Invariant λ-solitons

Irene Ortiz Sánchez
(Centro Universitario de la Defensa de San Javier)
Joint work with Antonio Bueno and Rafael López

Differential Geometry Workshop 2023

September 06-09 2023
University of Iasi

ALEXANDRU IOAN CUZA
UNIVERSITY of IAŞI

8th September, 2023

Supported by Fundación Séneca, REF. 21899/PI/22

Summary

1. Introduction
2. Characterizations of λ-solitons
3. Classification of rotational λ-solitons
4. Classification of cylindrical λ-solitons
5. Stability results of λ-solitons. Plateau-Rayleigh phenomenon

Summary

1. Introduction
2. Characterizations of λ-solitons
3. Classification of rotational λ-solitons
4. Classification of cylindrical λ-solitons
5. Stability results of λ-solitons. Plateau-Rayleigh phenomenon

Notation

- Throughout this work, we consider Σ an orientable surface immersed into $\mathbb{R}^3$.

Notation

- Throughout this work, we consider Σ an orientable surface immersed into $\mathbb{R}^3$.
- The function $N : \Sigma \to \mathbb{S}^2$ stands for the Gauss map of the surface.

Notation

- Throughout this work, we consider Σ an orientable surface immersed into $\mathbb{R}^3$.
- The function $N : \Sigma \to \mathbb{S}^2$ stands for the Gauss map of the surface.
- The principal curvatures of Σ are denoted by κ_1 and κ_2.

Notation

- Throughout this work, we consider Σ an orientable surface immersed into $\mathbb{R}^3$.
- The function $N : \Sigma \to \mathbb{S}^2$ stands for the Gauss map of the surface.
- The principal curvatures of Σ are denoted by κ_1 and κ_2.
- The mean and Gaussian curvature of Σ, are defined, respectively, by

$$H(p) = \kappa_1(p) + \kappa_2(p),\ K(p) = \kappa_1(p)\kappa_2(p),\ p \in \Sigma.$$

Notation

- Throughout this work, we consider Σ an orientable surface immersed into $\mathbb{R}^3$.
- The function $N : \Sigma \to \mathbb{S}^2$ stands for the Gauss map of the surface.
- The principal curvatures of Σ are denoted by κ_1 and κ_2.
- The mean and Gaussian curvature of Σ, are defined, respectively, by

$$H(p) = \kappa_1(p) + \kappa_2(p), \ K(p) = \kappa_1(p)\kappa_2(p), \ p \in \Sigma.$$

- The shape operator of Σ is denoted by A and it is satisfied that

$$|A|^2 = \kappa_1^2 + \kappa_2^2 = H^2 - 2K.$$

Translating λ-solitons

Definition (Translating λ-soliton)

Let $v \in \mathbb{R}^3$, $|v| = 1$, called the ***density vector***. An oriented surface Σ in $\mathbb{R}^3$ is a ***translating λ-soliton*** with respect to v if

$$H_\Sigma(p) = \langle N_p, v \rangle + \lambda \quad \forall p \in \Sigma. \tag{1.1}$$

Translating λ-solitons

Definition (Translating λ-soliton)

Let $v \in \mathbb{R}^3$, $|v| = 1$, called the ***density vector***. An oriented surface Σ in $\mathbb{R}^3$ is a ***translating λ-soliton*** with respect to v if

$$H_\Sigma(p) = \langle N_p, v \rangle + \lambda \quad \forall p \in \Sigma. \tag{1.1}$$

Observe that:

1. On the one hand, they are a particular case of prescribed mean curvature (PMC) surfaces.

$$H_\Sigma(p) = \mathcal{H}(N_p) \; \forall p \in \Sigma \text{ for a given } \mathcal{H} \in C^1(\mathbb{S}^2).$$

In this case, $\mathcal{H}(N_p) = \langle N_p, v \rangle + \lambda$.

Translating λ-solitons

Definition (Translating λ-soliton)

Let $v \in \mathbb{R}^3$, $|v| = 1$, called the ***density vector***. An oriented surface Σ in $\mathbb{R}^3$ is a ***translating λ-soliton*** with respect to v if

$$H_\Sigma(p) = \langle N_p, v \rangle + \lambda \quad \forall p \in \Sigma. \tag{1.1}$$

Observe that:

1. On the one hand, they are a particular case of prescribed mean curvature (PMC) surfaces.

   $$H_\Sigma(p) = \mathcal{H}(N_p)\ \forall p \in \Sigma \text{ for a given } \mathcal{H} \in C^1(\mathbb{S}^2).$$

   In this case, $\mathcal{H}(N_p) = \langle N_p, v \rangle + \lambda$.

2. On the other hand, if $\lambda = 0$ they generalize self-translating solitons of the mean curvature flow.

Translating λ-solitons

Definition (Translating λ-soliton)

Let $v \in \mathbb{R}^3$, $|v| = 1$, called the ***density vector***. An oriented surface Σ in $\mathbb{R}^3$ is a ***translating λ-soliton*** with respect to v if

$$H_\Sigma(p) = \langle N_p, v\rangle + \lambda \quad \forall p \in \Sigma. \tag{1.1}$$

Observe that:

1. On the one hand, they are a particular case of prescribed mean curvature (PMC) surfaces.

   $$H_\Sigma(p) = \mathcal{H}(N_p)\ \forall p \in \Sigma \text{ for a given } \mathcal{H} \in C^1(\mathbb{S}^2).$$

   In this case, $\mathcal{H}(N_p) = \langle N_p, v\rangle + \lambda$.

2. On the other hand, if $\lambda = 0$ they generalize self-translating solitons of the mean curvature flow.
3. Moreover, they are closely related to CMC surfaces.

Translating λ-solitons

Consequently, to obtain results for translating λ-solitons we take into account:

Translating λ-solitons

Consequently, to obtain results for translating λ-solitons we take into account:

1. **Results of PMC surfaces**:

Translating λ-solitons

Consequently, to obtain results for translating λ-solitons we take into account:

1. **Results of PMC surfaces**:
   - Motivation: Minkowski's problem. $K(p) = \mathcal{K}(N(p))$.

Translating λ-solitons

Consequently, to obtain results for translating λ-solitons we take into account:

1. **Results of PMC surfaces**:
    - Motivation: Minkowski's problem. $K(p) = \mathcal{K}(N(p))$.
    - Existence and uniqueness of PMC spheres: Alexandrov, Pogorelov, Hartman, Witner, B. Guan, P. Guan, Gálvez, Mira.

Translating λ-solitons

Consequently, to obtain results for translating λ-solitons we take into account:

1. **Results of PMC surfaces**:

   - Motivation: Minkowski's problem. $K(p) = \mathcal{K}(N(p))$.
   - Existence and uniqueness of PMC spheres: Alexandrov, Pogorelov, Hartman, Witner, B. Guan, P. Guan, Gálvez, Mira.
   - Other problems of PMC surfaces: properly embedded; rotational; half-space theorems; Björling problem, other ambient spaces.

Translating λ-solitons

Consequently, to obtain results for translating λ-solitons we take into account:

1. **Results of PMC surfaces**:
   - Motivation: Minkowski's problem. $K(p) = \mathcal{K}(N(p))$.
   - Existence and uniqueness of PMC spheres: Alexandrov, Pogorelov, Hartman, Witner, B. Guan, P. Guan, Gálvez, Mira.
   - Other problems of PMC surfaces: properly embedded; rotational; half-space theorems; Björling problem, other ambient spaces.
   - Related problems studied by the authors: prescribed Gaussian curvature and predetermined linear Weingarten curvature, ...

Translating λ-solitons

Consequently, to obtain results for translating λ-solitons we take into account:

1. **Results of PMC surfaces**:
   - Motivation: Minkowski's problem. $K(p) = \mathcal{K}(N(p))$.
   - Existence and uniqueness of PMC spheres: Alexandrov, Pogorelov, Hartman, Witner, B. Guan, P. Guan, Gálvez, Mira.
   - Other problems of PMC surfaces: properly embedded; rotational; half-space theorems; Björling problem, other ambient spaces.
   - Related problems studied by the authors: prescribed Gaussian curvature and predetermined linear Weingarten curvature, ...
   - A. Bueno, I. Ortiz, Invariant hypersurfaces with linear prescribed mean curvature, J. Math. Anal. Appl. 487 (2020).

