A Geometric Approach of Probability Distributions

Ioana RĂDULESCU (LĂZĂRESCU)
(joint work with Ioana Antonia BRANEA)

PhD Student, Transilvania University of Brasov

ioana.radulescu@unitbv.ro

Supervisor: Prof. Univ. Dr. Adela MIHAI

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Introduction

Statistical manifolds are geometric abstractions used to model information. They belong to the field of Information Geometry, which is a relatively new branch of mathematics that applies the tools of differential geometry to explore topics such as statistical inference, estimation, and information loss. This field originated from the differential geometric analysis of the manifold of probability density functions.

Statistical models consist of a family of probability distributions which can be given a geometric structure. They can be endowed with a Riemannian metric, specifically the Fisher information matrix, introduced by Rao ([7] *Information and accuracy attainable in estimation of statistical parameters*, Bulletin of the Calcutta Math. Soc. **37** (1945)) and Jeffreys ([6] *An invariant form for the prior probability in estimation problems*, Proceedings of the Royal Society of London, Series A, **186(1007)** (1946)) and they become Riemannian manifolds, as shown by Amari([1] S. Amari and H. Nagaoka, *Methods of Information Geometry*, American Mathematical Soc., Oxford Univ. Press **191**, 2000) and Călin and Udriște ([3] O. Călin and C. Udriște, *Geometric Modeling in Probability and Statistics*, Springer, 2014).

Introduction

We conduct a differential geometric study on the set of exponential and Bernoulli distributions by computing the Fisher matrix, Christoffel symbols, geodesics, Laplace-Beltrami operator and harmonic functions.

Our results reveal that the statistical models given by the exponential distribution and the one given by the Bernoulli distribution are 1-type curves in $\mathbb{R}^2$.

Definitions taken from [2] [G. Blom, *Probability and Statistics: Theory and Applications*, Springer, 1989].

Let $(\Omega, \mathcal{F}, P)$ be a *probability space*, where

- Ω is the set of all possible outcomes;
- σ-field $\mathcal{F}$ is a collection of subsets of Ω that is closed under countable many intersections, unions, and complements;
- P is a probability function, i.e. a measure on $\mathcal{F}$ for which $P(\Omega) = 1$.

A *random variable* X on $(\Omega, \mathcal{F}, P)$ is a function $X : \Omega \to \mathbb{R}$ that satisfies

$$X^{-1}(A) = \{\omega \in \Omega : X(\omega) \in A\} \in \mathcal{F}, \quad \forall A \in \mathcal{B}, \tag{1}$$

where $\mathcal{B}$ is the Borel algebra on the set of real numbers.

There are two classes of random variables:

1. *discrete random variables* $X : \Omega \to \chi = \{x_1, x_2, \dots\}$ for which the density function $p : \chi \to \mathbb{R}$ satisfies

$$p(x) = \begin{cases} P(X = x_i), & x = x_i, i = \{1, 2, \dots\}, \\ 0, & \text{otherwise} \end{cases} \qquad \text{and} \qquad \sum_k p(x_k) = 1. \tag{2}$$

2. *continuous random variables* $X : \Omega \to \chi \subset \mathbb{R}^n$ for which the density function $p : \chi \to \mathbb{R}$ satisfies

$$P(X \in D) = \int_D p(x)\,\mathrm{d}x \qquad \text{and} \qquad \int_\chi p(x)\,\mathrm{d}x = 1. \tag{3}$$

From the numerical characteristics of random variables, we recall the *expectation value*

$$\mathbf{E}(X) = \begin{cases} \displaystyle\sum_i x_i p(x_i), & \text{if } X \text{ is a discrete random variable,} \\ \displaystyle\int_{\chi} x p(x)\,\mathrm{d}x, & \text{if } X \text{ is a continuous random variable} \end{cases} . \qquad (4)$$

Differential Geometry

By investigating curves, surfaces, manifolds, and studying concepts like tangent vectors, curvature, and metrics, Differential Geometry reveals the intrinsic properties of geometric objects.

Let $(M, \mathcal{A})$ be a *differentiable manifold*, where M is a topological space and $\mathcal{A} = \{(U_i, h_i) : i \in I\}$ is the *atlas*, i.e. a collection of *charts* which are bijective mappings between open subsets of M and open subsets of $\mathbb{R}^m$. An *immersion* is a mapping $x : M \to \mathbb{R}^m$ that has rank $n = \dim M$.

