Gap results for biharmonic submanifolds in Euclidean spheres

Stefan Andronic

Alexandru Ioan Cuza University of Iași

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Article

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S.A. and Simona Nistor

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Harmonic and biharmonic maps

Let $\varphi : (M, g) \to (N, h)$ be a smooth map.

Energy functional

$$E(\varphi) = E_1(\varphi) = \frac{1}{2} \int_M |d\varphi|^2 \, \overline{v}_g$$

Euler-Lagrange equation

$$\begin{aligned} \tau(\varphi) &= \operatorname{trace} \nabla d\varphi \\ &= 0 \end{aligned}$$

Critical points of E:
harmonic maps

Bienergy functional

$$E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 \, \overline{v}_g$$

Euler-Lagrange equation

$$\begin{aligned} \tau_2(\varphi) &= -\Delta^{\varphi} \tau(\varphi) \\ &\quad - \operatorname{trace} R^N(d\varphi(\cdot), \tau(\varphi)) d\varphi(\cdot) \\ &= 0 \end{aligned}$$

Critical points of E_2:
biharmonic maps

Any harmonic map is biharmonic. We will say that a map is proper biharmonic if it is biharmonic but not harmonic.

Conventions

Sign Conventions:

- $\Delta\sigma = -\operatorname{trace}\left(\nabla^2\sigma\right)$,
- $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$.

For a submanifold M^m in N^n we will use the following notations:

- B is the second fundamental form of M in N,
- $H = \frac{1}{m}\operatorname{trace} B$ is the mean curvature vector field.

If M^m is a hypersurface in N^{m+1} we also have:

- A is the shape operator of M in the direction of $\eta \in C(NM)$, $|\eta| = 1$,
- $f = \frac{1}{m}\operatorname{trace} A$ is the mean curvature function.

Definitions

Let $\varphi : M^m \to N^n$ be an immersion.
Then φ is said to have

1. constant mean curvature (CMC) if $|H| = constant$;
2. parallel mean curvature vector field (PMC) if $\nabla^{\perp} H = 0$.

We say that φ is λ-biharmonic if there is $\lambda \in \mathbb{R}$ such that

$$\tau_2(\varphi) = \lambda \tau(\varphi).$$

Remark

Any PMC submanifold is CMC.

The biharmonic equation for submanifolds in space forms

Let $\varphi : M^m \to N^n(c)$ be a submanifold in a space form. Then

$$\tau(\varphi) = mH, \quad \tau_2(\varphi) = -m\Delta^\varphi H + m^2 cH,$$

thus φ is biharmonic if and only if $\Delta^\varphi H = mcH$.

1. The submanifold φ is biharmonic if and only if

$$\begin{cases} \Delta^\perp H + \operatorname{trace} B(\cdot, A_H(\cdot)) - mcH = 0 \\ 2 \operatorname{trace} A_{\nabla^\perp_{(\cdot)} H}(\cdot) + \frac{m}{2} \operatorname{grad} |H|^2 = 0 \end{cases}.$$

2. If M^m is a PMC submanifold of $N^n(c)$, then M is biharmonic if and only if

$$\operatorname{trace} B(\cdot, A_H(\cdot)) = mcH.$$

Main examples of biharmonic submanifolds in $\mathbb{S}^n$ (Caddeo, Montaldo, Oniciuc-2001, 2002)

The composition property

$$\mathbb{S}^{n-1}(a) \xrightarrow{\textit{p. biharmonic}} \mathbb{S}^n$$

$$\Updownarrow$$

$$a = \frac{1}{\sqrt{2}}$$

$$\begin{array}{ccc} M^m & \xrightarrow{\textit{minimal}} & \mathbb{S}^{n-1}\left(\frac{1}{\sqrt{2}}\right) \\ & \searrow^{\textit{p. biharmonic}} & \downarrow \varphi \\ & & \mathbb{S}^n \end{array}$$

Properties

- M has parallel mean curvature vector field, and $|H| = 1$.
- M is pseudo-umbilical in $\mathbb{S}^n$, i.e. $A_H = |H|^2 \operatorname{Id}$; $\nabla A_H = 0$.