Translating λ-solitons

Consequently, to obtain results for translating λ-solitons we take into account:

Translating λ-solitons

Consequently, to obtain results for translating λ-solitons we take into account:

2. **Results of self-translating solitons of the mean curvature flow**:

Translating λ-solitons

Consequently, to obtain results for translating λ-solitons we take into account:

2. **Results of self-translating solitons of the mean curvature flow**:
   - Widely studied during the last decades: Hoffman, Huisken, Ilmanen, F. Martín, Spruck, White, Xiao.

Translating λ-solitons

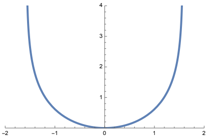
Consequently, to obtain results for translating λ-solitons we take into account:

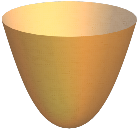
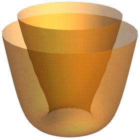
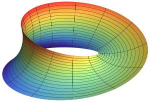
2. **Results of self-translating solitons of the mean curvature flow**:
   - Widely studied during the last decades: Hoffman, Huisken, Ilmanen, F. Martín, Spruck, White, Xiao.
   - They are minimal surfaces in $(\mathbb{R}^3, e^{2x_3}\langle \cdot, \cdot \rangle)$. In particular, the tangency principle is satisfied.

Translating λ-solitons

Consequently, to obtain results for translating λ-solitons we take into account:

2. **Results of self-translating solitons of the mean curvature flow**:
   - Widely studied during the last decades: Hoffman, Huisken, Ilmanen, F. Martín, Spruck, White, Xiao.
   - They are minimal surfaces in $(\mathbb{R}^3, e^{2x_3}\langle\cdot,\cdot\rangle)$. In particular, the tangency principle is satisfied.
   - Cylindrical solitons and rotationals. Non orientable examples.



Translating λ-solitons

Consequently, to obtain results for translating λ-solitons we take into account:

Translating λ-solitons

Consequently, to obtain results for translating λ-solitons we take into account:

3. **Analogy with CMC: solutions of a variational problem.**

Translating λ-solitons

Consequently, to obtain results for translating λ-solitons we take into account:

3. **Analogy with CMC: solutions of a variational problem.**

Remind that if Σ is a surface and $u \in C_0^\infty(\Sigma)$, $u = 0$ in $\partial\Sigma$, a normal variation of compact support is defined by

$$\psi(p,t) = p + tu(p)N(p), \quad |t| < \epsilon, \quad \Sigma_t = \{\psi(p,t) \colon p \in \Sigma\}.$$

Translating λ-solitons

Consequently, to obtain results for translating λ-solitons we take into account:

3. **Analogy with CMC: solutions of a variational problem.**

Remind that if Σ is a surface and $u \in C_0^\infty(\Sigma)$, $u = 0$ in $\partial\Sigma$, a normal variation of compact support is defined by

$$\psi(p,t) = p + tu(p)N(p), \quad |t| < \epsilon, \quad \Sigma_t = \{\psi(p,t) \colon p \in \Sigma\}.$$

CMC surfaces

- Area and volume functionals:

$$\mathcal{A}(t) = \int_\Sigma d\Sigma_t, \quad \mathcal{V}(t) = \int_{[0,t]\times\Sigma} |\mathrm{Jac}\psi| dV.$$

Translating λ-solitons

Consequently, to obtain results for translating λ-solitons we take into account:

3. **Analogy with CMC: solutions of a variational problem.**

Remind that if Σ is a surface and $u \in C_0^\infty(\Sigma)$, $u = 0$ in $\partial\Sigma$, a normal variation of compact support is defined by

$$\psi(p,t) = p + tu(p)N(p), \quad |t| < \epsilon, \quad \Sigma_t = \{\psi(p,t) \colon p \in \Sigma\}.$$

CMC surfaces

- Area and volume functionals:

$$\mathcal{A}(t) = \int_\Sigma d\Sigma_t, \quad \mathcal{V}(t) = \int_{[0,t]\times\Sigma} |\mathrm{Jac}\psi| dV.$$

- First variation of the area and volume functionals:

$$\mathcal{A}'(0) = -\int_\Sigma uH d\Sigma, \quad \mathcal{V}'(0) = \int_\Sigma u d\Sigma.$$

Translating λ-solitons

Consequently, to obtain results for translating λ-solitons we take into account:

3. **Analogy with CMC: solutions of a variational problem.**

Remind that if Σ is a surface and $u \in C_0^\infty(\Sigma)$, $u = 0$ in $\partial\Sigma$, a normal variation of compact support is defined by

$$\psi(p,t) = p + tu(p)N(p), \quad |t| < \epsilon, \quad \Sigma_t = \{\psi(p,t) \colon p \in \Sigma\}.$$

CMC surfaces

- Area and volume functionals:

$$\mathcal{A}(t) = \int_\Sigma d\Sigma_t, \quad \mathcal{V}(t) = \int_{[0,t]\times\Sigma} |\mathrm{Jac}\psi| dV.$$

- First variation of the area and volume functionals:

$$\mathcal{A}'(0) = -\int_\Sigma uH d\Sigma, \quad \mathcal{V}'(0) = \int_\Sigma u d\Sigma.$$

- Critical points of the area functional: minimal surfaces and CMC surfaces preserving the enclosed volume (i.e. $\int_\Sigma u d\Sigma = 0$).

Translating λ-solitons

Consequently, to obtain results for translating λ-solitons we take into account:

3. **Analogy with CMC: solutions of a variational problem.**

Translating λ-solitons

Translating λ-solitons

Consequently, to obtain results for translating λ-solitons we take into account:

3. **Analogy with CMC: solutions of a variational problem.**

Translating λ-solitons

- Given $\phi \in C^\infty(\mathbb{R}^3)$, we define $d\Sigma_\phi = e^\phi d\Sigma$ and $dV_\phi = e^\phi dV$.

Consequently, to obtain results for translating λ-solitons we take into account:

3. **Analogy with CMC: solutions of a variational problem.**

Translating λ-solitons

- Given $\phi \in C^\infty(\mathbb{R}^3)$, we define $d\Sigma_\phi = e^\phi d\Sigma$ and $dV_\phi = e^\phi dV$.
- Weighted area and volume functionals, $\mathcal{A}_\phi$ and $\mathcal{V}_\phi$, are defined.

Translating λ-solitons

Consequently, to obtain results for translating λ-solitons we take into account:

3. **Analogy with CMC: solutions of a variational problem.**

Translating λ-solitons

- Given $\phi \in C^\infty(\mathbb{R}^3)$, we define $d\Sigma_\phi = e^\phi d\Sigma$ and $dV_\phi = e^\phi dV$.
- Weighted area and volume functionals, $\mathcal{A}_\phi$ and $\mathcal{V}_\phi$, are defined.
- First variation of the weighted area and volume functionals:

$$\mathcal{A}'_\phi(0) = -\int_\Sigma u(H - \langle N, D\phi\rangle)d\Sigma_\phi = -\int_\Sigma uH_\phi d\Sigma_\phi,$$

$$\mathcal{V}'_\phi(0) = -\int_\Sigma u d\Sigma_\phi.$$

Translating λ-solitons

Consequently, to obtain results for translating λ-solitons we take into account:

3. **Analogy with CMC: solutions of a variational problem.**

Translating λ-solitons

- Given $\phi \in C^\infty(\mathbb{R}^3)$, we define $d\Sigma_\phi = e^\phi d\Sigma$ and $dV_\phi = e^\phi dV$.
- Weighted area and volume functionals, $\mathcal{A}_\phi$ and $\mathcal{V}_\phi$, are defined.
- First variation of the weighted area and volume functionals:

$$\mathcal{A}'_\phi(0) = -\int_\Sigma u(H - \langle N, D\phi\rangle)d\Sigma_\phi = -\int_\Sigma uH_\phi d\Sigma_\phi,$$

$$\mathcal{V}'_\phi(0) = -\int_\Sigma u d\Sigma_\phi.$$

- Critical points of the weighted area functional: surfaces with $H_\phi = 0$ and surfaces with $H_\phi = \lambda$ under variations preserving the enclosed weighted volume (i.e. $\int_\Sigma u d\Sigma_\phi = 0$).