A *Riemannian metric* g on a differentiable manifold is a symmetric, positive definite bilinear form on the tangent space. The pair (M, g) is called a *Riemannian manifold*.

The Riemannian metric allows for the definition of various geometric quantities such as the *Christoffel symbols of the first kind*

$$\Gamma_{ij,k} = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right), \tag{5}$$

and the *Christoffel symbols of the second kind*

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{l=1}^{n} g^{kl}\left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\right). \tag{6}$$

Using Christoffel symbols, we can define *geodesics*, which are the paths that locally minimize distance

$$\frac{d^2x^k}{dt^2} + \sum_{i,j=1}^{n} \Gamma_{ij}^{k} \cdot \frac{dx^i}{dt} \cdot \frac{dx^j}{dt} = 0, \qquad k = \overline{1, n}. \tag{7}$$

The *Laplace-Beltrami operator* is defined by

$$\Delta f := -\frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(g^{ij} \cdot \sqrt{\det g} \cdot \frac{\partial f}{\partial x_j} \right), \tag{8}$$

where g^{ij} is the inverse of the Riemannian metric g_{ij}.

A function satisfying $\Delta f = 0$ is called *harmonic*.

A *submanifold* is a subset of a manifold that itself possesses the structure of a manifold. It is well-known (see [4] B.-Y. Chen, *Total Mean Curvature and Submanifolds of Finite Type*, World Scientific Publishing Company, 1984) that an isometric immersion $x : M \to \mathbb{R}^m$, $x = (x^1, \ldots, x^m)$, $x^i \in C^\infty(M), i = \overline{1, m}$ satisfies

$$x^i = x^i_0 + \sum_{t=p_i}^{q_i} x^i_t, \quad i = \overline{1, m}, \tag{9}$$

with $x^i_0 \in \mathbb{R}$ and x^i_t eigenfunctions of the Laplace-Beltrami operator. Chen [4] defines *submanifolds of finite type* by denoting

$$p = \inf\{p_i : i = \overline{1, m}\} \in \mathbb{N}^* \quad \text{and} \quad q = \sup\{q_i : i = \overline{1, m}\} \in \mathbb{N}^* \cup \{\infty\} \tag{10}$$

as follows.

Definition 1

A compact submanifold M in $\mathbb{R}^m$ is said to be of finite type if q from (10) is finite. Otherwise, M is of infinite type.
If the set $\{t \in \{p, p+1, \ldots, q\} : x_t \neq 0\}$ has exactly k elements, then M is said to be of k-type.

Chen ([4], 1984) gives the following characterization for the submanifolds of finite type.

Theorem 2

Let $x : M \to \mathbb{R}^m$ be an isometric immersion of a compact, n-dimensional Riemannian manifold M. Then M is of finite type if and only if there is a non-trivial polynomial P such as

$$P(\Delta)H = \Delta^k H + c_1 \Delta^{k-1} H + \cdots + c_{k-1}\Delta H + c_k H = 0, \quad c_i \in \mathbb{R}, \quad i = \overline{1, k}, \tag{11}$$

where H is the **mean curvature vector** *defined by*

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i), \tag{12}$$

for any orthonormal frame $e_1, \ldots, e_n$.

Statistical Manifolds

The notions presented in the above subsections intertwine, resulting the theory behind statistical (or parametric) models. Explicitly, a family of probability distributions which depends on a finite number of parameters can be considered a parameterized surface. Denote the set of probability distributions on χ that depends on n parameters $\xi = \left(\xi^1, \ldots, \xi^n\right)$ by

$$\mathcal{S} = \{p_\xi = p(x;\xi)\}. \tag{13}$$

$\mathcal{S}$ is a subset of $\mathcal{P}(\chi) = \left\{f : \chi \to \mathbb{R} : f \geq 0, \int_\chi f dx = 1\right\}$. If the mapping $\xi \to p_\xi$ is an immersion, then the set $\mathcal{S}$ is a *statistical model* of dimension n.