Main examples of biharmonic submanifolds in $\mathbb{S}^n$ (Jiang-1986)

The product composition property

$$\mathbb{S}^{n_1}(r_1) \times \mathbb{S}^{n_2}(r_2) \xrightarrow{p.\ biharmonic} \mathbb{S}^n$$

$$\Updownarrow$$

$$r_1 = r_2 = \frac{1}{\sqrt{2}} \quad and \quad n_1 \neq n_2$$

$$n_1 + n_2 = n - 1, \quad r_1^2 + r_2^2 = 1$$

$$M_1^{m_1} \times M_2^{m_2} \xrightarrow{minimal} \mathbb{S}^{n_1}\left(\frac{1}{\sqrt{2}}\right) \times \mathbb{S}^{n_2}\left(\frac{1}{\sqrt{2}}\right) \xrightarrow{\varphi} \mathbb{S}^n$$

$$M_1^{m_1} \times M_2^{m_2} \xrightarrow{p.\ biharmonic} \mathbb{S}^n$$

$$n_1 + n_2 = n - 1 \quad and \quad m_1 \neq m_2$$

Properties

- $M_1 \times M_2$ has parallel mean curvature vector field and $|H| \in (0, 1)$.
- $M_1 \times M_2$ is not pseudo-umbilical in $\mathbb{S}^n$; $\nabla A_H = 0$.

The range of the mean curvature (Oniciuc-2003)

Theorem

Let $\varphi : M^m \to \mathbb{S}^n$ *be a* CMC *proper biharmonic immersion. Then*

$$|H| \in (0, 1].$$

Moreover, $|H| = 1$ *if and only if* M *is minimal in the hypersphere* $\mathbb{S}^{n-1}\left(1/\sqrt{2}\right) \subset \mathbb{S}^n$.

Gap Result (Balmuș, Oniciuc-2012)

Theorem

Let $\varphi : M^m \to \mathbb{S}^n$ be a PMC proper biharmonic immersion. Suppose that $m > 2$ and $|H| \in (0, 1)$. Then

$$|H| \in \left(0, \frac{m-2}{m}\right]$$

and $|H| = (m-2)/m$ if and only if locally $\varphi(M)$ is an open subset of a product

$$M_1 \times \mathbb{S}^1\left(\frac{1}{\sqrt{2}}\right) \subset \mathbb{S}^n,$$

where M_1 is a minimal submanifold embedded in $\mathbb{S}^{n-2}\left(1/\sqrt{2}\right)$ and the splitting $\mathbb{R}^{n+1} = \mathbb{R}^{n-1} \oplus \mathbb{R}^2$ does not depend on the point of M. Moreover, if M is complete, then the previous decomposition of $\varphi(M)$ is global, where M_1 is a complete minimal submanifold in $\mathbb{S}^{n-2}\left(1/\sqrt{2}\right)$.

The main problem

Next, we aim to further enlarge the gap for the range of the mean curvature of PMC biharmonic submanifolds in spheres, imposing additional properties to the submanifold. More precisely, we consider the submanifold M in $\mathbb{S}^n$ to be the extrinsic product, with respect to some splitting of $\mathbb{R}^{n+1}$, of two submanifolds, i.e.

We consider $M = M_1 \times M_2$ in $\mathbb{S}^n$ such that M_1 is PMC in $\mathbb{S}^{n_1}(r_1)$ and M_2 is PMC in $\mathbb{S}^{n_2}(r_2)$, where $n_1 + n_2 = n - 1$ and $r_1^2 + r_2^2 = 1$.

