$$\omega(X) = -\frac{1}{n-1}\sum_{i=1}^{n}[(\nabla_{E_i}\Omega)(E_i,JX) + (\nabla_{JE_i}\Omega)(JE_i,JX)],$$

  where $\{E_1,E_2,\ldots,E_n,JE_1,JE_2,\ldots,JE_n\}$ is a local orthonormal basis of M,
- the Lee vector field B on M defined by $\omega(X) = g(X,B)$,$\forall X \in \mathcal{X}(M)$

An almost Hermitian manifold (M, J, g) is called:

- *Kähler* if $d\Omega = 0$ and $N_J = 0$;
- *l.c.K* if $N_J = 0$, ω is closed and $d\Omega = \omega \wedge \Omega$
  or equivalently
  if ω is closed and for all $X, Y \in \mathcal{X}(M)$,

$$(\nabla_X J)Y = \frac{1}{2}(\omega(JY)X - \omega(Y)JX - g(X, JY)B + g(X, Y)JB).$$

A complex submanifolds of a l.c.K. manifold (which is not Kähler) is seldom minimal (see *Dragomir-Ornea*).

However, it is V-minimal for a suitable vector field V, as shown below.

Theorem

With the notation as above, let K be a complex submanifold of a l.c.K manifold M of complex dimension m, and V a smooth vector field on M. Then K is a V-minimal submanifold on M, for $V = -mB$.

Another example is obtained when considering l.c.K submersions.

Definition (cf. Dragomir-Ornea, Marrero-Rocha)

Let (M, J, g) and (M', J', g') be two almost Hermitian manifolds and $\varphi : M \to M'$ a Riemannian submersion, which is holomorphic (i.e. $d\varphi \circ J = J' \circ d\varphi$). Moreover, if (M, J, g) is a l.c.K manifold, then φ is called a *l.c.K submersion*.

Proposition

Let $\varphi : M \to M'$ *a l.c.K submersion, and* V *a vector field on* M*, where* M *is compact.*
Then, the fibers of φ *are* V*- minimal if and only if the Lee vector field* B *of* M *satisfies* $V + mB$ *is a vertical vector field, where* m *is the dimension of the fiber.*

Remark

Let $\varphi : M \to M'$, M *compact, a l.c.K submersion,* V *a vector field on* M. *Denote by:*

- ω, ω' *the Lee forms of* M *and* M',
- B *and* B' *the corresponding Lee vector fields.*

If the fibers of φ *are* V*-minimal, then* $V + mB$ *is a vertical vector and* $d\varphi(B) = -\frac{1}{m} d\varphi(V)$.

Then, the Lie vector field on M' *is* $B' = -\frac{1}{m} d\varphi(V)$.

Pseudo Harmonic Morphisms and Pseudo Horizontally Homothetic Maps

The notion of harmonic morphisms can be generalized when the target manifold is endowed with a Kähler structure (see Chen (Int. J. Math.,1997), Loubeau (Int. J. Math.,1997)).

Let $\varphi : (M^m, g) \to (N^{2n}, J, h)$ a smooth map from a Riemannian manifold to a Kähler manifold.

Definition (Loubeau (Int. J. Math.,1997))

The map φ is said to be a *pseudo-harmonic morphism* (PHM) if and only if it pulls back local holomorphic functions on N to local harmonic maps from M to $\mathbf{C}$.

For any $x \in M$, $d\varphi_x^* : T_{\varphi(x)}N \to T_xM$ the adjoint map of the tangent linear map $d\varphi_x : T_xM \to T_{\varphi(x)}N$, if X is a local section on the pull-back bundle $\varphi^{-1}TN$, then $d\varphi^*(X)$ is a local horizontal vector field on M.

Definition (Loubeau (Int. J. Math.,1997))

The map φ is called *pseudo-horizontally (weakly) conformal (shortening PHWC) at* $x \in M$ if $[d\varphi_x \circ d\varphi_x^*, J] = 0$.

The map φ is called *pseudo-horizontally (weakly) conformal* if it is pseudo-horizontally (weakly) conformal at every point of M.

Then, *pseudo-harmonic morphisms* can also be characterised as harmonic, pseudo-horizontally (weakly) conformal maps (see Chen, Loubeau).

