The geometry of harmonic maps into the unitary group

Differential Geometry Workshop 2023

Iași, Romania

John C. Wood, *University of Leeds, GB*

`j.c.wood@leeds.ac.uk`

`http://www1.maths.leeds.ac.uk/pure/staff/wood/wood.html`

Harmonic maps

Let $\varphi : (M, g) \to (N, h)$ be a smooth map between smooth Riemannian manifolds. For M compact, the **energy** or **Dirichlet integral** of φ is

$$E(\varphi) = \int_M e(\varphi)\,\omega_g = \int_M \tfrac{1}{2}|\mathrm{d}\varphi|^2\,\omega_g$$

where ω_g = volume measure and, for any $p \in M$,

$$|\mathrm{d}\varphi_p|^2 = \text{Hilbert–Schmidt square norm of } \mathrm{d}\varphi_p = g^{ij} h_{\alpha\beta}\,\varphi_i^\alpha \varphi_j^\beta\,.$$

The map φ is called **harmonic** if the first variation of E for variations φ_t of the map φ vanishes at φ, i.e., $\frac{\mathrm{d}}{\mathrm{d}t}E(\varphi_t)\big|_{t=0} = 0$.

$$\frac{\mathrm{d}}{\mathrm{d}t}E(\varphi_t)\big|_{t=0} = -\int_M \langle \tau(\varphi), v\rangle\,\omega_g$$

where $v = \partial\varphi_t/\partial t|_{t=0}$ is the **variation vector field** of (φ_t), and $\tau(\varphi)$ is called the **tension field of** φ.

Examples of harmonic maps

In general, a smooth map $\varphi : M \to N$ is harmonic iff it satisfies the **harmonic** (or **tension field**) **equation**: $\tau(\varphi) = 0$ where

$$\tau(\varphi) \;=\; \mathrm{div}\ \mathrm{d}\varphi = \mathrm{Trace}\,\nabla \mathrm{d}\varphi.$$

1. **Harmonic functions**: $\varphi : \mathbb{R}^m \supseteq U \to \mathbb{R}^n$ is harmonic iff $\Delta\varphi = 0$ where $\Delta =$ usual Laplacian on $\mathbb{R}^m$.

More generally, $\varphi : (M, g) \to \mathbb{R}^n$ is harmonic iff $\Delta^M \varphi = 0$ where $\Delta^M =$ is the (linear) Laplace–Beltrami operator on (M, g).

2. **Geodesics**: $\varphi : \mathbb{R} \supseteq U \to N$ or $S^1 \to N$ is harmonic iff it defines a geodesic parametrized linearly. 3. **Harmonic morphisms**, see ??

4. $\pm$-**Holomorphic** maps between Kähler manifolds; in fact they give absolute minima of the energy functional.

5. **Minimal submanifolds**; for maps from surfaces, we can allow **branch points** in the sense of [R.D. Gulliver, R. Osserman and H.L. Royden, *A theory of branched immersions of surfaces*][1].

[1] Amer. J. Math. **95** (1973), 750–812.

Harmonic maps from surfaces

For a map $\varphi : M \to N$ from a surface, the energy is unchanged by conformal change of metric on the domain, so we can talk about **harmonic maps from Riemann surfaces**. Then the harmonic map equation becomes, in a local complex coordinate z,

$$\nabla^{\varphi}_{\bar{z}} \frac{\partial \varphi}{\partial z} = 0, \quad \text{equivalently} \quad \nabla^{\varphi}_{z} \frac{\partial \varphi}{\partial \bar{z}} = 0.$$

where ∇^{φ} is the pull-back of the Levi-Civita connection on N. When $M = \mathbb{C}$ and $N = \mathbb{R}^n$ this reduces to the familiar way of writing Laplace's equation:

$$\frac{\partial}{\partial \bar{z}} \frac{\partial \varphi}{\partial z} = 0, \quad \text{equivalently} \quad \frac{\partial}{\partial z} \frac{\partial \varphi}{\partial \bar{z}} = 0.$$

These equations say that **the partial derivative $\frac{\partial \varphi}{\partial z}$ (resp. $\frac{\partial \varphi}{\partial \bar{z}}$) is holomorphic (resp. antiholomorphic)**.

