PNMC biconservative surfaces in 4-dimensional Euclidean sphere

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Articles

- S. N., C. Oniciuc, N.C. Turgay, R. Yeğin Şen, *Biconservative surfaces in the 4-dimensional Euclidean sphere*, Ann. di Mat. Pura ed Appl., https://doi.org/10.1007/s10231-023-01323-0, 2023.

Outline

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1. Introducing biconservative submanifolds
2. Biconservative surfaces
3. Biconservative surfaces in $\mathbb{S}^4$

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Biharmonic maps

Let (M^m, g) and (N^n, h) be two Riemannian manifolds. Assume that M is compact and consider

- Bienergy functional

$$E_2 : C^\infty(M^m, N^n) \to \mathbb{R}, \qquad E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 v_g$$

- Euler-Lagrange equation

$$\begin{aligned} \tau_2(\varphi) &= -\Delta^\varphi \tau(\varphi) - \operatorname{trace}_g R^N(d\varphi, \tau(\varphi))d\varphi \\ &= 0. \end{aligned}$$

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where

$$\Delta^\varphi = -\operatorname{trace}_g \left(\nabla^\varphi \nabla^\varphi - \nabla^\varphi_\nabla\right)$$

is the rough Laplacian on sections of $\varphi^{-1}TN^n$ and

$$R^N(X,Y)Z = \nabla^N_X \nabla^N_Y Z - \nabla^N_Y \nabla^N_X Z - \nabla^N_{[X,Y]} Z.$$

- is a fourth-order non-linear elliptic equation;
- any harmonic map is biharmonic;
- a non-harmonic biharmonic map is called proper-biharmonic;

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The stress-bienergy tensor

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$$\begin{aligned}\langle S_2(X),Y\rangle =&\frac{1}{2}|\tau(\varphi)|^2\langle X,Y\rangle + \langle d\varphi,\nabla\tau(\varphi)\rangle\langle X,Y\rangle\\ &-\langle d\varphi(X),\nabla_Y\tau(\varphi)\rangle - \langle d\varphi(Y),\nabla_X\tau(\varphi)\rangle.\end{aligned}$$

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If φ is a submersion, $\operatorname{div} S_2 = 0$ if and only if φ is biharmonic.

If $\varphi : M^m \to N^n$ is an isometric immersion then $(\operatorname{div} S_2)^\sharp = -\tau_2(\varphi)^\top$. In general, for an isometric immersion, $\operatorname{div} S_2 \neq 0$.

Biharmonic and biconservative submanifolds

Definition 3.1

A submanifold $\varphi : M^m \to N^n$ is called biharmonic if φ is a biharmonic map, i.e., $\tau_2(\varphi) = 0$.