Summary

First characterization of λ-solitons

λ-solitons $\leftrightarrow$ theory of manifolds with density.

First characterization of λ-solitons

λ-solitons $\leftrightarrow$ theory of manifolds with density.

Definition (Weighted mean curvature)

The ***weighted mean curvature*** H_ϕ of an oriented surface Σ in $\mathbb{R}^3$ with respect to the density $e^\phi \in C^\infty(\mathbb{R}^3)$ is defined by

$$H_\phi := H_\Sigma - \langle N, D\phi \rangle,$$

where D is the gradient operator in $\mathbb{R}^3$.

First characterization of λ-solitons

λ-solitons $\leftrightarrow$ theory of manifolds with density.

Definition (Weighted mean curvature)

The ***weighted mean curvature*** H_ϕ of an oriented surface Σ in $\mathbb{R}^3$ with respect to the density $e^\phi \in C^\infty(\mathbb{R}^3)$ is defined by

$$H_\phi := H_\Sigma - \langle N, D\phi \rangle,$$

where D is the gradient operator in $\mathbb{R}^3$.

- If we consider $\phi_v(x) = \langle x, v \rangle$,

$$\Sigma \text{ is a } \lambda\text{-soliton} \iff H_{\phi_v} = \lambda$$

Second characterization of λ-solitons

Second characterization of λ-solitons

- We have seen that λ-solitons appear as critical points of a variational problem.

Second characterization of λ-solitons

- We have seen that λ-solitons appear as critical points of a variational problem.

Definition (Weighted area and weighted volume)

Let $\Omega \subset \mathbb{R}^3$ be a measurable set with $\Sigma = \partial\Omega$. Then, the ***weighted area and volume*** with respect to ϕ_v are

$$\mathcal{A}_{\phi_v}(\Sigma) := \int_\Sigma e^{\phi_v} d\Sigma, \quad \mathcal{V}_{\phi_v}(\Omega) := \int_\Omega e^{\phi_v} dV,$$

where $d\Sigma$ and dV are the usual area and volume elements in $\mathbb{R}^3$.

Second characterization of λ-solitons

- We have seen that λ-solitons appear as critical points of a variational problem.

Definition (Weighted area and weighted volume)

Let $\Omega \subset \mathbb{R}^3$ be a measurable set with $\Sigma = \partial\Omega$. Then, the ***weighted area and volume*** with respect to ϕ_v are

$$\mathcal{A}_{\phi_v}(\Sigma) := \int_\Sigma e^{\phi_v} d\Sigma, \quad \mathcal{V}_{\phi_v}(\Omega) := \int_\Omega e^{\phi_v} dV,$$

where $d\Sigma$ and dV are the usual area and volume elements in $\mathbb{R}^3$.

- Σ is a λ-soliton $\iff$

  $\mathcal{A}'_{\phi_v}(0) = 0 \ \forall u \in C_0^\infty(\Sigma)$ s.t. $\int_\Sigma u d\Sigma_\phi = 0 \iff$

  Σ is a critical point under c.s.v of $L_{\phi_v} := \mathcal{A}_{\phi_v} - \lambda \mathcal{V}_{\phi_v}$.

Third characterization of λ-solitons

- Consider $\psi : \Sigma \to \mathbb{R}^3$ a λ-soliton.

Third characterization of λ-solitons

- Consider $\psi : \Sigma \to \mathbb{R}^3$ a λ-soliton.

- Then, the family of translations of ψ in the v direction given by $F(p, t) = \psi(p) + tv$ is the solution of the geometric flow

$$\left(\frac{\partial F}{\partial t}\right)^{\perp} = (H_\Sigma - \lambda)N,$$

  which corresponds to the mean curvature flow with a constant forcing term.

Third characterization of λ-solitons

- Consider $\psi : \Sigma \to \mathbb{R}^3$ a λ-soliton.
- Then, the family of translations of ψ in the v direction given by $F(p,t) = \psi(p) + tv$ is the solution of the geometric flow

$$\left(\frac{\partial F}{\partial t}\right)^{\perp} = (H_{\Sigma} - \lambda)N,$$

  which corresponds to the mean curvature flow with a constant forcing term.

- Thus,

  Σ is a λ-soliton $\iff$ Σ is a self-translating soliton of the mean curvature flow with a constant forcing term.

Characterizations of λ-solitons

Summing up, we get the following result.

Characterizations of λ-solitons

Summing up, we get the following result.

Proposition

The following conditions are equivalent:

1. Σ is a λ-soliton.
2. Σ has constant weighted mean curvature $H_{\phi_v} = \lambda$ for the density $e^{\phi} \in C^{\infty}(\mathbb{R}^3)$, where $\phi_v(x) = \langle x, v \rangle$.
3. Σ is a critical point of $\mathcal{A}_{\phi_v}$ under compactly supported variations preserving the enclosed weighted volume.
4. Σ is a critical point under compactly supported variations of the functional $L_{\phi_v} := \mathcal{A}_{\phi_v} - \lambda \mathcal{V}_{\phi_v}$.
5. Σ is a self-translating soliton of the mean curvature flow with a constant forcing term.

Summary

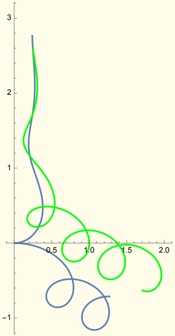
Classification of rotational λ-solitons

Theorem 1 (López, Bueno-Ortiz)

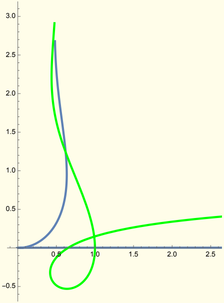
The rotational λ-solitons are given by:

- Either a vertical cylinder of radius $r_\lambda = 1/\lambda$,
- or one of the resulting surfaces after rotate the following profile curves around the axis x_3.

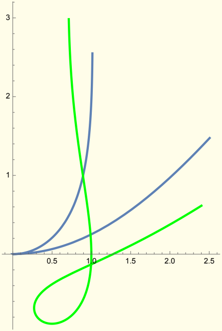
$\lambda > 1$



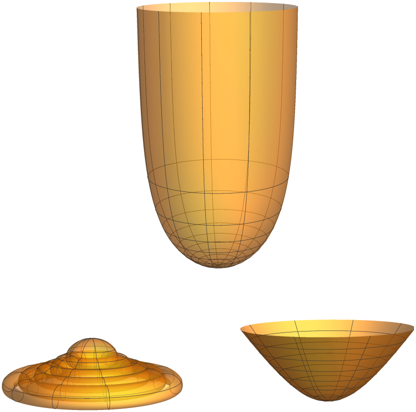
$\lambda = 1$



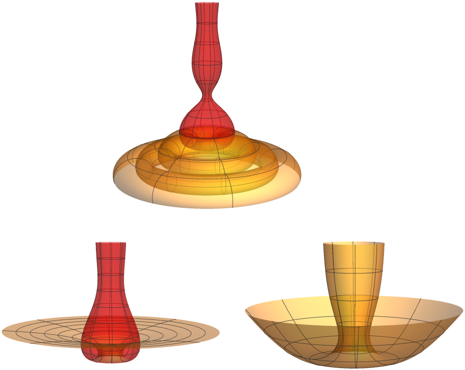
$\lambda < 1$



Rotational λ-solitons obtained rotating the profile curves



Rotational λ-solitons intersecting the rotation axis.