Theorem

Let $\varphi_1 : M_1^{m_1} \to \mathbb{S}^{n_1}(r_1)$ *and* $\varphi_2 : M_2^{m_2} \to S^{n_2}(r_2)$ *be two immersions, where* $r_1^2 + r_2^2 = 1$ *and* $n_1 + n_2 = n - 1$*. Then* $M = M_1 \times M_2$ *is a proper biharmonic submanifold in* $\mathbb{S}^n$ *if and only if*

$$\begin{cases} |\tau(\varphi_1)| > 0 \quad or \quad |\tau(\varphi_2)| > 0 \quad or \quad \frac{m_1}{r_1^2} \neq \frac{m_2}{r_2^2} \\ \tau_2(\varphi_1) = 2r_2^2 \left(\frac{m_1}{r_1^2} - \frac{m_2}{r_2^2} \right) \tau(\varphi_1) \\ \tau_2(\varphi_2) = 2r_1^2 \left(\frac{m_2}{r_2^2} - \frac{m_1}{r_1^2} \right) \tau(\varphi_2) \\ \frac{|\tau(\varphi_1)|^2}{r_1^2} - \frac{|\tau(\varphi_2)|^2}{r_2^2} + \left(r_2^2 - r_1^2 \right) \left(\frac{m_2}{r_2^2} - \frac{m_1}{r_1^2} \right)^2 = 0 \end{cases}.$$

Remark

- Note that φ_1 is λ_1-biharmonic and φ_2 is λ_2-biharmonic, where $\lambda_1 = 2r_2^2 \left(\frac{m_1}{r_1^2} - \frac{m_2}{r_2^2} \right)$ and $\lambda_2 = 2r_1^2 \left(\frac{m_2}{r_2^2} - \frac{m_1}{r_1^2} \right)$.
- Moreover, φ_1 and φ_2 are CMC.

Range of the mean curvature for CMC λ-biharmonic submanifolds

Theorem

Let $\varphi : M^m \to \mathbb{S}^n(r)$ be a CMC proper λ-biharmonic immersion. Then

$$\lambda < \frac{m}{r^2} \quad \textit{and} \quad |H| \in \left(0, \sqrt{\frac{1}{r^2} - \frac{\lambda}{m}}\right].$$

Moreover, $|H| = \sqrt{1/r^2 - \lambda/m}$ if and only if M is minimal in the small hypersphere $\mathbb{S}^{n-1}\left(\sqrt{mr^2/(2m - \lambda r^2)}\right) \subset \mathbb{S}^n(r)$.

Theorem

Let $\varphi : M^m \to \mathbb{S}^n(r)$ be a PMC proper λ-biharmonic immersion. Suppose that $m > 2$ and $|H|^2 \in (0, 1/r^2 - \lambda/m)$. Then

$$\lambda \leq \frac{m - 2\sqrt{m-1}}{r^2}$$

and

$$|H|^2 \in \begin{cases} (0, x_2]\,, & \text{if } \lambda \leq 0 \\ [x_1, x_2], & \text{if } \lambda \in \left(0, \frac{m-2\sqrt{m-1}}{r^2}\right) \\ \{x_2\}\,, & \text{if } \lambda = \frac{m-2\sqrt{m-1}}{r^2} \end{cases} ,$$

where x_1 and x_2 are the positive solutions of the following second degree equation

$$m^4x^2 + m^2\left(m\lambda - \frac{(m-2)^2}{r^2}\right)x + (m-1)\lambda^2 = 0.$$

Moreover, we have

- $|H|^2 = x_2$ if and only if locally $M = L \times I$, $\varphi = \varphi_1 \times \varphi_2$ is an extrinsic product, $\varphi_1 : L \to \mathbb{S}^{n-2}(r_1)$ is minimal and $\varphi_2 : I \to \mathbb{S}^1(r_2)$ is (an open subset of) the circle parametrized by arc length, where $I = (-\varepsilon, \varepsilon)$ and the radii are given by

$$\begin{cases} r_1^2 = \dfrac{3m - 2 - \lambda r^2 - \sqrt{(m - \lambda r^2)^2 - 4(m-1)}}{2(2m - \lambda r^2)} r^2 \\[2ex] r_2^2 = \dfrac{m + 2 - \lambda r^2 + \sqrt{(m - \lambda r^2)^2 - 4(m-1)}}{2(2m - \lambda r^2)} r^2 \end{cases}.$$

  If M is complete, the above decompositions of M and φ hold globally, $M = L \times \mathbb{R}$ and $\varphi_2(\mathbb{R}) = \mathbb{S}^1(r_2)$.