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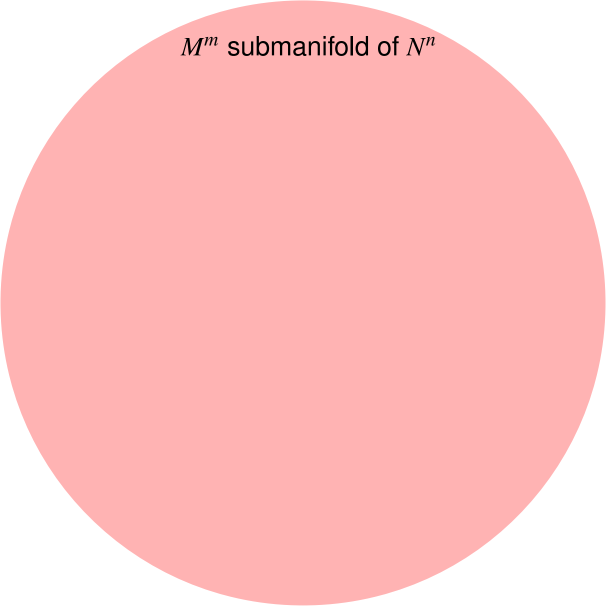
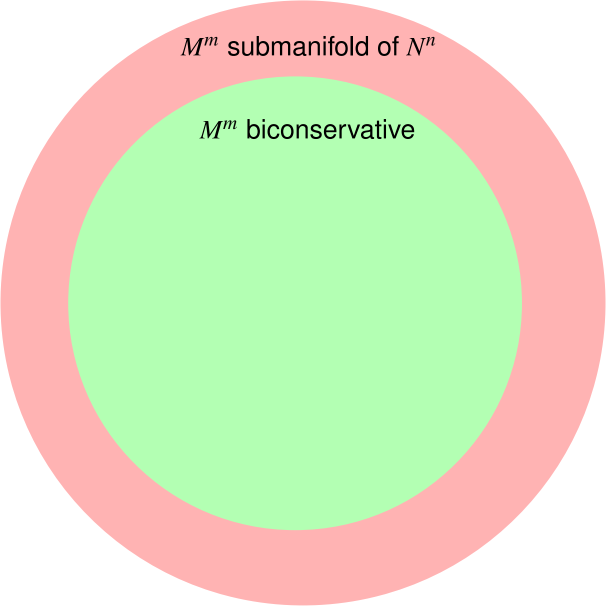
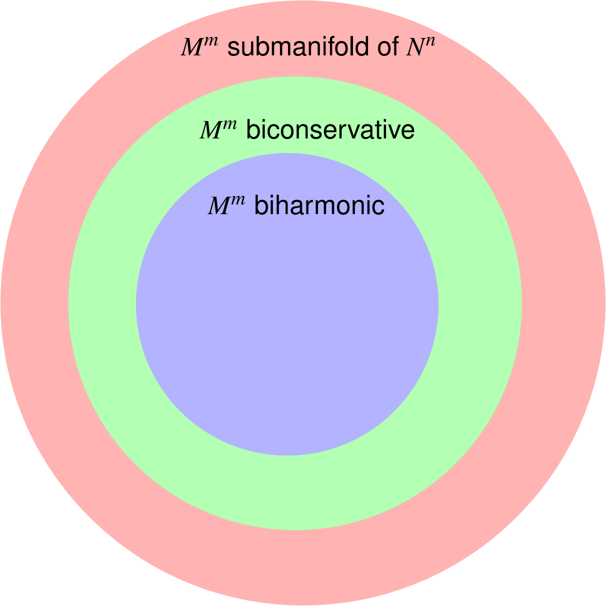
Definition 3.2 (Hasanis, Vlachos - 1995; Caddeo, Monataldo, Oniciuc, Piu - 2014)

A submanifold $\varphi : M^m \to N^n$ is called biconservative if $\operatorname{div} S_2 = 0$, i.e., $\tau_2(\varphi)^\top = 0$.


M^m submanifold of N^n


M^m submanifold of N^n
M^m biconservative


M^m submanifold of N^n
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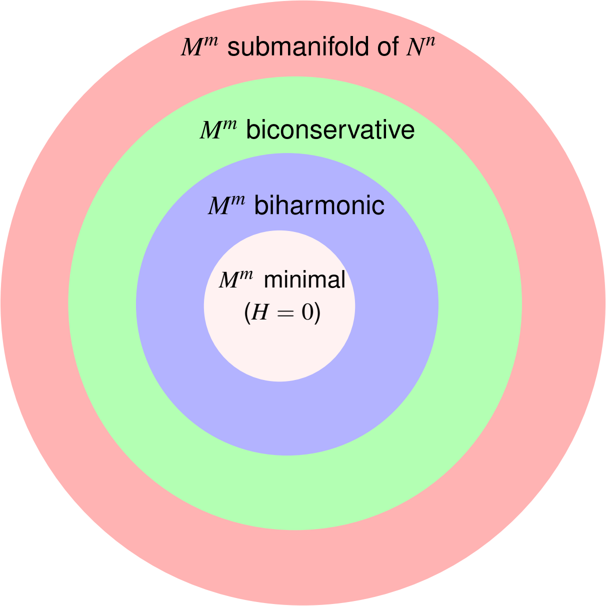
M^m submanifold of N^n
M^m biconservative
M^m biharmonic
M^m minimal
$(H = 0)$

Characterization results

Proposition 3.3

Let $\varphi : M^m \to N^n$ be a submanifold in the n-dimensional manifold. The following conditions are equivalent:

1. M *is biconservative;*
2. $\operatorname{trace} A_{\nabla^{\perp}_{(\cdot)} H}(\cdot) + \operatorname{trace}(\nabla A_H)(\cdot,\cdot) + \operatorname{trace}\left(R^N(\cdot,H)\cdot\right)^{\top} = 0$;
3. $4 \operatorname{trace} A_{\nabla^{\perp}_{(\cdot)} H}(\cdot) + m \operatorname{grad}\left(|H|^2\right) + 4 \operatorname{trace}\left(R^N(\cdot,H)\cdot\right)^{\top} = 0$;
4. $4 \operatorname{trace}(\nabla A_H)(\cdot,\cdot) - m \operatorname{grad}\left(|H|^2\right) = 0.$