Rotational λ-solitons non-intersecting the rotation axis.

Summary

1. Introduction

2. Characterizations of λ-solitons

3. Classification of rotational λ-solitons

4. Classification of cylindrical λ-solitons

5. Stability results of λ-solitons. Plateau-Rayleigh phenomenon

Cylindrical λ-solitons

Definition (Cylindrical surface)

A surface $\Sigma \subset \mathbb{R}^3$ is ***cylindrical*** if it can be parametrized by

$$\psi(s,t) = \alpha(s) + ta, \quad a \in \mathbb{R}^3,\ |a| = 1. \tag{4.1}$$

where α is a curve, called the ***base curve***, contained in a 2-dimensional plane Π orthogonal to the ruling direction a.

Cylindrical λ-solitons

Definition (Cylindrical surface)

A surface $\Sigma \subset \mathbb{R}^3$ is ***cylindrical*** if it can be parametrized by

$$\psi(s,t) = \alpha(s) + ta, \quad a \in \mathbb{R}^3,\ |a| = 1. \tag{4.1}$$

where α is a curve, called the ***base curve***, contained in a 2-dimensional plane Π orthogonal to the ruling direction a.

Observe that if Σ is a cylindrical λ-soliton:

Cylindrical λ-solitons

Definition (Cylindrical surface)

A surface $\Sigma \subset \mathbb{R}^3$ is ***cylindrical*** if it can be parametrized by

$$\psi(s,t) = \alpha(s) + ta, \quad a \in \mathbb{R}^3,\ |a| = 1. \tag{4.1}$$

where α is a curve, called the ***base curve***, contained in a 2-dimensional plane Π orthogonal to the ruling direction a.

Observe that if Σ is a cylindrical λ-soliton:

- The principal curvatures are 0 and κ_α and $N(\psi(s,t)) = n_\alpha(s)$.

Cylindrical λ-solitons

Definition (Cylindrical surface)

A surface $\Sigma \subset \mathbb{R}^3$ is ***cylindrical*** if it can be parametrized by

$$\psi(s,t) = \alpha(s) + ta, \quad a \in \mathbb{R}^3,\ |a| = 1. \tag{4.1}$$

where α is a curve, called the ***base curve***, contained in a 2-dimensional plane Π orthogonal to the ruling direction a.

Observe that if Σ is a cylindrical λ-soliton:

- The principal curvatures are 0 and κ_α and $N(\psi(s,t)) = n_\alpha(s)$.
- Then, from (1.1) we get the equation for cylindrical λ-solitons

$$\kappa_\alpha(s) = \langle n_\alpha(s), v\rangle + \lambda. \tag{4.2}$$

Cylindrical λ-solitons

Definition (Cylindrical surface)

A surface $\Sigma \subset \mathbb{R}^3$ is ***cylindrical*** if it can be parametrized by

$$\psi(s,t) = \alpha(s) + ta, \quad a \in \mathbb{R}^3,\ |a| = 1. \tag{4.1}$$

where α is a curve, called the ***base curve***, contained in a 2-dimensional plane Π orthogonal to the ruling direction a.

Observe that if Σ is a cylindrical λ-soliton:

- The principal curvatures are 0 and κ_α and $N(\psi(s,t)) = n_\alpha(s)$.
- Then, from (1.1) we get the equation for cylindrical λ-solitons

$$\kappa_\alpha(s) = \langle n_\alpha(s), v\rangle + \lambda. \tag{4.2}$$

- We assume that a and v are orthogonal and $\lambda \geq 0$. ($a = e_2$, $v = e_3$)

Cylindrical λ-solitons

Definition (Cylindrical surface)

A surface $\Sigma \subset \mathbb{R}^3$ is ***cylindrical*** if it can be parametrized by

$$\psi(s,t) = \alpha(s) + ta, \quad a \in \mathbb{R}^3, \ |a| = 1. \tag{4.1}$$

where α is a curve, called the ***base curve***, contained in a 2-dimensional plane Π orthogonal to the ruling direction a.

Observe that if Σ is a cylindrical λ-soliton:

- The principal curvatures are 0 and κ_α and $N(\psi(s,t)) = n_\alpha(s)$.
- Then, from (1.1) we get the equation for cylindrical λ-solitons

$$\kappa_\alpha(s) = \langle n_\alpha(s), v\rangle + \lambda. \tag{4.2}$$

- We assume that a and v are orthogonal and $\lambda \geq 0$. ($a = e_2$, $v = e_3$)
- Thus, by (4.1) Σ can be parametrized as $\Psi(s,t) = (\alpha_1(s), t, \alpha_3(s))$.

Cylindrical λ-solitons

Definition (Cylindrical surface)

A surface $\Sigma \subset \mathbb{R}^3$ is ***cylindrical*** if it can be parametrized by

$$\psi(s,t) = \alpha(s) + ta, \quad a \in \mathbb{R}^3,\ |a| = 1. \tag{4.1}$$

where α is a curve, called the ***base curve***, contained in a 2-dimensional plane Π orthogonal to the ruling direction a.

Observe that if Σ is a cylindrical λ-soliton:

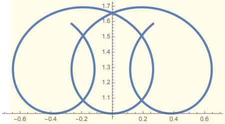
- The principal curvatures are 0 and κ_α and $N(\psi(s,t)) = n_\alpha(s)$.
- Then, from (1.1) we get the equation for cylindrical λ-solitons
$$\kappa_\alpha(s) = \langle n_\alpha(s), v\rangle + \lambda. \tag{4.2}$$
- We assume that a and v are orthogonal and $\lambda \geq 0$. ($a = e_2$, $v = e_3$)
- Thus, by (4.1) Σ can be parametrized as $\Psi(s,t) = (\alpha_1(s), t, \alpha_3(s))$.
- So, (4.2) can be written as
$$\kappa_\alpha(s) = \alpha_1'(s) + \lambda \Rightarrow \theta'(s) = \cos\theta(s) + \lambda.$$

Classification of cylindrical λ-solitons

Theorem 2 (López, Bueno-Ortiz)

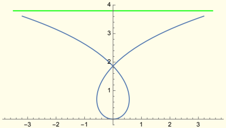
The base curve of a cylindrical λ-soliton parametrized by $\Psi(s,t) = (\alpha_1(s), t, \alpha_3(s))$ has the following behavior:

Case $\lambda > 1$



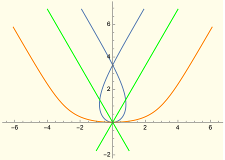
- Not embedded, not closed.
- Periodic (x_1-axis), $T = 2\pi/\sqrt{\lambda^2 - 1}$

Case $\lambda = 1$



- Symmetric (x_3-axis).
- Unique self-intersection point.
- $n_\alpha \to (0,0,-1)$ as $|s| \to \infty$.

Case $\lambda < 1$



- Symmetric (x_3-axis).
- Unique self-intersection point.
- $n_\alpha \to (\pm\sqrt{1-\lambda^2}, 0, -\lambda)$.

Classification of cylindrical λ-solitons

Theorem 2 (López, Bueno-Ortiz)

The explicit parametrizations of the base curves of a cilyndrical λ-soliton are:

- Case $\lambda > 1$:

$$\alpha(s) = \left(-\lambda s + 2\arctan\left(\sqrt{\frac{\lambda+1}{\lambda-1}}\tan\left(\frac{s}{2}\sqrt{\lambda^2-1}\right)\right), 0, \log(\lambda - \cos(s\sqrt{\lambda^2-1}))\right).$$

- Case $\lambda = 1$:

$$\alpha(s) = \left(-s + 2\arctan(s), 0, \log(1+s^2)\right).$$

- Case $\lambda < 1$:

$$\alpha(s) = \left(-\lambda s + 2\arctan\left(\sqrt{\frac{1+\lambda}{1-\lambda}}\tanh\left(\frac{s}{2}\sqrt{1-\lambda^2}\right)\right), 0, \log(-\lambda + \cosh(s\sqrt{1-\lambda^2}))\right).$$

Summary

Stability of λ-solitons

We define the stability for λ-solitons by **analogy with CMC surfaces**.