Statistical Manifolds

In our computations, we will make use of the *log-likelihood function* given by

$$\ell_x(\xi) = \ell(p_\xi)(x) = \ln p_\xi(x). \tag{14}$$

The *Fisher information matrix* is given by

$$g_{ij}(\xi) = \mathbf{E}\left[\frac{\partial \ell_x(\xi)}{\partial \xi^i} \cdot \frac{\partial \ell_x(\xi)}{\partial \xi^j}\right], \quad \forall i, j \in \{1, \ldots, n\}, \tag{15}$$

where $\xi = (\xi^1, \ldots, \xi^n) \in \mathbb{R}$. It can be proven (see e.g. [3] O. Călin and C. Udriște, *Geometric Modeling in Probability and Statistics*, Springer, 2014) that for any statistical model, the Fisher information matrix is a Riemannian metric. As a consequence, the pair $(\mathcal{S}, g)$ can be organized as a manifold.

Main Results

In the bellow paragraphs, we present a differential geometric study for the exponential distribution and the Bernoulli distribution.

For the rest of this paper, let $(\Omega, \mathcal{F}, P)$ be a probability space.

Exponential Distribution

We denote the family of exponential distributions by

$$\mathcal{S} = \{p_\xi(x) = \xi e^{-\xi x} : \xi > 0, x \geq 0\}. \tag{16}$$

Proposition 3

The Fisher information matrix of $\mathcal{S}$ is given by

$$g_{11}(\xi) = \frac{1}{\xi^2}. \tag{17}$$

Fisher Metric

Proof.

It is known (see e.g. [3]) that the Fisher information matrix can be written as:

$$g_{ij}(\xi) = -\mathbf{E}\left[\frac{\partial^2 \ell_x(\xi)}{\partial \xi^i \partial \xi^j}\right]. \tag{18}$$

For $\mathcal{S}$, the log-likelihood function is

$$\ell_x(\xi) = \ln\left(\xi e^{-\xi x}\right) = \ln \xi - \xi x,$$

hence

$$\frac{\partial \ell_x(\xi)}{\partial \xi} = \frac{1}{\xi} - x \implies \frac{\partial^2 \ell_x(\xi)}{\partial \xi^2} = -\frac{1}{\xi^2}. \tag{19}$$

Proof.

Finally, formulas (18) and (3) provide the Fisher information matrix

$$g_{11}(\xi) = -\mathbf{E}\left[-\frac{1}{\xi^2}\right] = \int_0^\infty \frac{1}{\xi^2} p_\xi(x)\,\mathrm{d}x = \frac{1}{\xi^2}\int_0^\infty p_\xi(x)\,\mathrm{d}x = \frac{1}{\xi^2}.$$

□

The Christoffel Symbols

We now use the Fisher information matrix to compute the Christoffel symbols for the manifold $(\mathcal{S}, g)$.

Proposition 4

The Christoffel symbols of the first and the second kind of $(\mathcal{S}, g)$ are given by:

$$\Gamma_{11,1} = -\frac{1}{\xi^3} \qquad \textit{and} \qquad \Gamma^1_{11} = -\frac{1}{\xi}. \tag{20}$$

The Christoffel symbols

Proof.

Applying (5) and (6), we obtain:

$$\Gamma_{11,1} = \frac{1}{2}\left(\frac{\partial g_{11}}{\partial \xi} + \frac{\partial g_{11}}{\partial \xi} - \frac{\partial g_{11}}{\partial \xi}\right) = -\frac{1}{2}\cdot\frac{2}{\xi^3} = -\frac{1}{\xi^3},$$

respectively

$$\Gamma_{11}^{1} = \frac{1}{2}g^{11}\left(\frac{\partial g_{11}}{\partial \xi} + \frac{\partial g_{11}}{\partial \xi} - \frac{\partial g_{11}}{\partial \xi}\right) = -\frac{1}{2}\cdot\xi^2\cdot\frac{2}{\xi^3} = -\frac{1}{\xi}.$$

□

The Geodesics

Using the Christoffel symbols in (7), we can compute the geodesics.

Proposition 5

The geodesics of $(\mathcal{S}, g)$ *are given by*

$$\xi(t) = e^{c_1 t + c_2}, \tag{21}$$

where $c_1, c_2 \in \mathbb{R}$ *are constants.*

Proof.

