Polynomial harmonic morphisms and eigenfamilies on spheres

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September 7, 2023

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Table of contents

Introduction - Harmonic morphisms

Let $(M, g), (N, h)$ be Riemannian manifolds.

Definition

A smooth map $\varphi : (M, g) \to (N, h)$ is called a *harmonic morphism* if for any open $U \subseteq N$ and harmonic map $f : U \to \mathbb{R}$ the composition

$$f \circ \varphi : \varphi^{-1}(U) \to \mathbb{R}$$

is again harmonic.

Theorem (Fuglede 1978, Ishihara 1979)

$\varphi : (M, g) \to (N, h)$ *is a harmonic morphism if and only if it is harmonic and weakly horizontally conformal, i.e. if and only if*

$$\tau(\varphi) = 0, \qquad \exists \lambda \in C(M) : \varphi^*(h) = \lambda g|_{\ker(D\varphi)^\perp}.$$

Introduction - Harmonic morphisms

The case $\dim(N) = 2$ is especially interesting:

- The regular level sets of φ are then minimal sub-manifolds (Baird-Eells, 1980).
- The property that a map is harmonic morphism is then invariant under conformal transformations of the codomain.
  In particular for local considerations one may assume $(N, h) = (\mathbb{C}, g_{\mathrm{Euc}})$.
- For maps $\varphi : (M, g) \to (\mathbb{C}, g_{\mathrm{Euc}})$ the conditions of harmonicity and weak horizontal conformality are equivalent to:

$$\Delta\varphi = 0, \qquad g_{\mathbb{C}}(\nabla\varphi, \nabla\varphi) = 0.$$

Where $g_{\mathbb{C}}$ is the $\mathbb{C}$-bilinear extension of g to $TM \otimes \mathbb{C}$.

Introduction - Harmonic morphisms

Some manifolds have an abundance of local harmonic morphisms with codomain $\mathbb{C}$:

Remark

Let (M, g, J) be a Kähler manifold, suppose that $f : M \to \mathbb{C}$ is holomorphic. Then f is a harmonic morphism.

While others do not:

Theorem (Baird-Wood 1992)

Let U be an open subset of the 3-dimensional homogeneous space Sol, then the constant map is the only harmonic morphism $U \to \mathbb{C}$.

Introduction - Eigenfamilies

One method of finding harmonic morphisms is via *eigenfamilies*, introduced by Gudmundsson and Sakovich 2008.

Definition

Let $\lambda, \mu \in \mathbb{C}$. A family $\mathcal{F}$ of smooth functions from M to $\mathbb{C}$ is called a (λ, μ)-*eigenfamily* on M if for all $\varphi, \psi \in \mathcal{F}$:

$$\Delta\varphi = \lambda\varphi, \tag{1}$$

$$g_{\mathbb{C}}(\nabla\varphi, \nabla\psi) = \mu\,\varphi\psi. \tag{2}$$

Functions that are elements of some (λ, μ)-eigenfamily are called (λ, μ)-*eigenfunctions*.

Introduction - Eigenfamilies

Eigenfamilies are machines that can produce harmonic morphisms:

Theorem (Gudmundsson-Sakovich 2008)

Let $\mathcal{F}$ be a (λ, μ)-eigenfamily on M, then for any $\varphi_1, ..., \varphi_k \in \mathcal{F}$ and homogeneous polynomials $P, Q \in \mathbb{C}[z_1, ..., z_k]$ of the same degree, the map

$$M \setminus \{x \in M \mid Q(\varphi_1(x), ..., \varphi_k(x)) = 0\} \to \mathbb{C},$$
$$x \mapsto \frac{P(\varphi_1(x), ..., \varphi_k(x))}{Q(\varphi_1(x), ..., \varphi_k(x))}$$

is a harmonic morphism.

Remark

$(0, 0)$-eigenfamilies are also called *orthogonal harmonic families* in the literature.

Introduction - Eigenfamilies

Remark

A lone eigenfunction is also useful for other goals:

- There is a universal formula for producing proper p-harmonic maps from an eigenfunction (Gudmundsson-Sobak 2020).
- The regular part of the 0-level set of an eigenfunction is a minimal submanifold (Gudmundsson-Munn 2023).