Maps into Kähler manifolds

From now on, M will denote a Riemann surface.

We can decompose any complexified tangent vector $v \in T^c N$ to a almost complex manifold into $(1,0)$ and $(0,1)$ parts, then

A smooth map $\varphi : M \to N$ into a Kähler manifold is harmonic iff

$$\nabla^{\varphi}_{\bar{z}} \frac{\partial^{(1,0)}\varphi}{\partial z} = 0, \quad \textit{equivalently} \quad \nabla^{\varphi}_{z} \frac{\partial^{(1,0)}\varphi}{\partial \bar{z}} = 0. \tag{1}$$

We view $\frac{\partial^{(1,0)}\varphi}{\partial z}$ as a (local) section of the pull-back bundle $\varphi^{-1} T^{(1,0)} N$. Then the first of equations (1) says:

The $(1,0)$-part of the partial derivative $\frac{\partial \varphi}{\partial z}$ is holomorphic w.r.t. the Koszul–Malgrange holomorphic structure[2] on $\varphi^{-1} T^{(1,0)} N$, i.e., the holomorphic structure with $\overline{\partial}$-operator $\nabla^{\varphi}_{\bar{z}}$.

---

[2] J. L. Koszul and B. Malgrange, *Sur certaines structures fibrées complexes*, Arch. Math. 9 (1958) 102–109.

Maps into Grassmannians[4]

A (smooth) map $\varphi : M \to G_k(\mathbb{C}^n)$ can be represented by a (smooth) subbundle $\underline{\varphi}$ of the trivial bundle $\underline{\mathbb{C}}^n := M \times \mathbb{C}^n$ given by the pullback via φ of the tautological bundle; $\underline{\varphi}$ has fibre at $x \in M$ given by the subspace $\varphi(x)$. Then

$$\varphi^{-1} T^{(1,0)} G_k(\mathbb{C}^n) \textit{ is naturally isomorphic to } L(\underline{\varphi}, \underline{\varphi}^{\perp})\ ^3.$$

The **derivatives** of φ w.r.t. z and to $\bar{z}$ correspond to locally defined linear bundle maps $A'_\varphi, A''_\varphi, : \underline{\varphi} \to \underline{\varphi}^{\perp}$ given by

$$A'_\varphi(s) = \pi_{\underline{\varphi}}^{\perp} \partial_z s \quad \text{and} \quad A''_\varphi(s) = \pi_{\underline{\varphi}}^{\perp} \partial_{\bar{z}} s \quad (s \in \Gamma(\underline{\varphi}));$$

we call these the **second fundamental forms of** φ **(in** $\underline{\mathbb{C}}^n$**)**. Then

φ is harmonic iff A'_φ is holomorphic iff A''_φ is antiholomorphic.

---

[3] F.E. Burstall and J.H. Rawnsley, *Twistor theory for Riemannian symmetric spaces*, Lecture Notes in Mathematics, 1424, Springer-Verlag, Berlin, Heidelberg (1990).

[4] F.E. Burstall and JCW, *The construction of harmonic maps into complex Grassmannians*, J. Diff. Geom. **23** (1986), 255–298.

Gauss transforms

If $\varphi : M \to G_k(\mathbb{C}^n)$ is a harmonic map, by the above $\pm$-holomorphicity, the images of A'_φ and A''_φ can be extended over points where A'_φ and A''_φ do not have their maximal rank giving globally well-defined smooth subbundles $G'(\varphi) := \operatorname{Im} A'_\varphi$ and $G''(\varphi) := \operatorname{Im} A''_\varphi$ of $\underline{\mathbb{C}}^n$ called the **∂'- and ∂''-Gauss bundles of φ**. These define maps into Grassmannians sometimes called **Gauss transforms** or **∂- and $\overline{\partial}$-transforms**[5].

If φ is harmonic, so are its Gauss transforms.