Applying (7), we have

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} + \Gamma^1_{11}\frac{\mathrm{d}\xi}{\mathrm{d}t}\cdot\frac{\mathrm{d}\xi}{\mathrm{d}t} = 0 \iff \frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} - \frac{1}{\xi}\cdot\left(\frac{\mathrm{d}\xi}{\mathrm{d}t}\right)^2 = 0. \tag{22}$$

We obtained the homogeneous differential equation

$$\xi'' - \frac{(\xi')^2}{\xi} = 0. \tag{23}$$

We divide the above equation by ξ $(\xi \neq 0)$.

Proof.

$$\frac{\xi''}{\xi} - \frac{(\xi')^2}{\xi^2} = 0 \iff \frac{\xi'' \cdot \xi - \xi' \cdot \xi'}{\xi^2} = 0 \iff \left(\frac{\xi'}{\xi}\right)' = 0,$$

and by integration, we have

$$\frac{\xi'}{\xi} = c_1 \iff \frac{\mathrm{d}\xi}{\mathrm{d}t} = c_1\xi \iff \frac{\mathrm{d}\xi}{\xi} = c_1\,\mathrm{d}t \iff \int \frac{\mathrm{d}\xi}{\xi} = \int c_1\,\mathrm{d}t$$

$$\iff \ln(\xi) = c_1 t + c_2 \implies \xi(t) = e^{c_1 t + c_2}, \quad c_1, c_2 \text{ constants,} \tag{24}$$

concluding the proof. □

Using (8), we will compute Δf with respect to g. We will find the harmonic functions.

Proposition 6

The Laplace-Beltrami operator acting on differentiable functions $f : \mathcal{S} \to \mathbb{R}$ has the following expression:

$$\Delta f = -\xi \left(\frac{\partial f}{\partial \xi} + \xi \frac{\partial^2 f}{\partial \xi^2} \right). \tag{25}$$

Proof.

We see that $\det g = \dfrac{1}{\xi^2}$ and $g^{11} = \xi^2$. Then

$$\Delta f = -\frac{1}{\sqrt{\frac{1}{\xi^2}}}\frac{\partial}{\partial \xi}\left(\xi^2 \cdot \frac{1}{\xi} \cdot \frac{\partial f}{\partial \xi}\right) = -\xi \frac{\partial}{\partial \xi}\left(\xi \frac{\partial f}{\partial \xi}\right) = -\xi\left(\frac{\partial f}{\partial \xi} + \xi \frac{\partial^2 f}{\partial \xi^2}\right).$$

□

Next we consider the case $\Delta f = 0$. We have the homogeneous differential equation:

$$f' + \xi f'' = 0.$$

We denote by $u(\xi) = f'(\xi)$. Then the above relation becomes

$$u + \xi u' = 0 \iff \xi \frac{\mathrm{d}u}{\mathrm{d}\xi} = -u \iff \frac{\mathrm{d}u}{u} = -\frac{\mathrm{d}\xi}{\xi}.$$

By integration, we have

$$\ln u = -\ln \xi + \ln c_1 \iff \ln u = \ln\left(\frac{c_1}{\xi}\right) \iff u = \frac{c_1}{\xi}$$

$$\iff \frac{\mathrm{d}f}{\mathrm{d}\xi} = \frac{c_1}{\xi} \iff \int \mathrm{d}f = \int \frac{c_1}{\xi}\,\mathrm{d}\xi \iff$$

$$f(\xi) = c_1 \ln \xi + c_2, \quad c_1, c_2 \in \mathbb{R} \text{ constants.} \tag{26}$$

Submanifold of Finite Type

In this subsection, we will use the framework provided by Chen in [4] to study the family of exponential distributions as a curve in $\mathbb{R}^2$.

Theorem 7

The set of exponential distributions is a 1-type curve in $\mathbb{R}^2$.

Proof.

We consider the immersion $x : \mathcal{S} \to \mathbb{R}^2$ defined by:

$$x(\xi) = (\cos(\ln \xi), \sin(\ln \xi)) \,. \tag{27}$$

Indeed, we have

$$\frac{\partial x}{\partial \xi} = \left(-\sin(\ln \xi) \cdot \frac{1}{\xi}, \cos(\ln \xi) \cdot \frac{1}{\xi} \right)$$

and

$$g_{11} = \left\langle \frac{\partial x}{\partial \xi}, \frac{\partial x}{\partial \xi} \right\rangle = \sin^2(\ln \xi) \cdot \frac{1}{\xi^2} + \cos^2(\ln \xi) \cdot \frac{1}{\xi^2} = \frac{1}{\xi^2}, \tag{28}$$

where $\langle \cdot, \cdot \rangle$ is the inner product.