Global eigenfamilies - S^n

One makes the following observation about eigenfamilies with domain all of S^n:

Theorem

Let $\mathcal{F}$ be a family of maps from S^n to $\mathbb{C}$, and $\lambda, \mu \in \mathbb{C}$. The following are equivalent:

(i) *$\mathcal{F}$ is a (λ, μ)-eigenfamliy.*

(ii) *There is a $(0,0)$-eigenfamily $\widetilde{\mathcal{F}}$ of homogeneous polynomials from $\mathbb{R}^{n+1}$ to $\mathbb{C}$ all of the same degree d, so that the map*

$$\widetilde{\mathcal{F}} \to \mathcal{F}, \qquad F \mapsto F|_{S^n},$$

is a well defined bijection and $\lambda = -d(d+n-1)$, $\mu = -d^2$.

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Remark

So: orthogonal families of homogeneous polynomial harmonic morphisms on $\mathbb{R}^{n+1}$ $\leftrightarrow$ eigenfamilies on S^n.

Proof.

Recall:
A function $F : S^n \to \mathbb{C}$ satsifies $\Delta^{S^n} F = \lambda F$ if and only if there is a harmonic homogeneous polynomial $\widetilde{F} : \mathbb{R}^{n+1} \to \mathbb{C}$ of some degree d so that $\widetilde{F}|_{S^n} = F$ and $\lambda = -d(d+n-1)$.

$$\nabla_x^{\mathbb{R}^{n+1}} \widetilde{F} = \|x\|^{d-1}(\nabla^{S^n}_{x/\|x\|} F + d \cdot F(\tfrac{x}{\|x\|})\partial_r)$$

so that for two lifts $\widetilde{F}, \widetilde{G}$ one has:

$$(\nabla_x^{\mathbb{R}^{n+1}} \widetilde{F})^T \nabla_x^{\mathbb{R}^{n+1}} \widetilde{G} = \|x\|^{2d-2} \left(g_{\mathbb{C}}(\nabla^{S^n}_{x/\|x\|} F, \nabla^{S^n}_{x/\|x\|} G) + d^2 F(\tfrac{x}{\|x\|}) G(\tfrac{x}{\|x\|}) \right).$$

If the lifts are a $(0,0)$-eigenfamily then the LHS is 0, implying that $g_{\mathbb{C}}(\nabla^{S^n}_{x/\|x\|} F, \nabla^{S^n}_{x/\|x\|} G) = -d^2 F(\frac{x}{\|x\|}) G(\frac{x}{\|x\|})$.
On the other hand if $\mathcal{F}$ is a (λ, μ)-eigenfamily then:

$$(\mu + d^2)\widetilde{F}(x)\widetilde{G}(x) = \|x\|^2 (\nabla_x \widetilde{F})^T \nabla_x \widetilde{G}.$$

If $\mu \neq -d^2$ this implies one of $\widetilde{F}, \widetilde{G}$ has an $\|x\|^2$ factor, which contradicts harmonicity. □

Other CROSSes

Remark

Let $X \in \{\mathbb{RP}^n, \mathbb{CP}^n, \mathbb{HP}^n\}$ and let $\pi : S^{m(X)} \to X$ be the standard quotient map. Then π is a Riemannian submersion with totally geodesic fibres. In particular for all $\varphi, \psi : X \to \mathbb{C}$ one has:

$$\Delta^{S^{m(X)}}(\varphi \circ \pi) = (\Delta^X \varphi) \circ \pi,$$

$$g_{\mathbb{C}}^{S^{m(X)}}(\nabla(\varphi \circ \pi), \nabla(\psi \circ \pi)) = g_{\mathbb{C}}^X(\nabla\varphi, \nabla\psi) \circ \pi.$$

Corollary

Let X be as above and let $\mathcal{F}$ be a family of functions $X \to \mathbb{C}$, $\lambda, \mu \in \mathbb{C}$. The following are equivalent:

(i) $\mathcal{F}$ is a (λ, μ)-eigenfamily.

(ii) $\pi^*(\mathcal{F}) = \{\varphi \circ \pi : S^{m(X)} \to \mathbb{C} \mid \varphi \in \mathcal{F}\}$ is a (λ, μ)-eigenfamily on $S^{m(X)}$.