This is an early example of K. Uhlenbeck's **adding a uniton**. Note that the two sorts of Gauss transform are mutually inverse when the bundles concerned are of the same rank. We can iterate this construction to define the ***r*-th ∂'- and ∂''-Gauss bundles** by

$$G^{(1)}(\varphi) = G'(\varphi), \quad G^{(r+1)}(\varphi) = G'(G^{(r)}(\varphi))$$

and

$$G^{(-1)}(\varphi) = G'(\varphi), \quad G^{(-r-1)}(\varphi) = G''(G^{(-r)}(\varphi)).$$

[5] S.-S. Chern and J. G. Wolfson, *Harmonic maps of the two-sphere in to a complex Grassmannm manifold*, Ann. Math. **125** (1987), 301–335.

'Harmonic' Diagrams

Generalizing the definitions of A'_φ and A''_φ, given any orthogonal subbundles $\underline{\varphi}$, $\underline{\psi}$, and local complex coordinate z, we define the **second fundamental form of $\underline{\varphi}$ in $\underline{\varphi} \oplus \underline{\psi}$** by

$$A'_{\varphi,\psi}(s) = \pi_\psi \partial_z s \quad \text{and} \quad A''_{\varphi,\psi}(s) = \pi_\psi \partial_{\bar{z}} s \quad (s \in \Gamma(\underline{\varphi})).$$

Note that $A'_{\varphi,\psi}\mathrm{d}z$ and $A''_{\varphi,\psi}\mathrm{d}\bar{z}$ are globally well-defined 1-forms.

Then, by a **diagram**[6] we mean a set of mutually orthogonal subbundles ψ_i of $\underline{\mathbb{C}}^n$ with sum $\underline{\mathbb{C}}^n$ together with second fundamental forms A'_{ψ_i,ψ_j} for some of the pairs (i,j), $i \neq j$.

We represent this by a directed graph with vertices representing the ψ_i and the arrow (i.e. directed edge) from ψ_i to ψ_j representing A'_{ψ_i,ψ_j}, with no arrow shown if this is known to be zero.

---

[6] See [Burstall–Wood]. Suggested by S. Salamon, cf. *Harmonic and holomorphic maps*, Geometry Seminar, Luigi Bianchi II, Lecture Notes in Math. **1164**, Springer.

Examples of diagrams

The simplest non-trivial diagram is

$$\begin{tikzcd} \varphi \arrow[r, "A'_{\varphi}"'] & \varphi^{\perp} \arrow[l, bend right, "A'_{\varphi^{\perp}}"'] \end{tikzcd} \tag{2}$$

Recall that *the second fundamental forms* A'_{φ} *and* $A'_{\varphi^{\perp}}$ *are holomorphic if and only if* φ *is harmonic*.

We can generalize the above diagram to a cyclic diagrams with 3 or more vertices, e.g. with 4 vertices:

$$\begin{tikzcd} \psi_4 \arrow[d] & \psi_3 \arrow[l] \\ \psi_1 \arrow[r] & \psi_2 \arrow[u] \end{tikzcd} \tag{3}$$

CLAIM: In a cyclic diagram with ℓ vertices, all the arrows (second fundamental forms) $A'_{\psi_i} = A'_{\psi_i, \psi_{i+1 \bmod \ell}}$ are holomorphic and all vertices are harmonic. We explain why:

Test for holomorphicity

Proposition (= Proposition 1.5, Burstall–Wood[7])

Given a diagram $\{\psi_i, A_{\psi_i,\psi_j}\}$, $A'_{\psi_i,\psi_j} : \psi_i \to \psi_j$ *is holomorphic if the diagram contains no configurations of the following forms:*

$$
\begin{tikzcd}
\psi_\ell & \\
\psi_i \arrow[u] \arrow[r] & \psi_j \arrow[ul]
\end{tikzcd}
\qquad
\begin{tikzcd}
\psi_\ell \arrow[d] \arrow[dr] & \\
\psi_i \arrow[r] & \psi_j
\end{tikzcd}
\qquad
\begin{tikzcd}
\psi_i \arrow[r, bend right] & \psi_j \arrow[l, bend right]
\end{tikzcd}
\tag{4}
$$

Note that *this test is a sufficient but not necessary condition*; see the 2-vertex example above. However, it is very useful test.