Stability of λ-solitons

We define the stability for λ-solitons by **analogy with CMC surfaces**.

Definition (Stability in CMC surfaces)

Second variation of the area functional:

$$\mathcal{A}''(0) = -\int_\Sigma u(\Delta u + |A|^2 u) d\Sigma.$$

Stability of λ-solitons

We define the stability for λ-solitons by **analogy with CMC surfaces**.

Definition (Stability in CMC surfaces)

Second variation of the area functional:

$$\mathcal{A}''(0) = -\int_\Sigma u(\Delta u + |A|^2 u) d\Sigma.$$

- A ***CMC surface Σ is strongly stable*** if $\mathcal{A}''(0) \geq 0, \forall u \in C_0^\infty(\Sigma)$.
- A ***CMC surface Σ is stable*** if $\mathcal{A}''(0) \geq 0, \forall u \in C_0^\infty(\Sigma)$ s.t. $\int_\Sigma u d\Sigma = 0$.

Stability of λ-solitons

We define the stability for λ-solitons by **analogy with CMC surfaces**.

Definition (Stability in CMC surfaces)

Second variation of the area functional:

$$\mathcal{A}''(0) = -\int_\Sigma u(\Delta u + |A|^2 u)d\Sigma.$$

- A ***CMC surface Σ is strongly stable*** if $\mathcal{A}''(0) \geq 0, \forall u \in C_0^\infty(\Sigma)$.
- A ***CMC surface Σ is stable*** if $\mathcal{A}''(0) \geq 0, \forall u \in C_0^\infty(\Sigma)$ s.t. $\int_\Sigma ud\Sigma = 0$.

Definition (Stability in λ-solitons)

Second variation of the weighted area functional:

$$\mathcal{A}_\phi''(0) = -\int_\Sigma u(\Delta u + \langle \nabla u, v\rangle + |A|^2 u)d\Sigma_\phi.$$

Stability of λ-solitons

We define the stability for λ-solitons by **analogy with CMC surfaces**.

Definition (Stability in CMC surfaces)

Second variation of the area functional:

$$\mathcal{A}''(0) = -\int_\Sigma u(\Delta u + |A|^2 u) d\Sigma.$$

- A ***CMC surface Σ is strongly stable*** if $\mathcal{A}''(0) \geq 0, \forall u \in C_0^\infty(\Sigma)$.
- A ***CMC surface Σ is stable*** if $\mathcal{A}''(0) \geq 0, \forall u \in C_0^\infty(\Sigma)$ s.t. $\int_\Sigma u d\Sigma = 0$.

Definition (Stability in λ-solitons)

Second variation of the weighted area functional:

$$\mathcal{A}''_\phi(0) = -\int_\Sigma u(\Delta u + \langle \nabla u, v \rangle + |A|^2 u) d\Sigma_\phi.$$

- A ***λ-soliton Σ is strongly stable*** if $\mathcal{A}''_\phi(0) \geq 0, \forall u \in C_0^\infty(\Sigma)$.
- A ***λ-soliton Σ is stable*** if $\mathcal{A}''_\phi(0) \geq 0, \forall u \in C_0^\infty(\Sigma)$ s.t. $\int_\Sigma u d\Sigma_\phi = 0$.

Stability of λ-solitons

Definition (Stability operator for λ-solitons)

The ***stability operator for λ-solitons*** is defined as a second order operator on $C_0^\infty(\Sigma)$ given by

$$J_\phi u = \Delta u + \langle \nabla u, v \rangle + |A|^2 u.$$

Stability of λ-solitons

Definition (Stability operator for λ-solitons)

The ***stability operator for λ-solitons*** is defined as a second order operator on $C_0^\infty(\Sigma)$ given by

$$J_\phi u = \Delta u + \langle \nabla u, v \rangle + |A|^2 u.$$

- J_ϕ is self-adjoint w.r.t. the weighted inner product $\int_\Sigma uvd\Sigma_\phi$.

Stability of λ-solitons

Definition (Stability operator for λ-solitons)

The ***stability operator for λ-solitons*** is defined as a second order operator on $C_0^\infty(\Sigma)$ given by

$$J_\phi u = \Delta u + \langle \nabla u, v \rangle + |A|^2 u.$$

- J_ϕ is self-adjoint w.r.t. the weighted inner product $\int_\Sigma uv d\Sigma_\phi$.
- Then, a quadratic form Q_ϕ can be defined as follows

$$Q_\phi(u) = -\int_\Sigma u J_\phi u d\Sigma_\phi = \int_\Sigma (|\nabla u|^2 - |A|^2 u^2) d\Sigma_\phi.$$

Stability of λ-solitons

Definition (Stability operator for λ-solitons)

The ***stability operator for λ-solitons*** is defined as a second order operator on $C_0^\infty(\Sigma)$ given by

$$J_\phi u = \Delta u + \langle \nabla u, v \rangle + |A|^2 u.$$

- J_ϕ is self-adjoint w.r.t. the weighted inner product $\int_\Sigma uv d\Sigma_\phi$.
- Then, a quadratic form Q_ϕ can be defined as follows

$$Q_\phi(u) = -\int_\Sigma u J_\phi u d\Sigma_\phi = \int_\Sigma (|\nabla u|^2 - |A|^2 u^2) d\Sigma_\phi.$$

- A λ-soliton Σ is strongly stable if $Q_\phi(u) \geq 0, \forall u \in C_0^\infty(\Sigma)$.

Stability of λ-solitons

Definition (Stability operator for λ-solitons)

The ***stability operator for λ-solitons*** is defined as a second order operator on $C_0^\infty(\Sigma)$ given by

$$J_\phi u = \Delta u + \langle \nabla u, v \rangle + |A|^2 u.$$

- J_ϕ is self-adjoint w.r.t. the weighted inner product $\int_\Sigma uv d\Sigma_\phi$.
- Then, a quadratic form Q_ϕ can be defined as follows

$$Q_\phi(u) = -\int_\Sigma u J_\phi u d\Sigma_\phi = \int_\Sigma (|\nabla u|^2 - |A|^2 u^2) d\Sigma_\phi.$$

- A λ-soliton Σ is strongly stable if $Q_\phi(u) \geq 0, \forall u \in C_0^\infty(\Sigma)$.
- A λ-soliton Σ is stable if $Q_\phi(u) \geq 0, \forall u \in C_0^\infty(\Sigma)$ s.t. $\int_\Sigma u d\Sigma_\phi = 0$.

Stability of λ-solitons

Definition (Stability operator for λ-solitons)

The ***stability operator for λ-solitons*** is defined as a second order operator on $C_0^\infty(\Sigma)$ given by

$$J_\phi u = \Delta u + \langle \nabla u, v \rangle + |A|^2 u.$$

- J_ϕ is self-adjoint w.r.t. the weighted inner product $\int_\Sigma uv d\Sigma_\phi$.
- Then, a quadratic form Q_ϕ can be defined as follows

$$Q_\phi(u) = -\int_\Sigma u J_\phi u d\Sigma_\phi = \int_\Sigma (|\nabla u|^2 - |A|^2 u^2) d\Sigma_\phi.$$

- A λ-soliton Σ is strongly stable if $Q_\phi(u) \geq 0, \forall u \in C_0^\infty(\Sigma)$.
- A λ-soliton Σ is stable if $Q_\phi(u) \geq 0, \forall u \in C_0^\infty(\Sigma)$ s.t. $\int_\Sigma u d\Sigma_\phi = 0$.

Example

Planes are strongly stable since $|A| = 0$.

First stability results of λ-solitons

Theorem 3 (Bueno, López, Ortiz)

Let Σ be a graphical λ-soliton. Then Σ is strongly stable.