Proof.

We have

$$\frac{\partial^2 x}{\partial \xi^2} = \left(\frac{-\cos(\ln \xi)\frac{1}{\xi}\xi + \sin(\ln \xi)}{\xi^2}, \frac{-\sin(\ln \xi)\frac{1}{\xi}\xi - \cos(\ln \xi)}{\xi^2} \right)$$

$$= \left(\frac{-\cos(\ln \xi) + \sin(\ln \xi)}{\xi^2}, \frac{-\sin(\ln \xi) - \cos(\ln \xi)}{\xi^2} \right).$$

Applying (25), we obtain:

$$\Delta x = -\xi \left(\frac{\partial x}{\partial \xi} + \xi \frac{\partial^2 x}{\partial \xi^2} \right)$$

$$= -\xi \left[\left(\frac{-\sin(\ln \xi)}{\xi}, \frac{\cos(\ln \xi)}{\xi} \right) + \xi \left(\frac{-\cos(\ln \xi) + \sin(\ln \xi)}{\xi^2}, \frac{-\sin(\ln \xi) - \cos(\ln \xi)}{\xi^2} \right) \right]$$

$$= -\xi \left(\frac{-\cos(\ln \xi)}{\xi}, \frac{-\sin(\ln \xi)}{\xi} \right) \implies$$

$$\Delta x = (\cos(\ln \xi), \sin(\ln \xi)). \tag{29}$$

Proof.

It is known (see Chen [4]) that H satisfies

$$\Delta x = -nH, \tag{30}$$

where n is the dimension of the submanifold and H is the mean curvature vector. Then

$$H = -\Delta x \implies H = (-\cos(\ln\xi), -\sin(\ln\xi)). \tag{31}$$

The first and second order partial derivatives are

$$\frac{\partial H}{\partial \xi} = \left(\frac{\sin(\ln\xi)}{\xi}, \frac{-\cos(\ln\xi)}{\xi}\right), \tag{32}$$

$$\frac{\partial^2 H}{\partial \xi^2} = \left(\frac{\cos(\ln\xi) - \sin(\ln\xi)}{\xi^2}, \frac{\sin(\ln\xi) + \cos(\ln\xi)}{\xi^2}\right). \tag{33}$$

Proof.

Hence

$$\Delta H = -\xi \left[\left(\frac{\sin(\ln \xi))}{\xi}, \frac{-\cos(\ln \xi)}{\xi} \right) + \xi \left(\frac{\cos(\ln \xi) - \sin(\ln \xi)}{\xi^2}, \frac{\sin(\ln \xi) + \cos(\ln \xi)}{\xi^2} \right) \right]$$

$$\Delta H = -\left(\cos(\ln \xi), \sin(\ln \xi)\right). \tag{34}$$

We showed that the mean curvature vector satisfies the following relation

$$\Delta H - H = 0, \tag{35}$$

so, by Theorem 2 we conclude that $\mathcal{S}$ is a 1-type curve in $\mathbb{R}^2$. □

As a consequence of (35), we have

Corollary 8

An eigenvalue of the Laplace-Beltrami operator is 1.

Bernoulli Distribution. Fisher Metric

In this section we study the manifold of Bernoulli distributions:

$$\mathcal{S} = \{p(\xi; k) = \xi^k (1-\xi)^{1-k} : \ 0 < \xi < 1, k \in \{0, 1\}\}. \tag{36}$$

Proposition 9

The Fisher information matrix has one element given by

$$g_{11}(\xi) = \frac{1}{\xi(1-\xi)}. \tag{37}$$

Proof.