For example, *in a cyclic diagram, all the arrows (second fundamental forms) are holomorphic and so all vertices are harmonic*. We give a famous example:

[7]Please note correction to [Burstall–Wood, Proposition 1.6], see `http://www1.maths.leeds.ac.uk/pure/staff/wood/Burstall-Wood-corrn.pdf`

Example: The Clifford torus

Suppose we have a cyclic diagram:

$$\begin{tikzcd} \varphi_2 \arrow[r] & \varphi_0 \arrow[r] & \varphi_1 \arrow[ll, bend right] \end{tikzcd} \tag{5}$$

As above, we can deduce from [Burstall–Wood, Proposition 1.5] that each arrow is holomorphic, so that each vertex φ_i is harmonic.

Example. Start with the map $\varphi = \varphi_0 : \mathbb{C} \to \mathbb{C}P^2$ with formula $\varphi_0(z) = \left[\mathrm{e}^{z-\overline{z}}, \mathrm{e}^{\omega z - \overline{\omega z}}, \mathrm{e}^{\omega^2 z - \overline{\omega^2 z}}\right]$, where $\omega = \mathrm{e}^{2\pi\mathrm{i}/3}$, and set $\varphi_i = G^{(i)}(\varphi)$ $(i = 0, 1, 2, \dots)$. This map factors to an isometric harmonic map of the torus $\mathbb{C}/\langle 2\pi/\sqrt{3},\ 2\pi\mathrm{i}\rangle$, and its image is a minimal torus called a **Clifford torus**.
Then, $\varphi_i(z) = \left[\mathrm{e}^{z-\overline{z}}, \omega^i \mathrm{e}^{\omega z - \overline{\omega z}}, \omega^{2i} \mathrm{e}^{\omega^2 z - \overline{\omega^2 z}}\right]$; in particular, $G^{(3)}(\varphi) = \varphi$ — such maps with cyclic Gauss transforms, so giving a cyclic diagram, are called **superconformal.**[8]

[8] J. Bolton, F. Pedit and L. M. Woodward, **Minimal surfaces and the affine Toda field model**, J. Reine Angew. Math. **459** (1995), 119–150.

Harmonic maps from the 2-sphere

The composition of holomorphic second fundamental forms $A'_{\varphi_i}\mathrm{d}z$ around a cycle gives a holomorphic differential which must be zero if the domain is the two-sphere. This means that there cannot be any such circuit, which leads to a slick proof of the following result, see Eells–Wood[9] and other papers cited there:

All harmonic maps from $S^2 \to \mathbb{C}P^n$ are obtained from holomorphic maps by applying the ∂'-Gauss transform up to n times.

Proof Since there are no circuits, the *harmonic sequence* given by forming ∂'- and ∂''-Gauss transforms must end in both directions:

$$G^{(-r)}(\varphi) \to \cdots \to G''(\varphi) \to \varphi \to G'(\varphi) \to \cdots \to G^{(s)}(\varphi) \quad (6)$$

with $f := G^{(-r)}(\varphi)$ holomorphic, $g := G^{(s)}(\varphi)$ antiholomorphic and $r + s \leq n$. Then $\varphi = G^{(r)}(f)$.

QUESTION: How does this work for other domains?

---

[9] J. Eells and J. C. Wood, *Harmonic maps from surfaces to complex projective spaces*, Advances in Math. **49** (1983), 217–263.

Harmonic maps into Lie groups

Let $\varphi : M \to G$ **be a smooth map from a Riemann surface to a Lie group.** Set $A = A^\varphi = \frac{1}{2}\varphi^{-1}\mathrm{d}\varphi$. Choose a local complex coordinate z and decompose A^φ into $(1,0)$- and $(0,1)$-parts:

$$A^\varphi = A^\varphi_z \mathrm{d}z + A^\varphi_{\bar{z}} \mathrm{d}\bar{z}\,.$$

Set $D^\varphi = \mathrm{d} + A^\varphi$, a derivation on the trivial bundle $M \times \mathfrak{g}$. Write $D^\varphi_z = \partial_z + A^\varphi_z$ and $D^\varphi_{\bar{z}} = \partial_{\bar{z}} + A^\varphi_{\bar{z}}$ where $\partial_z = \partial/\partial z$ and $\partial_{\bar{z}} = \partial/\partial\bar{z}$. Since A^φ_z corresponds to the partial derivative $\partial\varphi/\partial z$ under the identification of $M \times \mathfrak{g}$ with $\varphi^{-1}TG$, then[10]

φ is harmonic iff A^φ_z is holomorphic with respect to $D^\varphi_{\bar{z}} = \partial_{\bar{z}} + A^\varphi_{\bar{z}}$.