Eigenfamilies of homogeneous polynomials

Similar to harmonic morphisms:

Remark

Let (M, g, J) be a Kähler manifold, $\mathcal{F}$ a family of holmorphic maps $M \to \mathbb{C}$. Then $\mathcal{F}$ is a $(0,0)$-eigenfamily.

Holomorphic homogeneous polynomials then give a large amount of eigenfamilies on S^n (well known). Finding new examples leads to the following question:

Question

How can one tell if a family of functions is *not* holomorphic with respect to any Kähler structure?

Functions of complex type

For convenience define:

Definition

1. A family $\mathcal{F}$ of functions from $\mathbb{R}^n$ to $\mathbb{C}$ is said to be *uniformly of complex type* if there is a $\mathbb{R}$-linear isometric inclusion $a : \mathbb{C}^k \to \mathbb{R}^n$ so that for all $F \in \mathcal{F}$ one has that $F = F \circ (aa^*)$ and $F \circ a : \mathbb{C}^k \to \mathbb{C}$ is holomorphic.
2. A function $F : \mathbb{R}^n \to \mathbb{C}$ is said to be of *complex type* if the family $\{F\}$ is uniformly of complex type.

Theorem

A family $\mathcal{F}$ of smooth functions from $\mathbb{R}^n$ to $\mathbb{C}$ is uniformly of complex type if and only if for all $F, G \in \mathcal{F}$ and $x, y \in \mathbb{R}^n$:

$$(\nabla_x F)^T \nabla_y G = 0. \tag{3}$$

Functions of complex type

Weakly horizontally conformal polynomials $\mathbb{R}^n \to \mathbb{C}$ are automatically harmonic (Ababou-Baird-Brossard 1999). We are then interested in families $\mathcal{F}$ of homogeneous polynomials so that

$$(\nabla_x F)^T \nabla_x G = 0 \text{ for all } x \in \mathbb{R}^n \text{ and all } F, G \in \mathcal{F}$$

but for which there exist $x, y \in \mathbb{R}^n$ and $F, G \in \mathcal{F}$ so that

$$(\nabla_x F)^T \nabla_y G \neq 0.$$

Remark

For degree 2 maps (not families!) the two equations coincide: For $F(x) = x^T A x$, $A = A^T$ one has:

$$(\nabla_x F)^T \nabla_x F = 4x^T A^2 x = 0 \; \forall x \iff A^2 = 0$$
$$\iff 0 = 4x^T A^2 y = (\nabla_x F)^T \nabla_y F \; \forall x, y.$$

This is well known! It follows e.g. from the classification of homogeneous degree 2 polynomial harmonic morphisms $\mathbb{R}^n \to \mathbb{R}^m$. (Ou-Wood 1996, Ou 1997)

Example

The polynomial $\mathbb{R}^8 \to \mathbb{C}$ given by

$$\begin{aligned}
&x_3x_1^3 - ix_4x_1^3 + 3ix_2x_3x_1^2 + 3x_2x_4x_1^2 - 3x_2^2x_3x_1 \\
&+3ix_2^2x_4x_1 - x_3x_5x_7x_1 + ix_4x_5x_7x_1 + ix_3x_6x_7x_1 \\
&+x_4x_6x_7x_1 - ix_3x_5x_8x_1 - x_4x_5x_8x_1 - x_3x_6x_8x_1 \\
&+ix_4x_6x_8x_1 - ix_2^3x_3 - x_2^3x_4 - ix_2x_3x_5x_7 - x_2x_4x_5x_7 \\
&-x_2x_3x_6x_7 + ix_2x_4x_6x_7 + x_2x_3x_5x_8 - ix_2x_4x_5x_8 \\
&-ix_2x_3x_6x_8 - x_2x_4x_6x_8
\end{aligned}$$

is of complex type, but not holomorphic with respect to the standard Kähler structure on $\mathbb{R}^8 \cong \mathbb{C}^4$.

Examples - eigenfamilies

The following families are not uniformly of complex type:

(i) $F_1, F_2 : \mathbb{C}^4 \to \mathbb{C}$ given by

$$F_1(z,u,v,w) = zv + uw, \quad F_2(z,u,v,w) = z\overline{w} - u\overline{v}.$$

(ii) The product of the two polynomials above:

$$\mathbb{C}^4 \to \mathbb{C}, \quad (z,u,v,w) \mapsto z^2 vw - u^2 \overline{vw} + zu(|w|^2 - |v|^2).$$

(iii) As a, b, c, d vary over $\mathbb{C}$ the family of maps $\mathbb{C}^4 \to \mathbb{C}$ given by:

$$a(z^2 w + zu\overline{v}) + b(u^2\overline{w} - zuv) + c(z^2 v - zu\overline{w}) + d(u^2\overline{v} + zuw)$$

An element of the family is not of complex type unless $ab + cd = 0$.