Direct consequences

Corollary 3.4

- *If* $\varphi : M^m \to N^n$ *has* $\nabla A_H = 0$*, then it is biconservative.*
- *If* $\varphi : M^m \to N^n(c)$ *is a* PMC *submanifold, i.e.* $\nabla^\perp H = 0$*, in a space form of constant sectional curvature* c*, then it is biconservative.*
- *If* $\varphi : M^m \to N^{m+1}(c)$ *is a* CMC *hypersurface, i.e.* $|H|$ *is constant, then it is biconservative.*
- *If* $\varphi : M^m \to N^{m+1}(c)$ *is a hypersurface, then it is biconservative if and only if*

$$A(\operatorname{grad} f) = -\frac{m}{2} f \operatorname{grad} f.$$

Here, $f = \operatorname{trace} A / m$ *denotes the mean curvature function.*

Outline

Theorem 4.1 (Loubeau, Oniciuc - 2014; Montaldo, Oniciuc, Ratto - 2016; N. - 2017)

Let $\varphi : M^2 \to N^n$ *be a* CMC *surface. Then the following properties are equivalent:*

- M *is biconservative;*
- $\langle A_H(\partial_z), \partial_z \rangle$ *is holomorphic;*
- A_H *is a Codazzi tensor field.*

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Theorem 4.2 (Loubeau, Oniciuc - 2014; Montaldo, Oniciuc, Ratto - 2016)

Let $\varphi : M^2 \to N^n$ be a CMC biconservative surface. If M^2 is topologically a sphere $\mathbb{S}^2$, then it is pseudo-umbilical.

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Theorem 4.3 (Balmuş, Montaldo, Oniciuc - 2013)

Let $\varphi : M^2 \to N^n$ be a pseudo-umbilical surface. Then M^2 is biconservative if and only if M^2 is CMC.

Theorem 4.4 (Loubeau, Oniciuc - 2014; N. - 2017)

Let $\varphi : M^2 \to N^n$ be a compact CMC biconservative surface. If M^2 has no pseudo-umbilical points, then it is topologically a torus.

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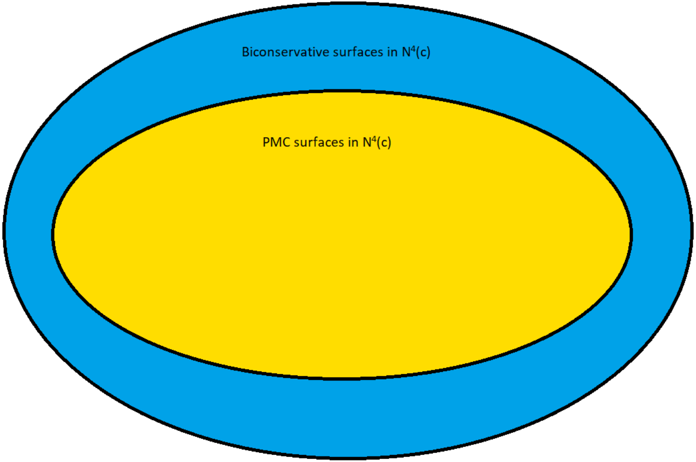
Theorem 4.5 (Loubeau, Oniciuc - 2014; N. - 2017)

Let $\varphi : M^2 \to N^n$ be a compact CMC biconservative surface. If the Gaussian curvature $K \geq 0$, then $\nabla A_H = 0$ and $K = 0$ or M^2 is pseudo-umbilical.

Outline

I. PMC surfaces in $N^4(c)$



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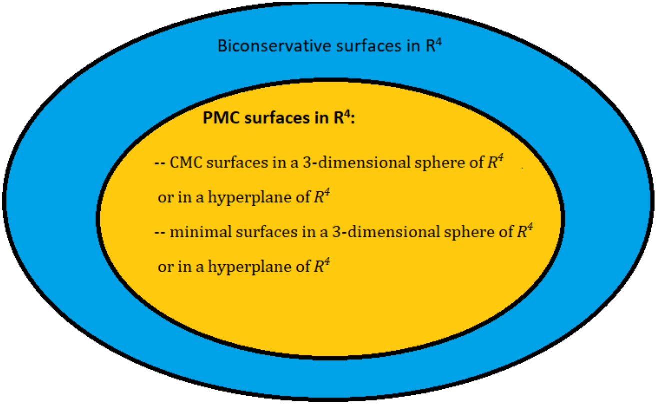