We can deal with **harmonic maps into symmetric spaces** G/K by embedding them in G by the totally geodesic Cartan embedding. For $G_k(\mathbb{C}^n)$, this is $G_k(\mathbb{C}^n) \ni \alpha \mapsto \pi_\alpha - \pi_\alpha^\perp \in \mathrm{U}(n)$.

[10] K. Uhlenbeck, *Harmonic maps into Lie groups: classical solutions of the chiral model*, J. Differential Geom. **30** (1989) 1–50.

Building harmonic maps from unitons

Given a harmonic map $\varphi : M \to \mathrm{U}(n)$, a **uniton** (for φ) is a map $\alpha : M \to G_k(\mathbb{C}^n)$ satisfying:

$$\text{(i) } A^{\varphi}_z(\sigma) \in \Gamma(\alpha)\,; \quad \text{(ii) } D^{\varphi}_{\bar z}(\sigma) \in \Gamma(\alpha) \quad \text{for all } \sigma \in \Gamma(\alpha).$$

(i) says that α is **closed** under the endomorphism A^{φ}_z.
(ii) says it is **holomorphic** w.r.t. $D^{\varphi}_{\bar z}$.
Given a harmonic map $\varphi : M \to \mathrm{U}(n)$ and a uniton α for it, the product $\varphi(\pi_\alpha - \pi^{\perp}_{\alpha})$ is a new harmonic map. A harmonic map which can be expressed as a product of a finite number of unitons is said to be **of finite uniton number**. Uhlenbeck showed that **all harmonic maps from S^2 to $\mathrm{U}(n)$ are of finite uniton number**. The uniton equations can be solved[11] giving an algebraic construction of all harmonic maps of finite uniton number from a surface; see Dai and Terng[12] for another approach.

[11] M. J. Ferreira, B. A. Simões and JCW, *All harmonic 2-spheres in the unitary group, completely explicitly* Math Z. **266** (2010), 953–978.

[12] B. Dai and C.-L. Terng, *Bäcklund transformations, Ward solitons, and unitons*, J. Differential Geom. **75** (2007) 57–108.

Maps into Grassmannians: relationship between A_z^φ and A'_φ

For any smooth map $\varphi : M \to G_k(\mathbb{C}^n)$, composing with the *Cartan embedding* $\iota : G_k(\mathbb{C}^n) \to \mathrm{U}(n)$, $\alpha \mapsto \pi_\alpha - \pi_\alpha^\perp$, gives a map which we shall still denote by $\varphi : M \to \mathrm{U}(n)$. What is the relationship between A_z^φ and A'_φ?

Answer: *A short calculation shows*:

$$A_z^\varphi = -A'_\varphi - A'_{\varphi^\perp}.$$

Recalling $D_z^\varphi = \partial_z + A_z^\varphi$, these give

$$A_z^\varphi(s) = -A'_\varphi(s) = -\pi_\varphi^\perp \partial_z s \text{ and } D_z^\varphi(s) = \pi_\varphi \partial_z s, \quad (s \in \Gamma(\varphi)) \tag{7}$$

and

$$A_z^\varphi(s) = -A'_{\varphi^\perp}(s) = -\pi_\varphi \partial_z s \text{ and } D_z^\varphi(s) = \pi_\varphi^\perp \partial_z s, \quad (s \in \Gamma(\varphi^\perp)) \tag{8}$$

Unitons for maps into Grassmannians

Let $\varphi : M \to G_k(\mathbb{C}^n)$ be a harmonic map. Let β be a subbundle of φ and γ a subbundle of $\varphi^\perp$ We say that the pair (β, γ) satisfies the **replacement conditions** [Burstall-W.] if

1. β is a holomorphic subbundle of φ,
2. γ is a holomorphic subbundle of $\varphi^\perp$,
3. $A'_\varphi(\beta) \subseteq \gamma$ and $A'_{\varphi^\perp}(\gamma) \subseteq \beta$.