- if $\lambda > 0$, then $|H|^2 = x_1$ if and only if locally $M = L \times I$, $\varphi = \varphi_1 \times \varphi_2$ is an extrinsic product, $\varphi_1 : L \to \mathbb{S}^{n-2}(r_1)$ is minimal and $\varphi_2 : I \to \mathbb{S}^1(r_2)$ is (an open subset of) the circle parametrized by arc length, where $I = (-\varepsilon, \varepsilon)$ and the radii are given by

$$\begin{cases} r_1^2 = \dfrac{3m - 2 - \lambda r^2 + \sqrt{(m - \lambda r^2)^2 - 4(m-1)}}{2(2m - \lambda r^2)} r^2 \\ r_2^2 = \dfrac{m + 2 - \lambda r^2 - \sqrt{(m - \lambda r^2)^2 - 4(m-1)}}{2(2m - \lambda r^2)} r^2 \end{cases}.$$

  If M is complete, the above decompositions of M and φ hold globally, $M = L \times \mathbb{R}$ and $\varphi_2(\mathbb{R}) = \mathbb{S}^1(r_2)$.

Recall that our main target is to find a gap for the range of the mean curvature of proper biharmonic submanifolds M of $\mathbb{S}^n$ such that

$M = M_1 \times M_2$, M_1 is PMC in $\mathbb{S}^{n_1}(r_1)$ and M_2 is PMC in $\mathbb{S}^{n_2}(r_2)$, where $n_1 + n_2 = n$ and $r_1^2 + r_2^2 = 1$.

Theorem

Let $\varphi_1 : M_1^{m_1} \to \mathbb{S}^{n_1}(r_1)$ *and* $\varphi_2 : M_2^{m_2} \to S^{n_2}(r_2)$ *be two immersions, where* $r_1^2 + r_2^2 = 1$ *and* $n_1 + n_2 = n - 1$. *Then* $M = M_1 \times M_2$ *is a proper biharmonic submanifold in* $\mathbb{S}^n$ *if and only if*

$$\begin{cases} |\tau(\varphi_1)| > 0 \quad or \quad |\tau(\varphi_2)| > 0 \quad or \quad \frac{m_1}{r_1^2} \neq \frac{m_2}{r_2^2} \\ \tau_2(\varphi_1) = 2r_2^2 \left(\frac{m_1}{r_1^2} - \frac{m_2}{r_2^2} \right) \tau(\varphi_1) \\ \tau_2(\varphi_2) = 2r_1^2 \left(\frac{m_2}{r_2^2} - \frac{m_1}{r_1^2} \right) \tau(\varphi_2) \\ \frac{|\tau(\varphi_1)|^2}{r_1^2} - \frac{|\tau(\varphi_2)|^2}{r_2^2} + \left(r_2^2 - r_1^2\right) \left(\frac{m_2}{r_2^2} - \frac{m_1}{r_1^2} \right)^2 = 0 \end{cases}.$$

Remark

- Note that φ_1 is λ_1-biharmonic and φ_2 is λ_2-biharmonic, where $\lambda_1 = 2r_2^2 \left(\frac{m_1}{r_1^2} - \frac{m_2}{r_2^2} \right)$ and $\lambda_2 = 2r_1^2 \left(\frac{m_2}{r_2^2} - \frac{m_1}{r_1^2} \right)$.
- Moreover, φ_1 and φ_2 are CMC.