First stability results of λ-solitons

Theorem 3 (Bueno, López, Ortiz)

Let Σ be a graphical λ-soliton. Then Σ is strongly stable.

Theorem 4 (Bueno, López, Ortiz)

Let Σ be a compact λ-soliton which is a graph over $\Omega \subset \{x_3 = 0\}$. Then Σ is a minimizer for $\mathcal{A}_\phi$ in the class of all surfaces in $\Omega \times \mathbb{R}$ with the same boundary and same weighted volume.

First stability results of λ-solitons

Theorem 3 (Bueno, López, Ortiz)

Let Σ be a graphical λ-soliton. Then Σ is strongly stable.

Theorem 4 (Bueno, López, Ortiz)

Let Σ be a compact λ-soliton which is a graph over $\Omega \subset \{x_3 = 0\}$. Then Σ is a minimizer for $\mathcal{A}_\phi$ in the class of all surfaces in $\Omega \times \mathbb{R}$ with the same boundary and same weighted volume.

- In the particular case $\lambda = 0$ we can drop the assumption on the constancy of the weighted volume.

Plateau-Rayleigh instability phenomenon

- In 1873, Plateau observed that a stream of water dropping vertically was divided into smaller packets with the same volume but less surface area.

Plateau-Rayleigh instability phenomenon

- In 1873, Plateau observed that a stream of water dropping vertically was divided into smaller packets with the same volume but less surface area.
- Later, Rayleigh proved that a stream of radius r breaks into drops when its length $L > 2\pi r$.

Plateau-Rayleigh instability phenomenon

- In 1873, Plateau observed that a stream of water dropping vertically was divided into smaller packets with the same volume but less surface area.
- Later, Rayleigh proved that a stream of radius r breaks into drops when its length $L > 2\pi r$.
- This fact can be explained by the theory of CMC surfaces:

A CMC cylinder C_r^L with length L and radius r is stable $\Leftrightarrow$ $L \leq 2\pi r$.

Plateau-Rayleigh instability phenomenon

- In 1873, Plateau observed that a stream of water dropping vertically was divided into smaller packets with the same volume but less surface area.
- Later, Rayleigh proved that a stream of radius r breaks into drops when its length $L > 2\pi r$.
- This fact can be explained by the theory of CMC surfaces:

A CMC cylinder C_r^L with length L and radius r is stable $\Leftrightarrow$ $L \leq 2\pi r$.

- It is known as Plateau-Rayleigh instability phenomenon.

Plateau-Rayleigh instability for λ-solitons

Our purpose: to study the Plateau-Rayleigh instability for λ-solitons.

Plateau-Rayleigh instability for λ-solitons

Our purpose: to study the Plateau-Rayleigh instability for λ-solitons.

- Let Σ be a cylindrical λ-soliton parametrized by

$$\Psi(s,t) = (\alpha_1(s), t, \alpha_3(s)),\ v = e_3,\ a = e_2.$$

Plateau-Rayleigh instability for λ-solitons

Our purpose: to study the Plateau-Rayleigh instability for λ-solitons.

- Let Σ be a cylindrical λ-soliton parametrized by

$$\Psi(s,t) = (\alpha_1(s), t, \alpha_3(s)), \ v = e_3, \ a = e_2.$$

- Fix an interval $[a, b]$ in the domain of α.

Plateau-Rayleigh instability for λ-solitons

Our purpose: to study the Plateau-Rayleigh instability for λ-solitons.

- Let Σ be a cylindrical λ-soliton parametrized by

$$\Psi(s,t) = (\alpha_1(s), t, \alpha_3(s)),\ v = e_3,\ a = e_2.$$

- Fix an interval $[a,b]$ in the domain of α.
- Since Σ is invariant along the a-direction, we take a compact piece Σ^* of length $L > 0$ in the t-parameter,

$$\Sigma^* = \Sigma(a,b,L) = \Psi([a,b] \times [0,L]).$$

Plateau-Rayleigh instability for λ-solitons

Our purpose: to study the Plateau-Rayleigh instability for λ-solitons.

- Let Σ be a cylindrical λ-soliton parametrized by

$$\Psi(s,t) = (\alpha_1(s), t, \alpha_3(s)),\ v = e_3,\ a = e_2.$$

- Fix an interval $[a, b]$ in the domain of α.
- Since Σ is invariant along the a-direction, we take a compact piece Σ^* of length $L > 0$ in the t-parameter,

$$\Sigma^* = \Sigma(a, b, L) = \Psi([a, b] \times [0, L]).$$

- Motivated by the Plateau-Rayleigh instability phenomenon, we answer the following question:

Plateau-Rayleigh instability for λ-solitons

Our purpose: to study the Plateau-Rayleigh instability for λ-solitons.

- Let Σ be a cylindrical λ-soliton parametrized by

$$\Psi(s,t) = (\alpha_1(s), t, \alpha_3(s)),\ v = e_3,\ a = e_2.$$

- Fix an interval $[a,b]$ in the domain of α.
- Since Σ is invariant along the a-direction, we take a compact piece Σ^* of length $L > 0$ in the t-parameter,

$$\Sigma^* = \Sigma(a,b,L) = \Psi([a,b] \times [0,L]).$$

- Motivated by the Plateau-Rayleigh instability phenomenon, we answer the following question:

Given $[a,b] \subset I$, does there exist $L_0 > 0$ s.t. Σ^* is unstable for any $L > L_0$?

Plateau-Rayleigh instability result for λ-solitons

- We analyze the Plateau-Rayleigh instability depending on the value of λ: $\lambda > 1$, $\lambda = 1$ and $\lambda < 1$.

Plateau-Rayleigh instability result for λ-solitons

- We analyze the Plateau-Rayleigh instability depending on the value of λ: $\lambda > 1$, $\lambda = 1$ and $\lambda < 1$.
- Note that for $\lambda = 0$, α corresponds to the grim reaper, then the surface is a graph on v^T and strongly stable by Theorem 3.

Plateau-Rayleigh instability result for λ-solitons

- We analyze the Plateau-Rayleigh instability depending on the value of λ: $\lambda > 1$, $\lambda = 1$ and $\lambda < 1$.
- Note that for $\lambda = 0$, α corresponds to the grim reaper, then the surface is a graph on v^T and strongly stable by Theorem 3.

Theorem 5 (Bueno, López, Ortiz)

Let Σ be a cylindrical translating λ-soliton.

Plateau-Rayleigh instability result for λ-solitons

- We analyze the Plateau-Rayleigh instability depending on the value of λ: $\lambda > 1$, $\lambda = 1$ and $\lambda < 1$.
- Note that for $\lambda = 0$, α corresponds to the grim reaper, then the surface is a graph on v^T and strongly stable by Theorem 3.

Theorem 5 (Bueno, López, Ortiz)

Let Σ be a cylindrical translating λ-soliton.

1. Case $\lambda > 1$. Let be $s_0 = T/2$, where T denotes the period of α. Then, $\Sigma^* = \Sigma(-s_0 + \sigma, s_0 + \sigma, L)$ with $\sigma \in [0, s_0]$ is unstable if

$$L > L_0 = \frac{4\pi}{\sqrt{3\sqrt{\lambda^2 - 1}(\lambda + \cos(\sigma\sqrt{\lambda^2 - 1}))}}.$$

Plateau-Rayleigh instability result for λ-solitons

- We analyze the Plateau-Rayleigh instability depending on the value of λ: $\lambda > 1$, $\lambda = 1$ and $\lambda < 1$.
- Note that for $\lambda = 0$, α corresponds to the grim reaper, then the surface is a graph on v^T and strongly stable by Theorem 3.

Theorem 5 (Bueno, López, Ortiz)

Let Σ be a cylindrical translating λ-soliton.