The log-likelihood for the Bernoulli probability density function is given by

$$\ell_k(\xi) = \ln p(\xi; k) = \ln\left(\xi^k (1-\xi)^{1-k}\right) = k \ln \xi + (1-k)\ln(1-\xi). \tag{38}$$

The first and second derivatives of the log-likelihood with respect to the parameter ξ are given by

$$\frac{\partial \ell_k(\xi)}{\partial \xi} = \frac{k-\xi}{\xi(1-\xi)}. \tag{39}$$

$$\frac{\partial^2 \ell_k(\xi)}{\partial \xi^2} = -\frac{k}{\xi^2} - \frac{1-k}{(1-\xi)^2}. \tag{40}$$

For computing the Fisher metric coefficients we use the formula from [3, Proposition 1.6.3]

$$g_{ij}(\xi) = -\mathbf{E}\left[\frac{\partial^2 \ell_x(\xi)}{\partial \xi^i \partial \xi^j}\right].$$

Proof.

In our case:

$$g_{11}(\xi) = \mathbf{E}\left[\frac{k}{\xi^2} + \frac{1-k}{(1-\xi)^2}\right]. \quad (41)$$

From the definition (4) of the expectation we have that

$$\mathbf{E}[k] = \xi$$
$$\mathbf{E}[1-k] = 1-\xi.$$

So we obtain:

$$g_{11}(\xi) = \frac{1}{\xi} + \frac{1}{1-\xi} = \frac{1}{\xi(1-\xi)}. \quad (42)$$

□

Using the Fisher information matrix, we can compute the Christoffel symbols for the manifold $(\mathcal{S}, g)$.

Proposition 10

The Christoffel symbols of first and second kind are given by

$$\Gamma_{11,1} = \frac{2\xi - 1}{2\xi^2(1-\xi)^2} \tag{43}$$

and

$$\Gamma_{11}^{1} = \frac{2\xi - 1}{2\xi(1-\xi)}. \tag{44}$$

Proof.

By applying formulas (5) and (6) we obtain

$$\Gamma_{11,1} = \frac{1}{2}\left(\frac{\partial g_{11}}{\partial \xi} + \frac{\partial g_{11}}{\partial \xi} - \frac{\partial g_{11}}{\partial \xi}\right) = \frac{1}{2}\left(-\frac{1-2\xi}{\xi^2(1-\xi)^2}\right) = \frac{2\xi - 1}{2\xi^2(1-\xi)^2},$$

$$\Gamma^1_{11} = \frac{1}{2} g^{11}\left(\frac{\partial g_{11}}{\partial \xi} + \frac{\partial g_{11}}{\partial \xi} - \frac{\partial g_{11}}{\partial \xi}\right),$$

where g^{11} is the inverse of the Fisher matrix, in our case $g^{11} = \xi(1-\xi)$.

$$\Gamma^1_{11} = \frac{1}{2}\xi(1-\xi)\frac{2\xi - 1}{\xi^2(1-\xi)^2} = \frac{2\xi - 1}{2\xi(1-\xi)}.$$

□

By replacing the formulas obtained in (44) for the Christoffel symbols in (7), we can compute the geodesics equations.

Proposition 11

The geodesics for the Bernoulli distribution model are given by

$$\xi(t) = \frac{1}{2}(1 + \sin(c_1 t + c_2)), \quad c_1, c_2 \in \mathbb{R}. \tag{45}$$

Proof.

From (7) we have

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} + \Gamma_{11}^1 \frac{\mathrm{d}\xi}{\mathrm{d}t} \cdot \frac{\mathrm{d}\xi}{\mathrm{d}t} = 0 \iff \frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} + \frac{2\xi - 1}{2\xi(1-\xi)} \frac{\mathrm{d}\xi}{\mathrm{d}t} \cdot \frac{\mathrm{d}\xi}{\mathrm{d}t} = 0.$$

We make the substitution $\frac{\mathrm{d}\xi}{\mathrm{d}t} = u \implies \frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} = \frac{\mathrm{d}u}{\mathrm{d}\xi}\frac{\mathrm{d}\xi}{\mathrm{d}t} \iff \xi'' = u\frac{\mathrm{d}u}{\mathrm{d}\xi}$. By replacing this in the equation above, we get

$$u\frac{\mathrm{d}u}{\mathrm{d}\xi} + \frac{2\xi - 1}{2\xi(1-\xi)} u^2 = 0. \tag{46}$$

Proof.

