On the Shape Operator of Biconservative Hypersurfaces in Euclid and Minkowski Spaces

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Section 1:

Preliminaries

Basic Notation

Let $\mathbb{E}^n_r$ denote the pseudo-Euclidean n space with the index r given by the metric

$$\tilde{g} = \langle \cdot, \cdot \rangle = -\sum_{i=1}^{r} dx_i^2 + \sum_{i=r+1}^{n} dx_i^2$$

Note: We drop r in the Riemannian case $r = 0$.

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Put

$$\mathbb{S}^n_r = \{x \in \mathbb{E}^{n+1}_r : \langle x, x \rangle = 1\}$$

and

$$\mathbb{H}^n_r = \{x \in \mathbb{E}^{n+1}_{r+1} : \langle x, x \rangle = -1\}.$$

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Further we use the notation

$$R^n_r(c) = \begin{cases} \mathbb{S}^n_r & \text{if } c = 1, \\ \mathbb{H}^n_r & \text{if } c = -1, \\ \mathbb{E}^n_r & \text{if } c = 0, \end{cases}$$

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Basic Definitions

Let (Ω, g) be a semi-Rimennian manifold and $f : (\Omega, g) \hookrightarrow R^n_r(c)$ an isometric immersion.

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Then, the Gauss and Weingarten formulas for the submanifold $M = f(\Omega)$ of $R^n_r(c)$:

$$\begin{aligned}
\widetilde{\nabla}_X Y &= \nabla_X Y + \alpha_f(X, Y), \\
\widetilde{\nabla}_X \xi &= -A^f_\xi X + \nabla^\perp_X \xi.
\end{aligned}$$

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The mean curvature vector field of f is defined by

$$H_f = \frac{1}{m} \operatorname{trace} \alpha_f.$$

Basic Equations

The following equations is satisfied for a given $f:(\Omega,g)\hookrightarrow R^n_r(c)$:

$$R(X,Y)Z = c(X\wedge Y)Z + A^f_{\alpha_f(Y,Z)}X - A^f_{\alpha_f(X,Z)}Y, \tag{G}$$

$$(\bar\nabla_X\alpha_f)(Y,Z) = (\bar\nabla_Y\alpha_f)(X,Z) \tag{C}$$

$$R^D(X,Y)\zeta = \alpha_f(X,A^f_\zeta Y) - \alpha_f(A^f_\zeta X,Y), \tag{R}$$

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where we put

$$(\bar{\nabla}_X \alpha_f)(Y,Z) = D^f_X \alpha_f(Y,Z) - \alpha_f(\nabla_X Y, Z) - \alpha_f(Y, \nabla_X Z)$$

and $(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y$.

Section 1.2:

Biconservative Submanifolds

Biharmonic Maps

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Biharmonic map

A mapping ψ is said to be biharmonic if it is a critical point of the energy functional E_2.

Biharmonic Mappings

In [1] G.Y. Jiang studied the first and second variation formulas of E_2 in order to understand its critical points, called biharmonic maps (See also [2]).

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[2]G. Y. Jiang, *2-harmonic isometric immersions between Riemannian manifolds*, 1986.

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He proved that a mapping $\psi : M \to N$ is biharmonic if and only if the associated Euler-Lagrange equation

$$\tau_2(\psi) = 0$$

is satisfied, where

$$\tau_2(\psi) = \Delta\tau(\psi) - \operatorname{trace} \tilde{R}(d\psi, \tau(\psi))d\psi$$

is the bi-tension field.

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Remark

It is obvious that a harmonic map is biharmonic.

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Biharmonic immersions

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Biharmonic immersions

Let $f:(\Omega,g)\hookrightarrow(N,\tilde g)$ be an isometric immersion. In this case, we have

$$\tau_2(f)=0\Leftrightarrow\begin{cases} m\nabla\|H\|^2+4\,\mathrm{trace}\,A_{\nabla^{\perp}_{\cdot}H}(\cdot)+4\,\mathrm{trace}\left(\tilde R(\cdot,H)\cdot\right)^T=0, & (\mathrm{T})\\ \mathrm{trace}\,\alpha_f(A_H(\cdot),\cdot)-\Delta^{\perp}H+\,\mathrm{trace}\,(\tilde R(\cdot,H)\cdot)^{\perp}=0 & (\perp),\end{cases}$$

where we simply put $A^f=A$ and $H=H_f$.