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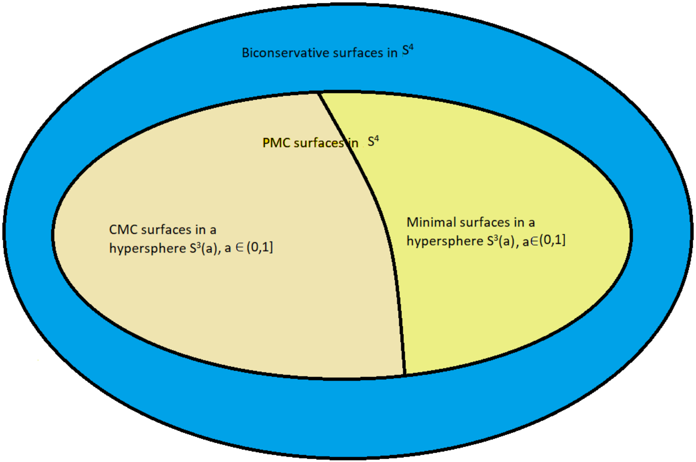

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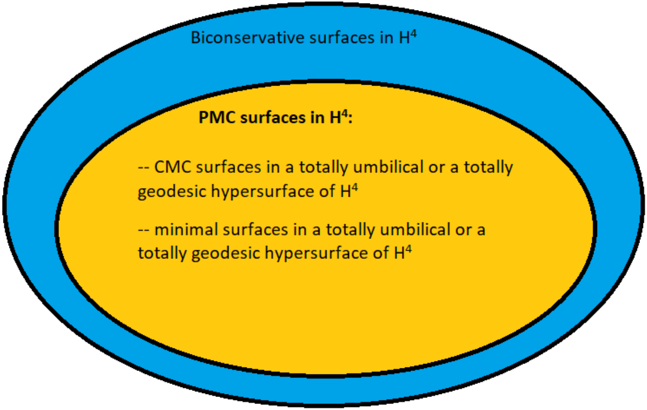
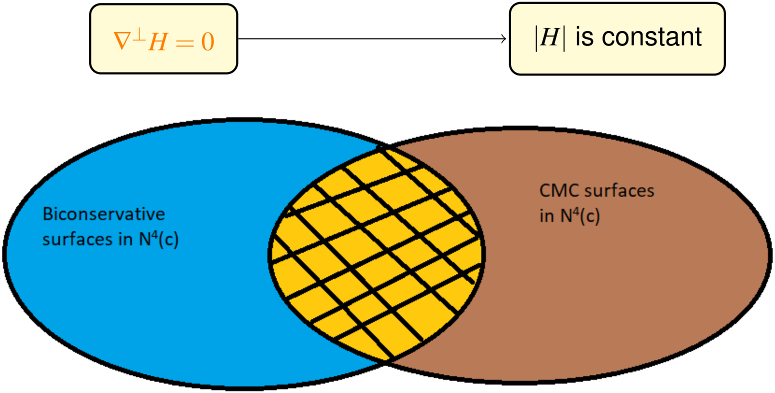
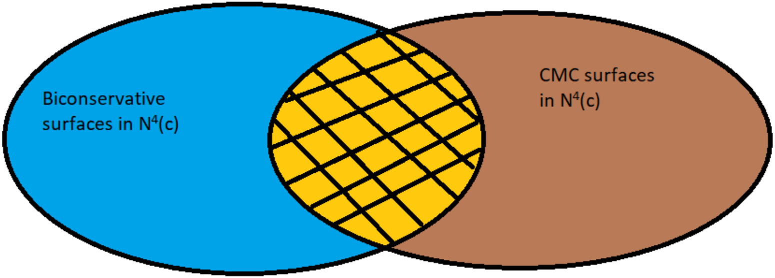


II. The study of *CMC* biconservative surfaces in $N^4(c)$



II. The study of CMC biconservative surfaces in $N^4(c)$

$\nabla^\perp H = 0$ → $|H|$ is constant



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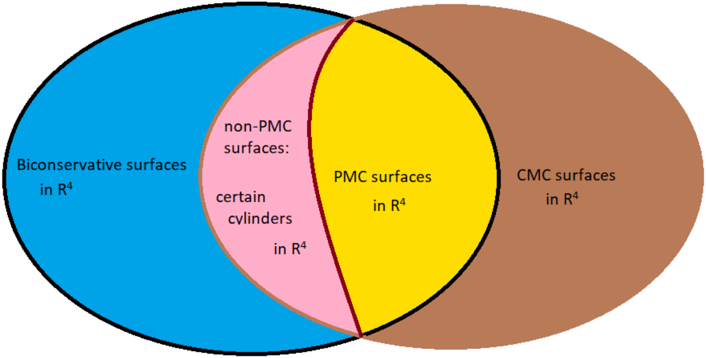