We distinguish 2 cases:

1. $u = 0 \iff \frac{\mathrm{d}\xi}{\mathrm{d}t} = 0 \iff \xi(t) = c,\ c \in \mathbb{R};$
2. $u \neq 0$. We divide (46) by u, which leads to

$$\frac{\mathrm{d}u}{\mathrm{d}\xi} = -\frac{2\xi - 1}{2\xi(1-\xi)}u \iff \frac{\mathrm{d}u}{u} = \frac{1-2\xi}{2\xi(1-\xi)}\,\mathrm{d}\xi.$$

   By integrating both sides, we get

$$\ln u = \ln\left(c\xi(1-\xi)\right)^{\frac{1}{2}}, \quad c \in \mathbb{R}.$$

Proof.

We obtain

$$u = (c\xi(1-\xi))^{\frac{1}{2}} \iff \frac{\mathrm{d}\xi}{\mathrm{d}t} = (c\xi(1-\xi))^{\frac{1}{2}} \iff \frac{\mathrm{d}\xi}{\sqrt{\xi(1-\xi)}} = c_1\,\mathrm{d}t, \quad c_1 \in \mathbb{R}.$$

By integrating both sides, we get

$$\begin{aligned} \arcsin(2\xi - 1) = c_1 t + c_2, \iff \\ \xi(t) = \frac{1}{2}(1 + \sin(c_1 t + c_2)), \quad c_1, c_2 \in \mathbb{R}. \end{aligned} \tag{47}$$

This ends the proof. □

Laplace-Beltrami Operator

Proposition 12

The formula for the Laplace-Beltrami operator acting on smooth functions $f : \mathcal{S} \to \mathbb{R}$ *is:*

$$\Delta f = \frac{2\xi - 1}{2}\frac{\partial f}{\partial \xi} - \xi(1-\xi)\frac{\partial^2 f}{\partial \xi^2}. \tag{48}$$

Proof.

Applying formula (8) for the Bernoulli distribution model, this becomes

$$\Delta f = -\sqrt{\xi(1-\xi)}\frac{\partial}{\partial \xi}\left(\xi(1-\xi)\sqrt{\frac{1}{\xi(1-\xi)}}\frac{\partial f}{\partial \xi}\right) \iff$$

$$\Delta f = -\sqrt{\xi(1-\xi)}\left(\frac{\partial}{\partial \xi}\cdot\frac{\partial f}{\partial \xi} + \sqrt{\xi(1-\xi)}\cdot\frac{\partial^2 f}{\partial \xi^2}\right) \iff$$

$$\Delta f = -\sqrt{\xi(1-\xi)}\left(\frac{1-2\xi}{2\sqrt{\xi(1-\xi)}}\cdot\frac{\partial f}{\partial \xi} + \sqrt{\xi(1-\xi)}\cdot\frac{\partial^2 f}{\partial \xi^2}\right) \iff$$

$$\Delta f = \frac{2\xi-1}{2}\frac{\partial f}{\partial \xi} - \xi(1-\xi)\frac{\partial^2 f}{\partial \xi^2}. \tag{49}$$

□

Proposition 13

Harmonic functions have the following expression

$$f(\xi) = c_1 \arcsin(2\xi - 1) + c_2, \qquad c_1, c_2 \in \mathbb{R}. \tag{50}$$

Proof.

Harmonic functions are those that satisfy $\Delta f = 0$

$$\frac{2\xi - 1}{2}\frac{\mathrm{d}f}{\mathrm{d}\xi} - \xi(1-\xi)\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2} = 0.$$

We make the substitution $\frac{\mathrm{d}f}{\mathrm{d}\xi} = u \iff f'' = \frac{\mathrm{d}u}{\mathrm{d}\xi}$ and we obtain, by rearranging the terms and also taking into account that $0 < \xi < 1$

$$\frac{2\xi - 1}{2}u - \xi(1-\xi)\frac{\mathrm{d}u}{\mathrm{d}\xi} = 0 \iff$$

$$\frac{\mathrm{d}u}{u} = \frac{2\xi - 1}{2\xi(1-\xi)}\,\mathrm{d}\xi.$$

Proof.