The radii r_1 and r_2 must obey the following constraints

$$\frac{m_1}{2m} < r_1^2 < \frac{2m_1 + m_2}{2m} \quad \text{and} \quad \frac{m_2}{2m} < r_2^2 < \frac{m_1 + 2m_2}{2m}.$$

Moreover, if M_1 and M_2 are PMC, we obtain even more restrictive ranges for the radii r_1 and r_2, that is

$$\begin{cases} \dfrac{m_1 + 2\sqrt{m_1 - 1}}{2m} \leq r_1^2 \leq \dfrac{2m_1 + m_2 - 2\sqrt{m_2 - 1}}{2m} \\ \dfrac{m_2 + 2\sqrt{m_2 - 1}}{2m} \leq r_2^2 \leq \dfrac{m_1 + 2m_2 - 2\sqrt{m_1 - 1}}{2m} \end{cases}.$$

Theorem

Let $\varphi_1 : M_1^{m_1} \to \mathbb{S}^{n_1}(r_1)$ *and* $\varphi_2 : M_2^{m_2} \to \mathbb{S}^{n_2}(r_2)$ *be two non-minimal immersions such that* $M^m = M_1^{m_1} \times M_2^{m_2}$ *is proper biharmonic in* $\mathbb{S}^n$*, where* $n_1 + n_2 = n - 1$*. Then*

$$|H^{\varphi_1}|^2 = \frac{1}{r_1^2} - \frac{\lambda_1}{m_1} \quad \textit{if and only if} \quad |H^{\varphi_2}|^2 = \frac{1}{r_2^2} - \frac{\lambda_2}{m_2}.$$

In this case, $|H^{\iota\circ\varphi}|^2 = 1$*.*

Remark

Since φ_1 and φ_2 are CMC, then $\frac{1}{r_1^2} - \frac{\lambda_1}{m_1}$ and $\frac{1}{r_2^2} - \frac{\lambda_2}{m_2}$ are their respective maximum values.

Theorem

Let $\varphi_1 : M_1^{m_1} \to \mathbb{S}^{n_1}(r_1)$ *and* $\varphi_2 : M_2^{m_2} \to \mathbb{S}^{n_2}(r_2)$ *be two PMC and non-pseudo-umbilical immersions,* $m_1 > 2$, $m_2 > 2$, *such that* $M^m = M_1^{m_1} \times M_2^{m_2}$ *is proper biharmonic in* $\mathbb{S}^n$, $r_1^2 + r_2^2 = 1$ *and* $n_1 + n_2 = n - 1$. *Then*

$$|H^{\iota\circ\varphi}| \in \left(0, \frac{m-4}{m}\right]$$

and $|H^{\iota\circ\varphi}| = (m-4)/m$ *if and only if* $r_1^2 = (3m_1 + m_2 - 4)/(4(m-2))$ *and locally, up to isometries of* $\mathbb{S}^n$, $\varphi(M)$ *is an open subset of the extrinsic product*

$$\tilde{M}_1 \times \mathbb{S}^1\left(\frac{1}{2}\right) \times \mathbb{S}^1\left(\frac{1}{2}\right) \subset \mathbb{S}^{n-4}\left(\frac{1}{\sqrt{2}}\right) \times \mathbb{S}^3\left(\frac{1}{\sqrt{2}}\right),$$

where $\tilde{M}_1$ *is minimal in* $\mathbb{S}^{n-4}(1/\sqrt{2})$ *and the splitting of* $\mathbb{R}^{n+1} = \mathbb{R}^{n-3} \times \mathbb{R}^4$ *does not depend on the point of* M. *Moreover, if* M *is complete, then the previous decomposition of* $\varphi(M)$ *is global, where* $\tilde{M}_1$ *is a complete minimal submanifold in* $\mathbb{S}^{n-4}\left(1/\sqrt{2}\right)$.

Proof sketch

Using a previous theorem, the expression of $|H^{\iota\circ\varphi}|$ and the fact that M_1 is PMC λ_1-biharmonic and M_2 is PMC λ_2-biharmonic, we deduce that

$$|H^{\varphi_1}|^2 \leq x_{2,1}, \quad |H^{\varphi_2}|^2 \leq x_{2,2}$$

and $|H^{\iota\circ\varphi}|$ reaches its maximum when $M_1 = L_1 \times \mathbb{S}^1(b_1)$ and $M_2 = L_2 \times \mathbb{S}^1(b_2)$, where L_1 is minimal in $\mathbb{S}^{n_1-2}(a_1)$ and L_2 is minimal in $\mathbb{S}^{n_2-2}(a_2)$, and a_1, b_1, a_2, b_2 satisfy the conditions $a_i^2 + b_i^2 = r_i^2$, $i \in \{1, 2\}$ and satisfy certain conditions.
Thus, it is sufficient to search for

$$M = L_1 \times \mathbb{S}^1(b_1) \times L_2 \times \mathbb{S}^1(b_2),$$

where $r_1^2 + r_2^2 = 1$.