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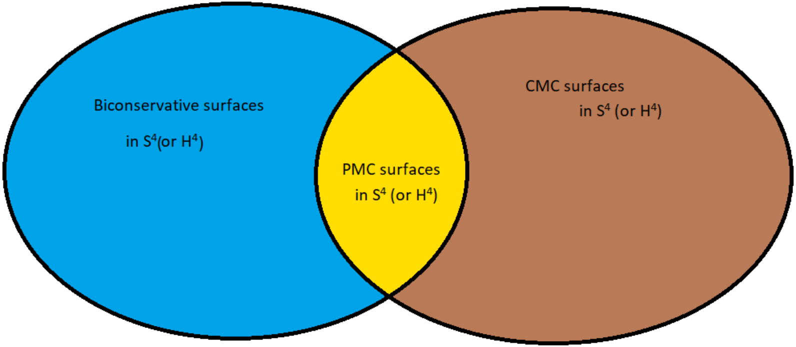
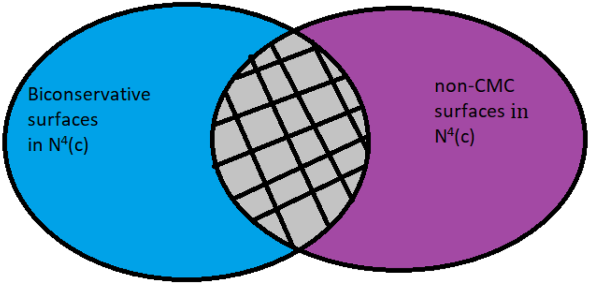


III. The study of non-CMC biconservative surfaces in $N^4(c)$



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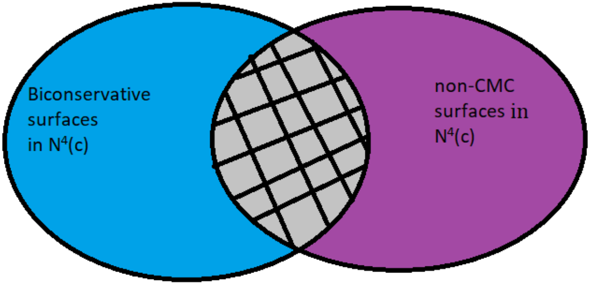
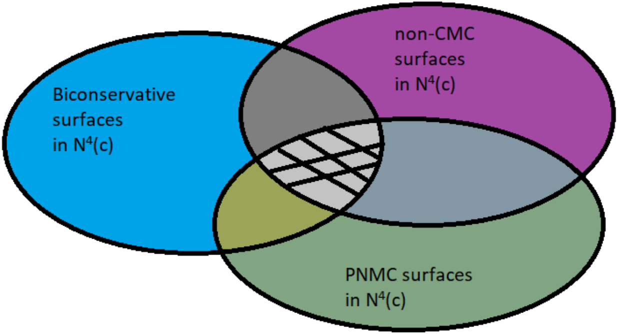


Very difficult to handle!

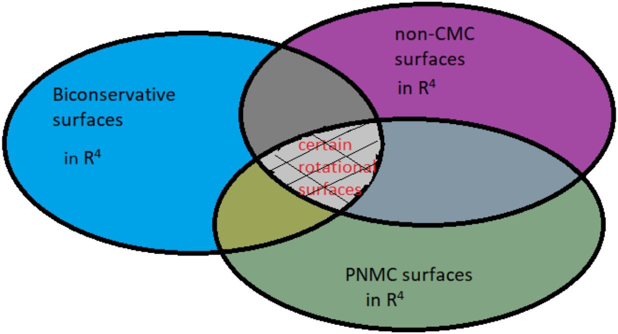
III. The study of *PNMC* (and non-*CMC*) biconservative surfaces in $N^4(c)$