Then if φ is harmonic, so is $\widetilde{\varphi} := (\varphi \ominus \beta) \oplus \gamma = (\beta^\perp \cap \varphi) \oplus \gamma$.
We say that $\widetilde{\varphi}$ is obtained from φ by *replacement of* α *by* β.
Example 1. Set $\beta = \varphi$ and $\gamma = \operatorname{Im} A'_\varphi$. These are subbundles of φ and $\varphi^\perp$ which satisfy the replacement conditions; they give $\widetilde{\varphi} = G'(\varphi)$.
Regarding the Grassmannian as embedded in $\mathrm{U}(n)$ via the totally geodesic Cartan embedding, *the replacement conditions say that* $\alpha = \beta \oplus \gamma$ *is a uniton*. We shall see that *all unitons are of this type*.

Nilpotency test for finite uniton number

For any cycle C in a diagram, the **corresponding operator C** is the composition of all second fundamental forms in the cycle.

We consider an arbitrary diagram, where a harmonic map $\varphi : M \to G_k(\mathbb{C}^n)$ is the sum of the subbundles represented by some of the vertices. A cycle is then called **external** if it includes vertices in φ and $\varphi^\perp$.

A cycle on a vertex ψ_0 is called **simple** if no vertices are repeated.

Proposition (Aleman, Pacheco, W.[13])

Suppose that, for some vertex, there is a unique simple external cycle C. If φ is of finite uniton number then the corresponding operator is nilpotent.

---

[13] A. Aleman, R. Pacheco and JCW. *Harmonic maps and shift-invariant subspaces*, Monatsh. Math. **194** (2021), no. 4, 625–656.

A more powerful nilpotency test

Again, we consider an arbitrary diagram, where a harmonic map $\varphi : M \to G_k(\mathbb{C}^n)$ is the sum of the subbundles represented by some of the vertices. We say that a cycle C has **degree** m if it has m external arrows and **type** (ℓ, m) if it has length ℓ and degree m.

Proposition (Pacheco, W.[14])

Let $\varphi : M \to G_k(\mathbb{C}^n)$ be a harmonic map of finite uniton number. Let α be a subbundle of φ (which may not be a vertex of the diagram). Let C be an external cycle of type (ℓ, m) on φ whose corresponding operator $\mathbf{C}$ sends (sections of) α to α.
Then $\mathbf{C}$ restricts to a nilpotent bundle map $\mathbf{C}|_\alpha : \alpha \to \alpha$ if, for each $j \in \mathbb{N}$, any cycle on φ of type $(j\ell, jm)$ is zero on α or is equal to C^j.

[14] R. Pacheco and JCW, *Diagrams and harmonic maps, revisited*, Ann. Mat. Pura Appl. (4)**202** (2023), no.3, 1051–1085.

A Negative Example

.

Recall the diagram for the Clifford torus:

$$\begin{tikzcd} \varphi_2 \arrow[r] & \varphi_0 \arrow[r] & \varphi_1 \arrow[ll, bend right] \end{tikzcd}$$

All the second fundamental forms in the diagram are isomorphisms, and so is their composition. Thus the cycle is not nilpotent so, by the above test:

The Clifford torus is not of finite uniton number.

More generally,

A superconformal map $M \to \mathbb{C}P^n$, i.e., one whose Gauss transforms give a cyclic diagram, is never of finite uniton number.

A positive example

Theorem (Aleman, Pacheco, W.) Any harmonic map $\varphi : M \to \mathbb{C}P^{n-1}$ of finite uniton number is given by $G^{(i)}(f)$ for some holomorphic map $f : M \to \mathbb{C}P^{n-1}$ and some $i \in \{0, 1, \ldots, n-1\}$.