Biconservative Immersions

Definition

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is satisfied.

Biharmonic Immersions

Every biharmonic immersion is biconservative.

Section 1.3:

Hypersurfaces of Riemannian Space Forms

Biconservative Hypersurfaces

In [3], we study biconservative hypersurfaces of non-flat Riemannian space forms.

- When $N = R^n_r(c)$, we have $\operatorname{trace}(\tilde{R}(\cdot, H)\cdot)^T = 0$

[3][nCt, Upadhyay, Mathematische Nachrichten, 2019]

Biconservative Hypersurfaces

In [3], we study biconservative hypersurfaces of non-flat Riemannian space forms.

- When $N = R^n_r(c)$, we have $\operatorname{trace}(\tilde{R}(\cdot, H)\cdot)^T = 0$
- Therefore, the biconservativity equation

$$m\nabla\|H\|^2 + 4\operatorname{trace} A_{D.H}(\cdot) + 4\operatorname{trace}\left(\tilde{R}(\cdot, H)\cdot\right)^T = 0, \qquad \text{(T)}$$

turns into

$$S(\nabla H) = -\frac{nH}{2}\nabla H. \qquad \text{(T)}$$

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Proposition

A hypersurface of $N = R^{n+1}(c)$ is biconservative if and only if

$$S = \begin{pmatrix} -\frac{nH}{2} & & & \\ & k_2 & & \\ & & \ddots & \\ & & & k_n \end{pmatrix}$$

for a function H such that $e_i(H) = 0,\ i = 2, 3, \ldots, n.$

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We have the following parametrization of the hypersurface M:

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Let M be a proper biconservative hypersurface of $R^{n+1}(c),\ c \in \{-1, 1\}$ with the mean curvature H and $m \in M$.

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$$\begin{aligned} x(s, t_1, t_2, \dots, t_{n-1}) =& \Theta(t_1, t_2, \dots, t_{n-1}) + \alpha_1(s)\xi_1(t_1, t_2, \dots, t_{n-1}) + \alpha_2(s)\xi_2(t_1, t_2, \dots, \\ & t_{n-1}) + \alpha_3(s)\xi_3(t_1, t_2, \dots, t_{n-1}) \end{aligned}$$

for any parallel, orthonormal base $\{\xi_1, \xi_2, \xi_3\}$ of the normal space of $\hat{M}$ in $E(n+1, c)$.

Biconservative hypersurfaces of $\mathbb{H}^4$

- $$x(s,t,u) = (\alpha_1(s), \alpha_2(s)\cos t, \alpha_2(s)\sin t, \alpha_3(s)\cos u, \alpha_3(s)\sin u),$$

- $$x(s,t,u) = (\alpha_1(s)\cosh u, \alpha_1(s)\sinh u, \alpha_2(s)\cos t, \alpha_2(s)\sin t, \alpha_3(s)),$$

- $$x(s,t,u) = \left(\frac{aA(s)^2 + a}{s} + asu^2 + \frac{s}{4a}, su, A(s)\cos t, A(s)\sin t, \frac{aA(s)^2 + a}{s} + asu^2 - \frac{s}{4a}\right)$$

- $$x(s,t,u) = \left(\frac{aA(s)^2}{s} + as\left(t^2+u^2\right) + \frac{s}{4a} + \frac{a}{s}, st, su, A(s), \frac{aA(s)^2}{s} + as\left(t^2+u^2\right) - \frac{s}{4a} + \frac{a}{s}\right)$$

Section 2:

Hypersurfaces of $\mathbb{E}^4_1$

Shape Operator of Hypersurfaces I

Let M be a Lorentzian surface, $p \in M$ and A be a symmetric endomorphism of T_pM. Then, by choosing an appropriated base for T_pM, A can put into one of the following four canonical forms:

$$\text{Case (i).} A \sim \begin{bmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{bmatrix},$$

with respect to an orthonormal base.

$$\text{Case (ii).} A \sim \begin{bmatrix} a_0 & & & & \\ -1 & a_0 & & & \\ & & a_1 & & \\ & & & \ddots & \\ & & & & a_{n-2} \end{bmatrix}$$

with respect to an pseudo-orthonormal base.

Shape Operator of Hypersurfaces II

$$\text{Case (iii)}.A \sim \begin{bmatrix} a_0 & 0 & 0 & & & \\ 0 & a_0 & 1 & & & \\ -1 & 0 & a_0 & & & \\ & & & a_1 & & \\ & & & & \ddots & \\ & & & & & a_{n-3} \end{bmatrix},$$

with respect to an pseudo-orthonormal base.

$$\text{Case (iv)}.A \sim \begin{bmatrix} a_0 & b_0 & & & \\ -b_0 & a_0 & & & \\ & & a_1 & & \\ & & & \ddots & \\ & & & & a_{n-2}. \end{bmatrix}$$

with respect to an orthonormal base

Some References

- Biconservative hypersurfaces with diagonalizable shape operator were classified in [4]
- Biharmonic hypersurfaces with non-diagonalizable shape operator were studied in [5]
- Biconservative hypersurfaces with complex principle curvature were studied by [6]

[4] [Fu, nCt, International Journal of Mathematics, 2015]

[5] [Arvanitoyeorgos, Kaimakamis, Magid, Illinois Journal of Mathematics, 2009]

[6] [Deepika, Mediterr. J. Math., 2017]

Section 2.1:

Uniqueness of Biconservative Hypersurfaces

Shape Operator of Biconservative Hypersurfaces

Assume that a hypersurface M in $\mathbb{E}^4_1$ with non-diagonalizable shape operator is proper biconservative. Then, we have

$$S(\operatorname{grad} H) = \frac{-3H}{2}\operatorname{grad} H. \tag{1}$$

Shape Operator of Biconservative Hypersurfaces

Assume that a hypersurface M in $\mathbb{E}^4_1$ with non-diagonalizable shape operator is proper biconservative. Then, we have

$$S(\mathrm{grad}\, H) = \frac{-3H}{2}\mathrm{grad}\, H. \tag{1}$$

We have the following two cases subject to the causality of $\mathrm{grad}\, H$:
Case (i). There exists a pseudo-orthonormal base field $\{e_1, e_2, e_3\}$ such that

$$S \sim \begin{bmatrix} -3H/2 & 1 & 0 \\ 0 & -3H/2 & 0 \\ 0 & 0 & 6H \end{bmatrix}, \tag{2}$$

where the null vector e_1 is proportional to $\mathrm{grad}\, H$.

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where the null vector e_1 is proportional to $\mathrm{grad}\, H$.
Case (ii). There exists a pseudo-orthonormal base field $\{e_1, e_2, e_3\}$ such that

$$S \sim \begin{bmatrix} 9H/4 & 1 & 0 \\ 0 & 9H/4 & 0 \\ 0 & 0 & -3H/2 \end{bmatrix}, \tag{3}$$

where the space-like vector e_3 is proportional to $\mathrm{grad}\, H$.

Sketch Proof

Lemma

Let M be a proper biconservative hypersurface of $\mathbb{E}^4_1$ with non-diagonalized shape operator. Then, its shape operator s has the matrix representation

$$S \sim \begin{bmatrix} 9H/4 & 1 & 0 \\ 0 & 9H/4 & 0 \\ 0 & 0 & -3H/2 \end{bmatrix}.$$

Main Theorem

Theorem

Let (Ω, g) be a 3-dimensional, connected Lorentzian manifold. If (Ω, g) admits two proper biconservative isometric immersion $x, \tilde{x} : (\Omega, g) \hookrightarrow \mathbb{E}^4_1$ with non-diagonalizable shape operator then there exists an isometry $\tau : \mathbb{E}^4_1 \to \mathbb{E}^4_1$ such that $\tilde{x} = \tau \circ x$.