***PNMC* surfaces in $N^4(c)$** = parallel normalized mean curvature vector field surfaces in $N^4(c)$

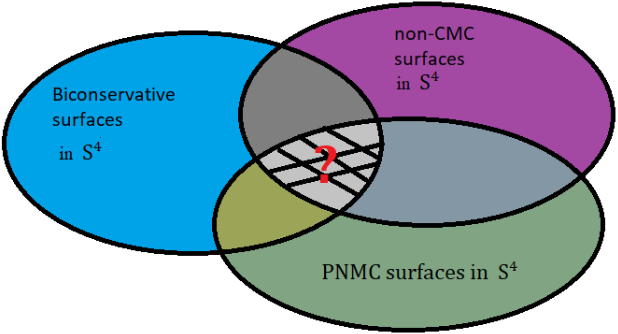
$$\nabla^\perp H = 0 \longrightarrow \nabla^\perp \frac{H}{|H|} = 0$$



III. The study of $PNMC$ (and non-CMC) biconservative surfaces in $\mathbb{R}^4$ ([Turgay, Yeğin Şen - 2018])



III. The study of *PNMC* (and non-*CMC*) biconservative surfaces in $\mathbb{S}^4$ ([N., Oniciuc, Turgay, Yeğin Şen - 2023])



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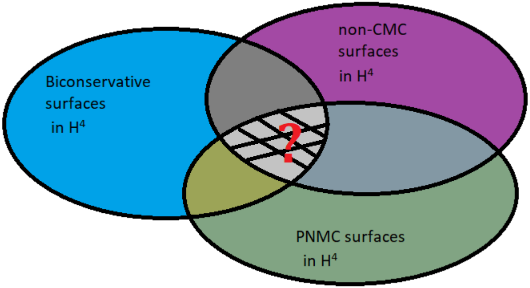


Work in progress!

The *PNMC* (and non-*CMC*) biconservative surfaces in $\mathbb{S}^4$

General context:

- connected, oriented, *PNMC*, non-*CMC* biconservative surfaces M^2 in $\mathbb{S}^4$
- $f = |H| > 0$, $\operatorname{grad} f \neq 0$ at any point of M^2
- M^2 is completely contained in $\mathbb{S}^4$, i.e., for any open subset of M^2 there exists no great hypersphere $\mathbb{S}^3$ of $\mathbb{S}^4$ such that it lies in $\mathbb{S}^3$.

M^2 is completely contained in $\mathbb{S}^4$ $\longleftrightarrow$ The rank of the first normal bundle of M^2 $N_1 = \operatorname{span} \operatorname{Im}(B)$ is 2

- positively oriented global orthonormal frame fields $\{E_1, E_2\}$ in the tangent bundle TM^2 and $\{E_3, E_4\}$ in the normal bundle NM^2

$$E_1 = \frac{\operatorname{grad} f}{|\operatorname{grad} f|} \qquad \text{and} \qquad E_3 = \frac{H}{f}.$$

- $E_2 f = 0$.
- Notations: $A_3 = A_{E_3}$ and $A_4 = A_{E_4}$.

Theorem 5.1

Let $\varphi : (M^2, g) \to \mathbb{S}^4$ be a $PNMC$ biconservative immersion. Then, the following hold:

(i)

$$\nabla_{E_1} E_1 = \nabla_{E_1} E_2 = 0, \quad \nabla_{E_2} E_1 = -\frac{3}{4}\frac{E_1 f}{f} E_2, \quad \nabla_{E_2} E_2 = \frac{3}{4}\frac{E_1 f}{f} E_1 \qquad (1)$$

and

$$\nabla^{\perp} E_3 = 0, \qquad \nabla^{\perp} E_4 = 0;$$

(ii)

$$A_3 = \begin{pmatrix} -f & 0 \\ 0 & 3f \end{pmatrix}, \quad A_4 = \begin{pmatrix} cf^{3/2} & 0 \\ 0 & -cf^{3/2} \end{pmatrix},$$

where c is a non-zero real constant;

Theorem (5.1 - continued)

(iii)
$$1 - K = 3f^2 + c^2 f^3, \tag{2}$$

thus $1 - K > 0$ *on* M^2*;*

(iv) *the level curves of K are circles of M^2 with positive constant signed curvature*
$$\kappa = \frac{3}{4}\frac{|\operatorname{grad} f|}{f} = \frac{|\operatorname{grad} K|}{4f^2\left(2 + c^2 f\right)} > 0; \tag{3}$$

(v) f *satisfies*
$$f\Delta f + |\operatorname{grad} f|^2 + \frac{4}{3}f^2 - 4f^4 - \frac{4}{3}c^2 f^5 = 0; \tag{4}$$