Suppose that there is an $r \geq 1$ such that $G^{(r+1)}(\varphi)$ not orthogonal to φ. We obtain a diagram

$$
\begin{tikzcd}
\varphi \arrow[r, "A'_{\varphi}"'] & G'(\varphi) \arrow[r, "A'_{G^{(1)}(\varphi)}"'] & \cdots & {} \arrow[r, "A'_{G^{(r-2)}(\varphi)}"'] & G^{(r-1)}(\varphi) \arrow[r, "A'_{G^{(r-1)}(\varphi)}"'] & G^{(r)}(\varphi) \arrow[r, "A'_{G^{(r)}(\varphi),R}"'] \arrow[lllll, bend right] & R \arrow[llllll, bend right]
\end{tikzcd}
$$

The second fundamental forms on the inner cycle are all non-zero so that their composition is also. But by our tests, this composition must be nilpotent and so zero.
Hence, there is no such r, so all the Gauss bundles are mutually orthogonal and we get the same diagram, and so the same result, as for the S^2 case.

New approach: $\mathrm{U}(n)$ as a Grassmannian[15]

For any m, let $\mathbb{C}^m$ be equipped with its standard Hermitian metric $\langle\cdot,\cdot\rangle$. Write $\mathbb{C}^{2n} = \mathbb{C}^n \oplus \mathbb{C}^n$, so that, for $X = (X_1, X_2)$, $Y = (Y_1, Y_2) \in \mathbb{C}^n \oplus \mathbb{C}^n$, $\langle X, Y\rangle = \langle X_1, Y_1\rangle + \langle X_2, Y_2\rangle$.

Define $J : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ by $J(X_1, X_2) = (-X_1, X_2)$ and

$$\omega(X, Y) = \langle JX, Y\rangle = \langle X_2, Y_2\rangle - \langle X_1, Y_1\rangle.$$

Define the **complex Lagrangian Grassmannian** $G^L = G_n^{Lag}(\mathbb{C}^{2n})$ by

$$G^L = \{U \in G_n(\mathbb{C}^{2n}) : \omega(U, U) = 0,\ \text{equiv. } (JU)^\perp = U\}.$$

This is a totally geodesic submanifold of $G_n(\mathbb{C}^{2n})$.

The map $K : A \mapsto$ graph of $A = \{(X_1, X_2) \in \mathbb{C}^n \oplus \mathbb{C}^n : X_2 = AX_1\}$ defines an isometry of $\mathrm{U}(n)$ to G^L.

---

[15] V.I. Arnol'd, *The complex Lagrangian Grassmannian* (Russian) Funktsional. Anal. i Prilozhen. **34** (2000), no.3, 63–65; translation in Funct. Anal. Appl. **34** (2000), no.3, 208–210.

$\mathrm{U}(n)$ as a Grassmannian: interpretation of adding a uniton[16]

Let $g : M \to \mathrm{U}(n)$ be harmonic. Recall the definition of a *uniton* for g: Skip to Unitons

Let $\varphi : M \to G_n^{Lag}(\mathbb{C}^{2n})$ be the corresponding harmonic map via the isometry $K : \mathrm{U}(n) \to G_n^{Lag}(\mathbb{C}^{2n})$. Recall the *replacement conditions*: Skip to Replacements

Let γ be a subbundle of $\underline{\mathbb{C}}^n$ and set $\alpha = \{(s_1, gs_1) : s_1 \in \gamma\}$ and $\beta = \{(-s_1, gs_1) : s_1 \in \gamma\}$. Note that α is a subbundle of φ and β a subbundle of $\varphi^\perp$.

Then (α, β) *satisfies the replacement conditions if and only if* γ *is a uniton*. Thus:

We can interpret adding a uniton to a harmonic map $g : M \to \mathrm{U}(n)$ as doing a replacement to the corresponding harmonic map $\varphi : M \to G_n^{Lag}(\mathbb{C}^{2n})$.

---

[16] R. Pacheco and JCW, work in progress.

$\mathrm{U}(n)$ as a Grassmannian: an application

A harmonic map is said to be of **finite type** if it can be obtained by using integrable systems methods from a certain Lax-type equation It is known[17] that *all non-constant harmonic tori in the Euclidean sphere S^n or complex projective space $\mathbb{C}P^n$ are either of finite type or of finite uniton number*. Further [Pacheco-W.] *a harmonic map from a 2-torus T^2 to a complex Grassmannian which is simultaneously of finite type and finite uniton number is constant*.