(iv) Let $\gamma \in \mathbb{C}$, the map

$$\mathbb{C}^3 \oplus \mathbb{R} \mapsto \mathbb{C}, \qquad ((z,u,w),t) \mapsto z^2 w + 2\gamma\, zut - \gamma^2 u^2 \overline{w}$$

is not of complex type unless $\gamma = 0$ (inflation of an example from Ababou-Baird-Brossard 1999).

Axis of holomorphicity

A feature of all examples on the previous slide is that they are holomorphic in some variables.
It turns out to be useful to investigate this further:

Definition

Let $\mathcal{F}$ be a family of functions $\mathbb{R}^n \to \mathbb{C}$. A vector subspace $V \subseteq \mathbb{R}^n$ is said to be a *uniform axis of holomorphicity* of $\mathcal{F}$ if the family

$$\{V \to \mathbb{C}, v \mapsto F(x+v) \mid x \in \mathbb{R}^n, F \in \mathcal{F}\}$$

is uniformly of complex type.

Theorem

Let $n \in \{5, 6\}$ and suppose $P : \mathbb{R}^n \to \mathbb{C}$ is a homogeneous harmonic morphism. If P has an axis of holomorphicity of (real) dimension at least 2, then P is of complex type.

Axis of holomorphicity

Homogeneous polynomials without an axis of holomorphicity contain information that, in a certain sense, appears for the first time in a given dimension. For example the proof of the previous Theorem implies:

Corollary

Let $n \leq 9$ and suppose $P : \mathbb{R}^n \to \mathbb{C}$ is a homogeneous polynomial harmonic morphism not admitting an axis of holomorphicity. Then there is no non-trivial decomposition $\mathbb{R}^n = V \oplus V^{\perp}$ with respect to which P is a harmonic morphism in each variable seperately.

Remark

The speaker does not know of *any* homogeneous polynomial harmonic morphisms without an axis of holomorphicity! However numerics indicate the existence of homogeneous degree 3 examples on $\mathbb{R}^6$.

Eigenfamilies consisting of homogeneous degree 2 polynomials

Degree 2 eigenfamilies always admit an axis of holomorphicity:

Theorem

Let $\mathcal{F}$ be an eigenfamily of homogeneous degree 2 polynomial harmonic morphisms $\mathbb{R}^n \to \mathbb{C}$. Then $\mathcal{F}$ admits a uniform axis of holomorphicity of (real) dimension at least $\min(2, n)$.

Eigenpairs of degree 2 polynomials

Definition

Let $\mathcal{F}$ be a family of smooth functions $\mathbb{R}^n \to \mathbb{C}$. A uniform axis of holomorphicity V of $\mathcal{F}$ is called *maximal* if it is not contained in any other uniform axis of holomorphicity.

Definition

1. We call a triple (m, k, δ) of natural numbers a *subspace type* if $m \geq k$, k is even, and $\delta \in \{0, 1\}$.
2. We call a triple (P_1, P_2, A) *polynomial data* of a subspace type (m, k, δ) if $P_1, P_2 \in \mathbb{C}[z_1, ..., z_m]$ are homogenous complex polynomials of degree 2 and A is a complex $m \times k$ matrix of rank k.
3. We call a triple (Y, C, v) *twisting data* of a subspace type (m, k, δ) if Y, C are anti-symmetric $k \times k$ matrices, Y is invertible, and if $v \in \mathbb{C}^m$ with $v = 0$ if and only if $\delta = 0$.