From the biharmonic equation for PMC submanifolds, we obtain the following set of algebraic equations.

$$\begin{cases} \alpha_1\left(d-2m^2\right)+2m-\dfrac{1}{\alpha_1}=0 \\ \alpha_2\left(d-2m^2\right)+2m-\dfrac{1}{\alpha_2}=0 \\ \alpha_3\left(d-2m^2\right)+2m-\dfrac{1}{\alpha_3}=0 \\ \alpha_4\left(d-2m^2\right)+2m-\dfrac{1}{\alpha_4}=0 \end{cases},$$

where

$$\alpha_1=\frac{a_1^2}{m_1-1},\quad \alpha_2=b_1^2,\quad \alpha_3=\frac{a_2^2}{m_2-1},\quad \alpha_4=b_2^2$$

and

$$d=\frac{(m_1-1)^2}{a_1^2}+\frac{1}{b_1^2}+\frac{(m_2-1)^2}{a_2^2}+\frac{1}{b_2^2}.$$

Solving that system of equations, we obtain the following solutions.

1. $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1/m$,
2. $\alpha_1 = \alpha_2 = 1/(2m_1)$ and $\alpha_3 = \alpha_4 = 1/(2m_2)$,
3. $\alpha_1 = \alpha_3 = 1/(2(m-2))$ and $\alpha_2 = \alpha_4 = 1/4$,
4. $\alpha_1 = \alpha_4 = 1/(2m_1)$ and $\alpha_2 = \alpha_3 = 1/(2m_2)$,
5. $\alpha_1 = \alpha_2 = \alpha_3 = 1/(2(m-1))$ and $\alpha_4 = 1/2$,
6. $\alpha_1 = \alpha_2 = \alpha_4 = 1/(2(m_1+1))$ and $\alpha_3 = 1/(2(m_2-1))$,
7. $\alpha_1 = \alpha_3 = \alpha_4 = 1/(2(m-1))$ and $\alpha_2 = 1/2$,
8. $\alpha_2 = \alpha_3 = \alpha_4 = 1/(2(m_2+1))$ and $\alpha_1 = 1/(2(m_1-1)$.

Taking into account that M_1 and M_2 are not minimal, i.e. $\alpha_1 \neq \alpha_2$ and $\alpha_3 \neq \alpha_4$, this list narrows down to 3 and 4.

In case 3, we obtain

$$r_1^2 = \frac{3m_1 + m_2 - 4}{4(m-2)} \quad \text{and} \quad |H^{\iota\circ\varphi}|^2 = \frac{(m-4)^2}{m^2}.$$

In case 4, we obtain

$$r_1^2 = \frac{m_1 m_2 - m_2 + m_1}{2m_1 m_2} \quad \text{and} \quad |H^{\iota\circ\varphi}|^2 = \frac{(m_1 - m_2)^2}{m^2}.$$

Next, we show that $\frac{(m-4)^2}{m^2}$ is the maximum value of $|H^{\iota\circ\varphi}|^2$ for any r_1.