1. Case $\lambda > 1$. Let be $s_0 = T/2$, where T denotes the period of α. Then, $\Sigma^* = \Sigma(-s_0 + \sigma, s_0 + \sigma, L)$ with $\sigma \in [0, s_0]$ is unstable if

$$L > L_0 = \frac{4\pi}{\sqrt{3\sqrt{\lambda^2 - 1}(\lambda + \cos(\sigma\sqrt{\lambda^2 - 1}))}}.$$

2. Case $\lambda = 1$. $\Sigma^* = \Sigma(-\sigma, \sigma, L)$ with $\sigma > s_0 \sim 1.0213$ is unstable if

$$L > L_0 = 8\pi\sigma^{5/2}\left(15(\frac{\sigma^3}{3} + 9\sigma - \left(9 + 6\sigma^2 - 3\sigma^4\right)\tan^{-1}(\sigma))\right)^{-1/2}.$$

Plateau-Rayleigh instability result for λ-solitons

- We analyze the Plateau-Rayleigh instability depending on the value of λ: $\lambda > 1$, $\lambda = 1$ and $\lambda < 1$.
- Note that for $\lambda = 0$, α corresponds to the grim reaper, then the surface is a graph on v^T and strongly stable by Theorem 3.

Theorem 5 (Bueno, López, Ortiz)

Let Σ be a cylindrical translating λ-soliton.

1. Case $\lambda > 1$. Let be $s_0 = T/2$, where T denotes the period of α. Then, $\Sigma^* = \Sigma(-s_0 + \sigma, s_0 + \sigma, L)$ with $\sigma \in [0, s_0]$ is unstable if
$$L > L_0 = \frac{4\pi}{\sqrt{3\sqrt{\lambda^2 - 1}(\lambda + \cos(\sigma\sqrt{\lambda^2 - 1}))}}.$$
2. Case $\lambda = 1$. $\Sigma^* = \Sigma(-\sigma, \sigma, L)$ with $\sigma > s_0 \sim 1.0213$ is unstable if
$$L > L_0 = 8\pi\sigma^{5/2}\left(15(\frac{\sigma^3}{3} + 9\sigma - \left(9 + 6\sigma^2 - 3\sigma^4\right)\tan^{-1}(\sigma))\right)^{-1/2}.$$
3. Case $\lambda < 1$. We have numerical evidences of instability.

Plateau-Rayleigh instability result for λ-solitons

Proof.

- To get the results, we must find a test function $u \in C_0^\infty(\Sigma^*)$ s.t.

$$Q_\phi(u) < 0,\ u = 0 \text{ at } \partial\Sigma^* \int_{\Sigma^*} u d\Sigma_\phi = 0,$$

where $\partial\Sigma^* = (\alpha([a, b]) \times \{0, L\}) \cup (\{\alpha(a), \alpha(b)\} \times [0, L])$.

Plateau-Rayleigh instability result for λ-solitons

Proof.

- To get the results, we must find a test function $u \in C_0^\infty(\Sigma^*)$ s.t.

$$Q_\phi(u) < 0, \ u = 0 \text{ at } \partial\Sigma^* \int_{\Sigma^*} u d\Sigma_\phi = 0,$$

where $\partial\Sigma^* = (\alpha([a,b]) \times \{0, L\}) \cup (\{\alpha(a), \alpha(b)\} \times [0, L])$.

- We consider $u(s,t) = f(s)g(t) = f(s)\sin\left(\frac{2\pi t}{L}\right)$, satisfying:

Plateau-Rayleigh instability result for λ-solitons

Proof.

- To get the results, we must find a test function $u \in C_0^\infty(\Sigma^*)$ s.t.

$$Q_\phi(u) < 0, \ u = 0 \text{ at } \partial\Sigma^* \ \int_{\Sigma^*} u d\Sigma_\phi = 0,$$

where $\partial\Sigma^* = (\alpha([a,b]) \times \{0, L\}) \cup (\{\alpha(a), \alpha(b)\} \times [0, L])$.

- We consider $u(s,t) = f(s)g(t) = f(s) \sin\left(\frac{2\pi t}{L}\right)$, satisfying:
  - $f(a) = f(b) = 0$, $g(0) = g(L) = 0$.

Plateau-Rayleigh instability result for λ-solitons

Proof.

- To get the results, we must find a test function $u \in C_0^\infty(\Sigma^*)$ s.t.

$$Q_\phi(u) < 0, \; u = 0 \text{ at } \partial\Sigma^* \; \int_{\Sigma^*} u d\Sigma_\phi = 0,$$

  where $\partial\Sigma^* = (\alpha([a,b]) \times \{0, L\}) \cup (\{\alpha(a), \alpha(b)\} \times [0, L])$.

- We consider $u(s,t) = f(s)g(t) = f(s) \sin\left(\frac{2\pi t}{L}\right)$, satisfying:
  - $f(a) = f(b) = 0$, $g(0) = g(L) = 0$.
  - $\int_{\Sigma^*} u d\Sigma_\phi = \int_a^b f(s) e^{\alpha_3(s)} ds \int_0^L g(t) dt = 0$.

Plateau-Rayleigh instability result for λ-solitons

Proof.

- To get the results, we must find a test function $u \in C_0^\infty(\Sigma^*)$ s.t.

$$Q_\phi(u) < 0, \; u = 0 \text{ at } \partial\Sigma^* \; \int_{\Sigma^*} u d\Sigma_\phi = 0,$$

where $\partial\Sigma^* = (\alpha([a,b]) \times \{0, L\}) \cup (\{\alpha(a), \alpha(b)\} \times [0, L])$.

- We consider $u(s,t) = f(s)g(t) = f(s)\sin\left(\frac{2\pi t}{L}\right)$, satisfying:
  - $f(a) = f(b) = 0$, $g(0) = g(L) = 0$.
  - $\int_{\Sigma^*} u d\Sigma_\phi = \int_a^b f(s) e^{\alpha_3(s)} ds \int_0^L g(t) dt = 0$.
  - $Q_\phi(u) = \frac{L}{2}\int_a^b \left(f'^2 - \left((\alpha_1' + \lambda)^2 - \frac{4\pi^2}{L^2}\right) f^2\right) e^{\alpha_3(s)} ds$, since
    - $\langle \nabla u, v \rangle = f' g' \alpha_3'$,
    - $\Delta u = f'' g` + f g''$, and
    - $K = 0$.

Plateau-Rayleigh instability result for λ-solitons

Proof.

- To get the results, we must find a test function $u \in C_0^\infty(\Sigma^*)$ s.t.

$$Q_\phi(u) < 0,\ u = 0 \text{ at } \partial\Sigma^* \ \int_{\Sigma^*} u d\Sigma_\phi = 0,$$

where $\partial\Sigma^* = (\alpha([a,b]) \times \{0, L\}) \cup (\{\alpha(a), \alpha(b)\} \times [0, L])$.

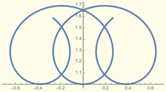
- We consider $u(s,t) = f(s)g(t) = f(s)\sin\left(\frac{2\pi t}{L}\right)$, satisfying:
    - $f(a) = f(b) = 0$, $g(0) = g(L) = 0$.
    - $\int_{\Sigma^*} u d\Sigma_\phi = \int_a^b f(s)e^{\alpha_3(s)}ds \int_0^L g(t)dt = 0$.
    - $Q_\phi(u) = \frac{L}{2}\int_a^b \left(f'^2 - \left((\alpha_1' + \lambda)^2 - \frac{4\pi^2}{L^2}\right) f^2\right) e^{\alpha_3(s)}ds$, since
        - $\langle \nabla u, v \rangle = f'g'\alpha_3'$,
        - $\Delta u = f''g` + fg''$, and
        - $K = 0$.

- How to define $f(s)$?

Plateau-Rayleigh instability result for λ-solitons

Proof.

Case $\lambda > 1$

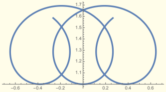


- α is periodic with $T = 2\pi/\sqrt{\lambda^2 - 1}$, $s_0 = T/2$.

Plateau-Rayleigh instability result for λ-solitons

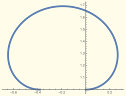
Proof.