By integrating both sides, we get

$$\ln u = \ln \frac{c_1}{\sqrt{\xi(1-\xi)}} \iff u = \frac{c_1}{\sqrt{\xi(1-\xi)}} \iff \frac{\mathrm{d}f}{\mathrm{d}\xi} = \frac{c_1}{\sqrt{\xi(1-\xi)}} \iff$$
$$\mathrm{d}f = c_1 \frac{1}{\sqrt{\xi(1-\xi)}}\mathrm{d}\xi \iff f(\xi) = c_1 \arcsin(2\xi - 1) + c_2, \qquad c_1, c_2 \in \mathbb{R}.$$

□

We use again the framework provided by Chen in [4] to study the type of the family of Bernoulli distributions.

Theorem 14

The family of Bernoulli distributions is 1*-type curve in* $\mathbb{R}^2$.

Proof.

Consider the immersion $x : \mathcal{S} \to \mathbb{R}^2$,

$$x(\xi) = (2\xi^{\frac{1}{2}}, -2(1-\xi)^{\frac{1}{2}}), \qquad 0 < \xi < 1. \tag{51}$$

We have

$$\frac{\partial x}{\partial \xi} = (\xi^{-\frac{1}{2}}, (1-\xi)^{-\frac{1}{2}}),$$

$$g_{11} = \left\langle \frac{\partial x}{\partial \xi}, \frac{\partial x}{\partial \xi} \right\rangle = \frac{1}{\xi} + \frac{1}{1-\xi} = \frac{1}{\xi(1-\xi)},$$

the same as the Fisher matrix coefficient.

Proof.

The second order derivative is

$$\frac{\partial^2 x}{\partial \xi^2} = \left(-\frac{1}{2}\xi^{-\frac{3}{2}}, \frac{1}{2}(1-\xi)^{-\frac{3}{2}}\right).$$

By using (48) we obtain

$$\Delta x = \frac{2\xi - 1}{2}\left(\xi^{-\frac{1}{2}}, (1-\xi)^{-\frac{1}{2}}\right) - \xi(1-\xi)\left(-\frac{1}{2}\xi^{-\frac{3}{2}}, \frac{1}{2}(1-\xi)^{-\frac{3}{2}}\right).$$

After doing the computations we obtain

$$\Delta x = \left(\frac{1}{2}\sqrt{\xi}, -\frac{1}{2}\sqrt{1-\xi}\right). \tag{52}$$

Proof.

We know (see Chen [4]) that the mean curvature vector H satisfies

$$\Delta x = -nH, \tag{53}$$

where n is the dimension of the submanifold; in our case $n = 1$. This implies $\Delta x = -H \implies H = -\Delta x$, which means

$$H = \left(-\frac{1}{2}\sqrt{\xi}, \frac{1}{2}\sqrt{1-\xi}\right). \tag{54}$$

We compute the first and second order derivatives of H

$$\frac{\partial H}{\partial \xi} = \left(-\frac{1}{4\sqrt{\xi}}, -\frac{1}{4\sqrt{1-\xi}}\right). \tag{55}$$

$$\frac{\partial^2 H}{\partial \xi^2} = \left(\frac{1}{8\xi^{\frac{3}{2}}}, -\frac{1}{8(1-\xi)^{\frac{3}{2}}}\right). \tag{56}$$

Proof.

By using the formula (48) for the Laplace-Beltrami operator, (55), and (56) we obtain

$$\Delta H = \frac{2\xi - 1}{2}\frac{\partial H}{\partial \xi} - \xi(1-\xi)\frac{\partial^2 H}{\partial \xi^2} \iff$$

$$\Delta H = \left(-\frac{1}{8}\sqrt{\xi}, \frac{1}{8}\sqrt{1-\xi}\right). \tag{57}$$

From (54) and (57) we see that the following is true

$$-4\Delta H + H = 0. \tag{58}$$

By the characterization Theorem 2 for submanifolds of finite type, we conclude that $\mathcal{S}$ is a 1-type curve in $\mathbb{R}^2$. □

As a consequence of (58) we have

Corollary 15

An eigenvalue of the Laplace-Beltrami operator is $1/4$.

Conclusions

Inspired by the geometric study of the set of normal distributions presented by Călin and Udriște in [3], we have conducted a similar study for the family of exponential distributions and for the family of Bernoulli distributions. In addition to the mentioned work, we have found appropriate immersions in order to apply the characterization theorem for submanifolds of finite type given by Chen in [4] and conclude that the exponential and Bernoulli statistical models are both 1-type curves in $\mathbb{R}^2$.

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