Sketch Proof

Assume the existence of $x : (\Omega, g) \hookrightarrow \mathbb{E}^4_1$ with the shape operator

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We first prove that H, e_1, e_2, e_3 can be obtained intrinsically:

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- It turns out that the scalar curvature of is $S = \frac{27}{16}H^2$

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- It turns out that the scalar curvature of is $S = \frac{27}{16} H^2$
- Then, we obtain $e_3 = x_* E_3$, where

$$E_3 = \frac{\operatorname{grad} S}{\| \operatorname{grad} S \|}.$$

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- Then, we obtain $e_3 = x_* E_3$, where

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- Finally, we proved that $e_1 = x_* E_1,\ e_2 = x_* E_2$, where E_1, E_2 are the only vectors satisfying

$$R(E_1, E_3)E_1 = 0, \qquad R(E_3, E_2)E_3 = -\frac{3H}{2}E_1,$$
$$R(E_2, E_3)E_2 = \frac{3H}{2}E_3 \qquad .$$

Section 2.2:

Local Classification

Construction of Hypersurface

Construction of Hypersurface

Let α be a null curve which admits a pseudo-orthonormal base $\{T, U, \alpha_1, \alpha_2\}$ such that

$$\begin{aligned}
\alpha' &= T,\\
T' &= A_2\alpha_2 + A_1 T,\\
U' &= A_3\alpha_1 + A_4\alpha_2 - A_1 U,\\
\alpha_1' &= A_3 T,\\
\alpha_2' &= A_2 U + A_4 T
\end{aligned}$$

for some smooth functions A_i.

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\alpha_2' &= A_2 U + A_4 T
\end{aligned}$$

for some smooth functions A_i.
Consider the hypersurface

$$g(s,u,w) = \alpha(s) + uT(s) + w\alpha_1(s) + f(w)\alpha_2(s),$$

where f is a smooth function.

Construction of Hypersurface

$$g(s,u,w)=\alpha(s)+uT(s)+w\alpha_1(s)+f(w)\alpha_2(s),$$

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$$g(s,u,w)=\alpha(s)+uT(s)+w\alpha_1(s)+f(w)\alpha_2(s),$$

The shape operator of this hypersurface has the form

$$\begin{pmatrix} \frac{1}{f(w)\sqrt{f'(w)^2+1}} & -\frac{A_3(s)\left(w+f(w)f'(w)\right)+1}{f(w)^2A_2(s)\sqrt{f'(w)^2+1}} & 0 \\ 0 & \frac{1}{f(w)\sqrt{f'(w)^2+1}} & 0 \\ 0 & 0 & -\frac{f''(w)}{\left(f'(w)^2+1\right)^{3/2}} \end{pmatrix}.$$

Therefore, this hypersurface is biconservative if and only if

$$-3f(w)f''(w)+2f'(w)^2+2=0$$

Local Classfication Result

Theorem

A hypersurface M in $\mathbb{E}^4_1$ with non-diagonalizable shape operator is proper biconservative if and only if it is locally congruent to the hypersurface previously constructed

Section 3:

Biconservative Hypersurfaces in Euclidean Spaces

Biconservative Hypersurfaces

Let $f : (\Omega, g) \to \mathbb{E}^{n+1}$ be a proper biconservative immersion, where (Ω, g) is an n dimensional Riemannian manifold. Then, we have

$$S(\mathrm{grad}\, H) = \frac{-nH}{2}\mathrm{grad}\, H.$$

Biconservative Hypersurfaces

Let $f : (\Omega, g) \to \mathbb{E}^{n+1}$ be a proper biconservative immersion, where (Ω, g) is an n dimensional Riemannian manifold. Then, we have