Theorem

Let $F_1, F_2 : \mathbb{R}^n \to \mathbb{C}$ be two homogenous degree 2 polynomials so that $\{F_1, F_2\}$ is a full eigenfamily. Then there are subspace data (m, k, δ) as well as polynomial and twisting data (P_1, P_2, A), (Y, C, v) so that:

1. *Up to an isometry of the domain one can decompose $\mathbb{R}^n \cong \mathbb{C}^m \oplus \mathbb{C}^k \oplus \mathbb{R}^\delta$ so that $\mathbb{C}^m$ is a maximal uniform axis of holomorphicity for $\{F_1, F_2\}$.*
2. *Let $X = (\frac{1}{2}vv^T + C)Y^{-1}$. One has, with respect to the above decomposition:*

$$F_1((z_1, ..., z_m), (w_1, ..., w_k), t) = P_1(z_1, ..., z_m) + \sum_{ij} z_i A_{ij} w_j$$

$$F_2((z_1, ..., z_m), (w_1, ..., w_k), t) = P_2(z_1, ..., z_m) + \sum_{ij} z_i A_{ij} \left(\sum_l X_{jl} w_l + \sum_l Y_{jl} \overline{w}_l + v_j t \right)$$

Eigenairs of degree 2 polynomials - examples

The following are minimalistic examples of such eigenpairs:

(i) $F_1, F_2 : \mathbb{C}^4 \to \mathbb{C}$ given by

$$F_1(z,u,v,w) = zv + uw, \quad F_2(z,u,v,w) = z\overline{w} - z\overline{v}.$$

Here $(m,k,\delta) = (2,2,0)$, $(P_{\mathbf{1}}, P_{\mathbf{2}}, A) = (0, 0, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})$, $(Y, C, v) = (\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 0, 0)$.

(ii) $F_1, F_2 : \mathbb{C}^4 \oplus \mathbb{R} \to \mathbb{C}$ given by

$$F_1(z,u,v,w,t) = zv+uw, \quad F_2(z,u,v,w,t) = z(\overline{w}+w+2it)-u\overline{v}.$$

Here $\delta = 1$ and $v = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, the other data are as in the previous example.

Other topics - Eigenfamilies induced by polynomial harmonic morphisms $\mathbb{R}^n \to \mathbb{R}^m$

Consider a homogeneous polynomial harmonic morphism $P : \mathbb{R}^n \to \mathbb{R}^m$, $x \mapsto (P_1(x), ..., P_m(x))$. Then

$$\mathcal{E}(P) := \{P_{2k-1} + iP_{2k} \mid 1 \leq k \leq \lfloor m/2 \rfloor\}$$

is a $(0, 0)$-eigenfamily.

Example

Let $P : \mathbb{H}^3 \to \mathbb{H}$ be the multiplication of 3 quaternions. Then $\mathcal{E}(P)$ is congruent to the following two maps from $\mathbb{C}^6 \to \mathbb{C}$:

$$(z_1, z_2, u_1, u_2, w_1, w_2) \mapsto z_1(u_1 w_1 - u_2\overline{w_2}) - z_2(\overline{u_1 w_2} + \overline{u_2} w_1),$$
$$(z_1, z_2, u_1, u_2, w_1, w_2) \mapsto z_1(u_1 w_2 + u_2\overline{w_1}) + z_2(\overline{u_1 w_1} - \overline{u_2} w_2).$$

This family is not uniformly of complex type.

Other topics - Eigenfamilies induced by polynomial harmonic morphisms $\mathbb{R}^n \to \mathbb{R}^m$

Using the classification of degree 2 polynomial harmonic morphisms $\mathbb{R}^n \to \mathbb{R}^m$ (Wood and Ou 1996, Ou 1997):

Theorem

Let $P : \mathbb{R}^n \to \mathbb{R}^m$ be a non-zero homogeneous degree 2 polynomial harmonic morphism.

1. *If $m \geq 4$ then $\mathcal{E}(P)$ is not uniformly of complex type.*
2. *For any $B \in O(m)$ the families $\mathcal{E}(P)$ and $\mathcal{E}(B \circ P)$ are congruent.*

Definition

Say that two families $\mathcal{F}_1$ and $\mathcal{F}_2$ of maps $\mathbb{R}^n \to \mathbb{C}$ are *congruent* if there is an isometry $\Phi \in O(n)$ so that $\operatorname{span}_{\mathbb{C}} \mathcal{F}_1 = \operatorname{span}_{\mathbb{C}} \Phi^*(\mathcal{F}_2)$.

Here $\Phi^*(\mathcal{F}) = \{F \circ \Phi \mid F \in \mathcal{F}\}$.

Thank you!