Theorem (5.1 - continued)

(vi) *around any point of* M^2 *there exists a positively oriented local chart* $X^f = X^f(u,v)$ *such that*

$$\left(f \circ X^f\right)(u,v) = f(u,v) = f(u)$$

and f *satisfies the following second order* ODE

$$f''f - \frac{7}{4}\left(f'\right)^2 - \frac{4}{3}f^2 + 4f^4 + \frac{4}{3}c^2 f^5 = 0 \tag{5}$$

and the condition $f' > 0$. *The first integral of the above second order ODE is*

$$\left(f'\right)^2 - 2C^2 f^{7/2} + \frac{16}{9}f^2 + 16f^4 + \frac{16}{9}c^2 f^5 = 0, \tag{6}$$

where C *is a non-zero real constant.*

Remark

We note that the constant c *is unique, depends on* $\left(M^2, g\right)$ *and it is not an indexing constant.*

- Intrinsic approach
- Extrinsic approach

Intrinsic approach

- Prove that if there are two *PNMC* biconservative immersions from a given abstract surface $\varphi_1, \varphi_2 : (M^2, g) \to \mathbb{S}^4$, then they are congruent;
- The main result: given an abstract surface (M^2, g), then it admits a (unique) *PNMC* biconservative immersion $\varphi : (M^2, g) \to \mathbb{S}^4$ iff. the abstract surface (M^2, g) satisfies certain intrinsic conditions.

Studying the *PNMC* biconservative immersions $\varphi : (M^2, g) \to \mathbb{S}^4$	$\longleftrightarrow$	Studying the abstract surfaces (M^2, g) satisfying certain conditions

Objective: To prove the existence and the classification of all abstract surfaces that satisfy those conditions.
Idea: To work with the curvature κ of the level curves of K; to show that κ satisfies a third order *ODE* and κ determines uniquely the abstract surface.

The intrinsic characterization theorem

Theorem 5.2

Let (M^2, g) *be an abstract surface. Then* M^2 *admits locally a (unique)* $PNMC$ *biconservative embedding in* $\mathbb{S}^4$ *if and only if the metric* g *is given by*

$$g(u,s) = du^2 + e^{2\int_0^u \kappa(\tau)\, d\tau} ds^2,$$

where κ *is a positive solution of the following* ODE

$$3\kappa\kappa''' + 26\kappa^2\kappa'' - 3\kappa'\kappa'' + 72\kappa^3\kappa' + 32\kappa^3 + 32\kappa^5 = 0, \tag{7}$$

which bears certain conditions.

Initial data:

$$\begin{cases} \kappa(0)=\kappa_0>0, \quad (\kappa \text{ is positive}) \\ \kappa'(0)=\kappa_0'>-1-\kappa_0^2, \quad (1-K>0) \\ -4\kappa_0-6\kappa_0\kappa_0'-4\kappa_0^3<\kappa''(0)=\kappa_0''<\frac{1}{3}\left(-8\kappa_0-14\kappa_0\kappa_0'-8\kappa_0^3\right). \end{cases} \tag{8}$$

Therefore, the curvature κ of the level curves of K determines the abstract surface which we look for. The correspondence

$$\kappa=\kappa(u)\longrightarrow g=g(u,s)$$

is bijective (up to isometries of the metric and translations of the argument for κ).

Question: How many non-isometric *PNMC* biconservative surfaces in $\mathbb{S}^4$ are?

Answer: We can perform a change of coordinates such that we can see that the abstract surfaces (M^2, g) that admit a *PNMC* biconservative immersion in $\mathbb{S}^4$ form a family indexed by 2 parameters.

The intrinsic characterization theorem

Theorem 5.3

Let (M^2, g) be an abstract surface. Then M^2 admits locally a (unique) $PNMC$ biconservative embedding in $\mathbb{S}^4$ if and only if the metric g is given by

$$g(f,t) = \frac{1}{2C^2 f^{7/2} - \frac{16}{9}c^2 f^5 - 16f^4 - \frac{16}{9}f^2} df^2 + \frac{1}{f^{3/2}} dt^2,$$

where C and c are arbitrary non-zero real constants.

Extrinsic approach

We will present a geometric description of the biconservative surface M^2 in $\mathbb{S}^4$, but viewed in $\mathbb{R}^5$.