Case $\lambda > 1$

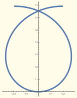


- α is periodic with $T = 2\pi/\sqrt{\lambda^2 - 1}$, $s_0 = T/2$.
- $\alpha([-s_0 + \sigma, s_0 + \sigma])$, $\sigma \in [0, s_0]$, generates a fundamental piece:

$\alpha([0, 2s_0])$



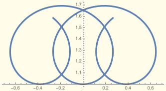
$\alpha([-s_0, s_0])$



Plateau-Rayleigh instability result for λ-solitons

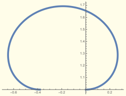
Proof.

Case $\lambda > 1$

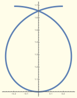


- α is periodic with $T = 2\pi/\sqrt{\lambda^2 - 1}$, $s_0 = T/2$.
- $\alpha([-s_0 + \sigma, s_0 + \sigma])$, $\sigma \in [0, s_0]$, generates a fundamental piece:

$\alpha([0, 2s_0])$



$\alpha([-s_0, s_0])$

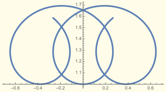


- We define $f(s) = \sin\left(\frac{\pi}{2s_0} s + \frac{\pi}{2}(1 - \frac{\sigma}{s_0})\right) e^{-\alpha_3(s)/2}$.

Plateau-Rayleigh instability result for λ-solitons

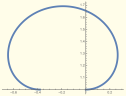
Proof.

Case $\lambda > 1$

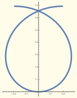


- α is periodic with $T = 2\pi/\sqrt{\lambda^2 - 1}$, $s_0 = T/2$.
- $\alpha([-s_0 + \sigma, s_0 + \sigma])$, $\sigma \in [0, s_0]$, generates a fundamental piece:

$\alpha([0, 2s_0])$



$\alpha([-s_0, s_0])$



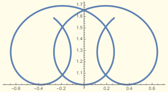
- We define $f(s) = \sin\left(\frac{\pi}{2s_0}s + \frac{\pi}{2}(1 - \frac{\sigma}{s_0})\right) e^{-\alpha_3(s)/2}$.
- Then,

$$Q_\phi(u) = \frac{\pi}{8L\sqrt{\lambda^2-1}}\left(16\pi^2 - 3L^2\sqrt{\lambda^2 - 1}(\lambda + \cos(\sigma\sqrt{\lambda^2 - 1}))\right).$$

Plateau-Rayleigh instability result for λ-solitons

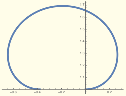
Proof.

Case $\lambda > 1$

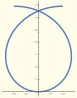


- α is periodic with $T = 2\pi/\sqrt{\lambda^2 - 1}$, $s_0 = T/2$.
- $\alpha([-s_0 + \sigma, s_0 + \sigma])$, $\sigma \in [0, s_0]$, generates a fundamental piece:

$\alpha([0, 2s_0])$



$\alpha([-s_0, s_0])$

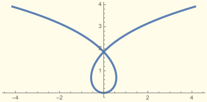


- We define $f(s) = \sin\left(\frac{\pi}{2s_0}s + \frac{\pi}{2}(1 - \frac{\sigma}{s_0})\right) e^{-\alpha_3(s)/2}$.
- Then,
  $Q_\phi(u) = \frac{\pi}{8L\sqrt{\lambda^2-1}}\left(16\pi^2 - 3L^2\sqrt{\lambda^2 - 1}(\lambda + \cos(\sigma\sqrt{\lambda^2 - 1}))\right)$.
- Equaling to 0 the above parenthesis we get the expected result.

Plateau-Rayleigh instability result for λ-solitons

Proof.

Case $\lambda = 1$

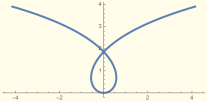


- α is a graph in $(-1, 1)$ and its projection on the x_1-axis is $(1 - \frac{\pi}{2}, -1 + \frac{\pi}{2})$. Thus, for any $s_0 < 1$, $\Sigma(s_0; L)$ is stable.

Plateau-Rayleigh instability result for λ-solitons

Proof.

Case $\lambda = 1$

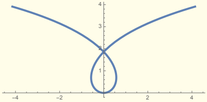


- α is a graph in $(-1, 1)$ and its projection on the x_1-axis is $(1 - \frac{\pi}{2}, -1 + \frac{\pi}{2})$. Thus, for any $s_0 < 1$, $\Sigma(s_0; L)$ is stable.
- Consequently, we study the instability of symmetric compact pieces of $\alpha(s)$ for $s \in [-s_0, s_0],\ s_0 > 1$.

Plateau-Rayleigh instability result for λ-solitons

Proof.

Case $\lambda = 1$

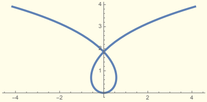


- α is a graph in $(-1, 1)$ and its projection on the x_1-axis is $(1 - \frac{\pi}{2}, -1 + \frac{\pi}{2})$. Thus, for any $s_0 < 1$, $\Sigma(s_0; L)$ is stable.
- Consequently, we study the instability of symmetric compact pieces of $\alpha(s)$ for $s \in [-s_0, s_0],\ s_0 > 1$.
- We define $f(s) = (s^2 - s_0^2)e^{-\alpha_3(s)/2}$.

Plateau-Rayleigh instability result for λ-solitons

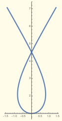
Proof.

Case $\lambda = 1$



- α is a graph in $(-1, 1)$ and its projection on the x_1-axis is $(1 - \frac{\pi}{2}, -1 + \frac{\pi}{2})$. Thus, for any $s_0 < 1$, $\Sigma(s_0; L)$ is stable.
- Consequently, we study the instability of symmetric compact pieces of $\alpha(s)$ for $s \in [-s_0, s_0]$, $s_0 > 1$.
- We define $f(s) = (s^2 - s_0^2)e^{-\alpha_3(s)/2}$.

Case $\lambda < 1$



- We give a numerical analysis with $f(s) = \cos\left(\frac{\pi s}{2s_0}\right) e^{-\alpha_3(s)/2}$.

Plateau-Rayleigh instability result for λ-solitons

- Now, we can obtain similar results for the problem of strong stability.

Plateau-Rayleigh instability result for λ-solitons

- Now, we can obtain similar results for the problem of strong stability.
- It is expectable that for strong stability the length of Σ reduces compared to the stable case.

Plateau-Rayleigh instability result for λ-solitons

- Now, we can obtain similar results for the problem of strong stability.
- It is expectable that for strong stability the length of Σ reduces compared to the stable case.
- In this case, we remove the assumption $\int_\Sigma u d\Sigma_\phi = 0$, and consider $g(t) = \sin(\pi t/L)$.

Plateau-Rayleigh instability result for λ-solitons

- Now, we can obtain similar results for the problem of strong stability.
- It is expectable that for strong stability the length of Σ reduces compared to the stable case.
- In this case, we remove the assumption $\int_\Sigma u d\Sigma_\phi = 0$, and consider $g(t) = \sin(\pi t/L)$.

Corollary 6 (Bueno, López, Ortiz)

Let Σ be a λ-soliton. Then, for any case of λ, Σ^* is strongly stable if

$$L > L_0/2.$$

where L_0 is the corresponding bound of Theorem 5 depending on the value of λ.

References

- **A. Bueno, R. López, I. Ortiz, The Plateau-Rayleigh instability of translating λ-solitons, preprint, arXiv:2301.06042.**
- **A. Bueno, I. Ortiz, Invariant hypersurfaces with linear prescribed mean curvature. J. Math. Anal. Appl. 487 (2020), 124033.**
- R. López, Invariant surfaces in Euclidean space with a log-linear density. Adv. Math. 339 (2018) 285–309.
- R. López, Compact λ-translating solitons with boundary, Mediterranean J. Math. **15** (2018).
- R. López, Plateau-Rayleigh instability of singular minimal surfaces. Commun. Pure Appl. Anal. 21 (2022), 2981–2997.

Thank you very much
for your attention!