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Therefore,

- $e_1 = \frac{\mathrm{grad}\, H}{\|\mathrm{grad}\, H\|}$ is a principle direction of f
- with the corresponding principle curvature

$$k_1 = \frac{-nH}{2}.$$

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$$k_1 = \frac{-nH}{2}.$$

Consider an orthonormal frame field $\{e_1, e_2, \dots, e_n\}$ of $T\Omega$ such that

$$Se_i = k_i e_i.$$

Corollary

f is biconservative if and only if

$$-3k_1 + k_2 + \dots + k_n = 0 \text{ and } e_a(k_1) = 0,\ a = 2, 3, \dots, n.$$

Uniqueness Theorem

Theorem

Let (Ω, g) be an n-dimensional, connected Riemannian manifold. If (Ω, g) admits two proper biconservative isometric immersion $f, \tilde{f} : (\Omega, g) \hookrightarrow \mathbb{E}^{n+1}$, then there exists an isometry $\tau : \mathbb{E}^{n+1} \to \mathbb{E}^{n+1}$ such that $\tilde{f} = \tau \circ f$.

Sketch Proof

We try to determine $e_1, e_2, \ldots, e_n$ intrinsically.
Consider the Ricci tensor Ric of (Ω, g) defined to be

$$\mathrm{Ric}\,(X) = \mathrm{tr}\,\left(R(\cdot, X)\cdot\right).$$

Sketch Proof

We try to determine $e_1, e_2, \ldots, e_n$ intrinsically.
Consider the Ricci tensor Ric of (Ω, g) defined to be

$$\mathrm{Ric}\,(X) = \mathrm{tr}\ (R(\cdot, X)\cdot)\,.$$

Then, by the Gauss equation, we have

- $\mathrm{Ric}\,(e_1) = -3k_1^2 e_1$
- $\mathrm{Ric}\,(e_a) = (2k_1 k_1 + k_a^2)e_a$

Sketch Proof

We try to determine $e_1, e_2, \ldots, e_n$ intrinsically.
Consider the Ricci tensor Ric of (Ω, g) defined to be

$$\mathrm{Ric}\,(X) = \mathrm{tr}\,\left(R(\cdot, X)\cdot\right).$$

Then, by the Gauss equation, we have

- $\mathrm{Ric}\,(e_1) = -3k_1^2 e_1$
- $\mathrm{Ric}\,(e_a) = (2k_1 k_1 + k_a^2)e_a$

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ denote the eigenvectors of Ric.

Sketch Proof

Note that $X = e_1$ satisfies

$$\begin{aligned} \nabla_X X = 0, \qquad \mathrm{Ric}\,(X) &= \lambda X, \\ Y(\lambda) &= 0 \qquad \text{whenever } g(X, Y) = 0, \\ \nabla_Y X &= a_Y Y \qquad \text{whenever } \mathrm{Ric}\,(Y) = \zeta Y. \end{aligned}$$

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By a computation we proved that

Claim

If X is a unit tangent vector field satisfying the properties above, then either $X = e_1$ or $X = -e_1$.

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Consequently, e_1 and $k_1 = \sqrt{-\lambda_1/3}$ can be determined intrinsically.

Sketch Proof

Next, in order to determine k_a, we consider the map

$$L : (\langle e_1 \rangle)^\perp \to (\langle e_1 \rangle)^\perp, LX = \nabla_X e_1.$$

Note that, by using Codazzi equations, we have

$$e_a \text{ is a principle curvature} \Leftrightarrow Le_a = \mu_a e_a.$$

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By a further computation we observe that k_a can be determined by using the equations

$$\begin{aligned} k_a &= -k_1 + \varepsilon\sqrt{k_1^2 + \lambda_a} \\ 2k_a\mu_a + e_1(k_1) &= \varepsilon\left(\mu_a\sqrt{k_1^2 + \lambda_a} + e_1(\sqrt{k_1^2 + \lambda_a})\right). \end{aligned}$$

THANK YOU