- Find some properties of the integral curves of E_1 (for example, E_2 is constant along these curves viewed in $\mathbb{R}^5$).
- Find some properties of the integral curves of E_2 (for example, these curves are circles in $\mathbb{R}^5$).
- Find the local parametrization of M^2 in $\mathbb{R}^5$. This parametrization will rely on a solution f of a second order ODE and on a certain curve in $\mathbb{S}^3$, uniquely determined by f and the condition that its position vector has to make a specific angle with a constant direction.

Theorem 5.4

Let $\varphi : (M^2, g) \to \mathbb{S}^4$ *be a* $PNMC$ *biconservative immersion and denote* $\Phi = i \circ \varphi : M^2 \to \mathbb{R}^5$, *where* $i : \mathbb{S}^4 \to \mathbb{R}^5$ *is the canonical inclusion. We identify* M^2 *with its image, and then,* M^2 *can be locally parametrized by*

$$\Phi(u,t) = \hat{\gamma}(u) + \frac{1}{\hat{\kappa}(u)} \left((\cos(t) - 1)\, c_1 + \sin(t) c_2 \right),$$

where

(i) $\hat{\kappa}(u) = \frac{3|C|}{2\sqrt{2}} f^{3/4}(u)$, *with* $f = f(u)$ *is a positive solution of the second order ODE* (5), *with* $f' > 0$, *and whose first integral is* (6). *The non-zero constant* C *is given in* (6);

(ii) c_1 *and* c_2 *are constant orthonormal vectors in* $\mathbb{R}^5$;

Theorem (5.4 - continued)

(iii) $\hat{\gamma} = \hat{\gamma}(u)$ *is a curve in* $\mathbb{R}^5$ *such that* $\hat{\gamma} = i \circ \tilde{\gamma}$, *where* $\tilde{\gamma}$ *is a curve parametrized by arc-length which lies in a great hypersphere* $\mathbb{S}^3 = \mathbb{S}^4 \cap \Pi$; *the hyperplane* Π *contains the origin and is orthogonal to* c_2. *Moreover, the curvature and torsion of* $\tilde{\gamma}$, *as a curve in* $\mathbb{S}^3$, *are given by*

$$\mathrm{k}(u) = f(u)\sqrt{1 + c^2 f(u)} \tag{9}$$

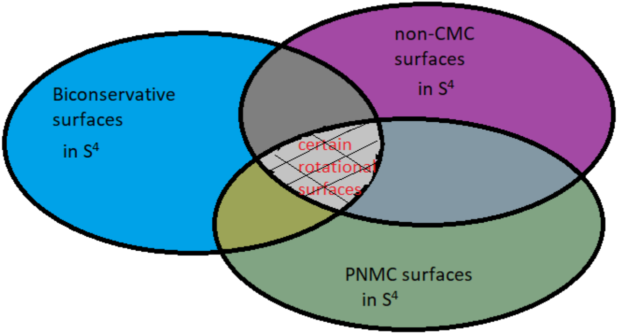
and

$$\tau(u) = \frac{|c|\sqrt{2C^2 f^{7/2}(u) - \frac{16}{9} f^2(u) - 16 f^4(u) - \frac{16}{9} c^2 f^5(u)}}{2\sqrt{f(u)}\,(1 + c^2 f(u))}, \tag{10}$$

and the curve $\hat{\gamma}$ *must satisfy*

$$\langle \hat{\gamma}(u), c_1 \rangle = \frac{1}{\hat{\kappa}(u)}. \tag{11}$$

Final answer



The codimension reduction for $PNMC$ biconservative surfaces

Theorem 5.5

Let $\varphi : (M^2, g) \to \mathbb{S}^n$, $n \geq 5$, be a $PNMC$ biconservative surface. Assume that the rank of the first normal bundle is 2 or 3. Then, M^2 lies in some 4-dimensional great sphere $\mathbb{S}^4$ of $\mathbb{S}^n$.

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- $\{E_1, E_2\}$ diagonalizes all the A_α, $\alpha \in \{3, 4, \ldots, n\}$, simultaneously and therefore M^2 has flat normal bundle.
- the first normal bundle of M^2 in $\mathbb{S}^n$ is given by $N_1 = \operatorname{span} \operatorname{Im}(B) = \operatorname{span}\{B(E_1, E_1), B(E_2, E_2)\}$
- N_1 is a parallel with respect to the normal connection, i.e., $\nabla^\perp_{E_i} B(E_j, E_j) \in C(N_1)$, for $i, j = 1, 2$.
- the codimension reduction result follows from Erbacher - 1971.

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Thank you for your attention!