Using the characterization of biharmonic submanifolds, the expression of $\left|H^{\iota\circ\varphi}\right|^2$ and the bounds of $\left|H^{\varphi_1}\right|^2$ and $\left|H^{\varphi_2}\right|^2$ we obtain that

$$\left|H^{\iota\circ\varphi}\right|^2 \leq h_1\left(r_1^2\right) \quad \text{and} \quad \left|H^{\iota\circ\varphi}\right|^2 \leq h_2\left(r_1^2\right),$$

where h_1 is an increasing function and h_2 is a decreasing function and satisfy

$$h_1\left(\frac{3m_1+m_2-4}{4(m-2)}\right) = h_2\left(\frac{3m_1+m_2-4}{4(m-2)}\right) = \frac{(m-4)^2}{m^2}.$$

From here we find a better upper bound, that is, for any r_1 we have

$$\left|H^{\iota\circ\varphi}\right|^2 \leq \min\left\{h_1\left(r_1^2\right), h_2\left(r_1^2\right)\right\} \leq \frac{(m-4)^2}{m^2}.$$

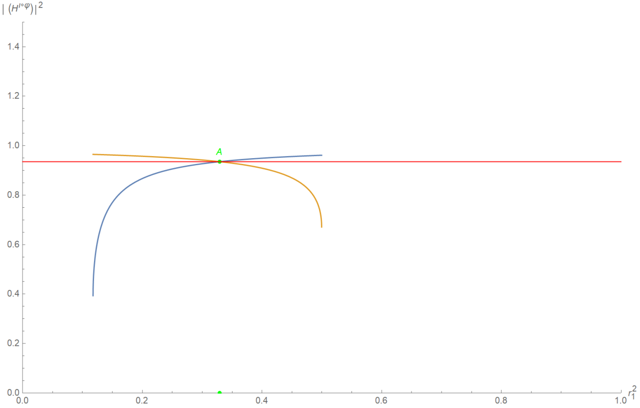

Figure: $m_1 = 20$ and $m_2 = 102$

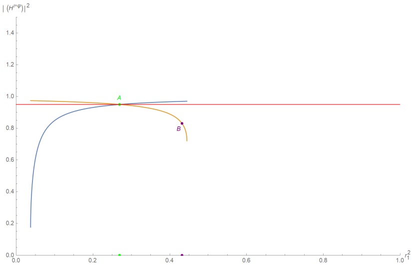

Figure: $m_1 = 7$ and $m_2 = 151$

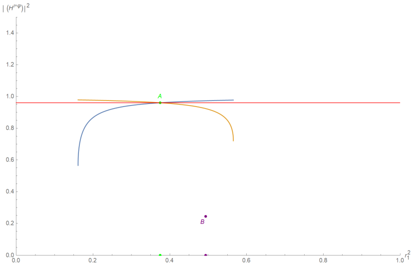

Figure: $m_1 = 51$ and $m_2 = 151$

Using the classification of parallel surfaces in Euclidean spheres, we can treat the cases when M_1 or M_2 is a surface. Thus, we obtain the following theorem which summarizes the results presented above.

Theorem

Let $\varphi_1 : M_1^{m_1} \to \mathbb{S}^{n_1}(r_1)$, $m_1 \geq 2$, and $\varphi_2 : M_2^{m_2} \to \mathbb{S}^{n_2}(r_2)$, $m_2 \geq 2$, be two non-minimal and PMC submanifolds such that $M^m = M_1^{m_1} \times M_2^{m_2}$ is proper biharmonic in $\mathbb{S}^n$, $r_1^2 + r_2^2 = 1$ and $n_1 + n_2 = n - 1$.

1. *If $m_1 = m_2 = 2$, then φ_1 and φ_2 are pseudo-umbilical, that is*

$$|H^{\iota\circ\varphi}| = 1.$$

2. *If $m_1 > 2$ or $m_2 > 2$, then*

$$|H^{\iota\circ\varphi}| \in \left(0, \frac{m-4}{m}\right] \cup \{1\}.$$

Open Problem

Prove that for any r_1 such that

$$r_1^2 \in \left[\frac{m_1 + 2\sqrt{m_1 - 1}}{2m}, \frac{2m_1 + m_2 - 2\sqrt{m_2 - 1}}{2m}\right]$$

there exists $M = M_1 \times M_2$ proper biharmonic in $\mathbb{S}^n$ such that M_i are PMC and non-pseudo-umbilical in $\mathbb{S}^{n_i}(r_i)$, $i \in \{1, 2\}$, $r_1^2 + r_2^2 = 1$, $m_1 > 2$, $m_2 > 2$.

Thank you!