A special class of pseudo-horizontally weakly conformal maps, are pseudo-horizontally homothetic maps.

Definition (Aprodu-Aprodu-Brînzănescu (Int. J. Math., 2000))

A map $\varphi : (M^m, g) \to (N^{2n}, J, h)$ is called *pseudo-horizontally homothetic at* x (shortening PHH) if is PHWC at a point $x \in M$ and satisfies:

$$d\varphi_x \left((\nabla^M_v d\varphi^*(JY))_x \right) = J_{\varphi(x)} d\varphi_x \left((\nabla^M_v d\varphi^*(Y))_x \right), \tag{6}$$

for any horizontal tangent vector $v \in T_x M$ and any vector field Y locally defined on a neighbourhood of $\varphi(x)$.
A PHWC map φ is called *pseudo-horizontally homothetic*, if and only if

$$d\varphi(\nabla^M_X d\varphi^*(JY)) = J d\varphi(\nabla^M_X d\varphi^*(Y)), \tag{7}$$

for any horizontal vector field X on M and any vector field Y on N.

One of the basic properties of pseudo-horizontally homothetic maps (cf. M.A.Aprodu-M.Aprodu-Brînzănescu):

a PHH submersion is a harmonic map if and only if it has minimal fibres.

Also, pseudo-horizontally homothetic maps are good tools to construct minimal submanifolds.

V-Pseudo Harmonic Morphisms

Generalizing the class of:
harmonic maps and morphisms, respectively *to V-harmonic maps and pseudo harmonic morphisms*
we obtain
V-pseudo harmonic morphisms with a description similar to pseudo harmonic morphisms.

Definition

Let (M^m, g) be a Riemannian manifold of real dimension m, (N^{2n}, J, h) a Hermitian manifold of complex dimension n, $\varphi : M \to N$ a smooth map and V a smooth vector field on M.
The map φ is called *V-pseudo harmonic morphism* (shortening V-PHM) if φ is *V-harmonic* and *pseudo horizontally weakly conformal.*

The characterization of *pseudo harmonic morphism* (see Loubeau) remains true in the general case of V*-harmonic* maps.

Theorem

Let $\varphi : M \to \mathrm{N}$ *be a smooth map from a Riemannian manifold* (M^m, g) *to a Kähler one* (N^{2n}, J, h) *and* V *a smooth vector field on* M*.*
Then φ *is* V*-pseudo harmonic morphism (*V*-PHM) if and only if it pulls back local complex valued holomorphic functions on* N *to local* V*-harmonic functions on* M*.*

Relation between:
V-harmonic morphisms, *pseudo harmonic morphisms* and *V-pseudo harmonic morphisms.*

Proposition

Let (M, g) and (N, h) be two Riemannian manifolds, V a smooth vector field on M, (P, J, p) a Kähler manifold and $\psi : M \to N$ and $\varphi : N \to P$ two smooth maps. If ψ is V-harmonic morphism and φ is pseudo harmonic morphism (PHM), then $\varphi \circ \psi$ is V-pseudo harmonic morphism (V-PHM).

Theorem

Let $\varphi : (M^m, g) \to (N^{2n}, J, h), n \geq 2,$ *be a pseudo horizontally homothetic submersion (PHH). Then* φ *is* V*-harmonic if and only if* φ *has* V*-minimal fibres.*

The construction of minimal submanifolds was done for:

- horizontally homothetic harmonic morphisms (see Baird-Gudmundsson (Math. Ann., 1992)) and
- adapted for pseudo-horizontally homothetic harmonic submersions (see Aprodu-Aprodu-Brînzănescu).

Replacing harmonicity by V-harmonicity, a similar result can be proved.

Theorem

Let (M^m, g) be a Riemannian manifold, (N^{2n}, J, h) be a Kähler manifold, V a smooth vector field on M and $\varphi : M \to \mathrm{N}$ be a pseudo-horizontally homothetic (PHH), V-harmonic submersion.
If $P^{2p} \subset \mathrm{N}^{2n}$ is a complex submanifold of N, then $\varphi^{-1}(P) \subset M$ is a V-minimal submanifold of M.

Thank you for your attention!

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