What about maps into the unitary group $\mathrm{U}(n)$?

Theorem (Pacheco-W.)

(i) *A harmonic map $T^2 \to \mathrm{U}(n)$ which is simultaneously of finite type and finite uniton number is constant.*

(ii) *Let $g : T^2 \to \mathrm{U}(n)$ be harmonic. Suppose that $A := A_z^g = g^{-1}\partial_z g$ is invertible and semisimple on a dense subset of T^2. Then g is of finite type.*

---

[17] R. Pacheco, see next frame

The proofs

(i) follows immediately from the interpretation of $\mathrm{U}(n)$ as a Grassmannian.

(ii) For a harmonic map $g : M \to \mathrm{U}(n)$, let $\varphi : M \to G_n^{Lag}(\mathbb{C}^{2n})$ be the corresponding harmonic map via the isometry $K : \mathrm{U}(n) \to G_n^{Lag}(\mathbb{C}^{2n})$. Then for $s = (s_1, gs_1) \in \Gamma(\varphi)$, simple calculations show that

$$A'_\varphi(s) = (-As, gAs), \qquad A''_\varphi(s) = (-\bar{A}s, g\bar{A}s)$$

hence the *first return map* $c_1(\varphi) := A'_{\varphi^\perp} \circ A'_\varphi$ is given by

$$c_1(\varphi)(s) = \tfrac{1}{2}(A^2 s, gA^2 s).$$

This is clearly invertible and semisimple on a dense subset of T^2, therefore, by Theorem 4.1 of [S. Udagawa, *Harmonic maps from a two-torus into a complex Grassmann manifold*, Internat. J. Math. **6** (1995), no.3, 447–459], φ, and so g has finite type.

Further developments

According to [Y. Huang and N.C Leung, *A uniform description of compact symmetric spaces as Grassmannians using the magic square*, Math. Ann. **350** (2011), no.1, 79–106], **every compact symmetric space, in particular, every compact Lie group, is a Grassmannian**. See also
[J.H. Eschenburg and S. Hosseini, *Symmetric spaces as Grassmannians*, Manuscripta Math. **141** (2013), no.1-2, 51–62].

Interpreting $G_2/\mathrm{SO}(4)$ as a Grassmannian, we can study the twistor theory of harmonic maps, see [M. Svensson and JCW, *Harmonic maps into the exceptional symmetric space* $G_2/\mathrm{SO}(4)$, J. Lond. Math. Soc. **91** (2015) 291–319].

For a similar approach for harmonic maps into $F_4/\mathrm{Spin}(9)$, see [N. Correia, R. Pacheco and M. Svensson, *Harmonic surfaces in the Cayley plane*, J. Lond. Math. Soc. (2) **103** (2021), no.2, 353–371].

Mulțumesc!

Thank you for your attention!

Short Bibliography

- F.E. Burstall and J.C. Wood, *The construction of harmonic maps into complex Grassmannians*, J. Diff. Geom. **23** (1986), 255–298.
- K. Uhlenbeck, *Harmonic maps into Lie groups: classical solutions of the chiral model*, J. Differential Geom. **30** (1989) 1–50.
- A. Aleman, R. Pacheco and JCW. *Harmonic maps and shift-invariant subspaces*, Monatsh. Math. **194** (2021), no. 4, 625–656.
- R. Pacheco and JCW, *Diagrams and harmonic maps, revisited*, Ann. Mat. Pura Appl. (4)**202** (2023), no.3, 1051–1085.
- V.I. Arnol'd, *The complex Lagrangian Grassmannian* (Russian) Funktsional. Anal. i Prilozhen. **34** (2000), no.3, 63–65; translation in Funct. Anal. Appl. **34** (2000), no.3, 208–210.
- Y. Huang and N.C Leung, *A uniform description of compact symmetric spaces as Grassmannians using the magic square*, Math. Ann. **350** (2011), no.1, 79–106.
- J.H. Eschenburg and S. Hosseini, *Symmetric spaces as Grassmannians*, Manuscripta Math. **141** (2013), no.1-2, 51–62.