Harmonic Morphisms and Minimal Conformal Foliations on Lie Groups

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Slides available at:
https://www.matematik.lu.se/matematiklu/personal/munn/slides/2023-09-06-Iasi.pdf

General Setting

Let (M, g) be a Riemannian manifold, $\mathcal{V}$ be an integrable distribution on M and $\mathcal{H}$ its orthogonal complement distribution. We will also denote by $\mathcal{V}, \mathcal{H}$ the orthogonal projections onto the corresponding subbundles of TM and by $\mathcal{F}$ the foliation tangent to $\mathcal{V}$.

Second Fundamental Forms

The second fundamental form $B^{\mathcal{V}}$ of $\mathcal{V}$ is given by

$$B^{\mathcal{V}}(V, W) = \mathcal{H}(\nabla_V W) = \frac{1}{2} \cdot \mathcal{H}(\nabla_V W + \nabla_W V), \quad \text{where } V, W \in \mathcal{V}.$$

The corresponding second fundamental form $B^{\mathcal{H}}$ of $\mathcal{H}$ satisfies

$$B^{\mathcal{H}}(X, Y) = \frac{1}{2} \cdot \mathcal{V}(\nabla_X Y + \nabla_Y X), \quad \text{where } X, Y \in \mathcal{H}.$$

The foliation $\mathcal{F}$ tangent to $\mathcal{V}$ is said to be *conformal* if there exists a vector field $V \in \mathcal{V}$ such that

$$B^{\mathcal{H}} = g \otimes V$$

and $\mathcal{F}$ is said to be *semi-Riemannian* if $V = 0$. Furthermore, $\mathcal{F}$ is *minimal* if $\operatorname{trace} B^{\mathcal{V}} = 0$ and *totally geodesic* if $B^{\mathcal{V}} = 0$. This is equivalent to the leaves of $\mathcal{F}$ being minimal and totally geodesic submanifolds of M, respectively.

Motivation

Why should we care about minimal conformal foliations? Key result by Baird and Eells:

Theorem 1 (Baird, Eells (1981) [1])

Let $\phi : (M, g) \to (N^2, h)$ *be a horizontally conformal submersion from a Riemannian manifold to a surface. Then* ϕ *is harmonic if and only if* ϕ *has minimal fibres.*

A foliation is conformal if and only if its corresponding (local) submersions are conformal, so the leaves of minimal conformal foliations are locally fibres of harmonic morphisms into $\mathbb{C}$.

Conjectures

Conjecture 1 (Ghandour, Gudmundsson, Turner (2021) [2])

Let (G, g) be a Lie group equipped with a left-invariant Riemannian metric. Further let K be a subgroup generating a left-invariant conformal foliation $\mathcal{F}$, of codimension two, of G. Then

1. If K is semisimple then the foliation $\mathcal{F}$ is minimal.
2. If, additionally, K is compact, then $\mathcal{F}$ is totally geodesic.

K	Compact	Non-compact
Semisimple	$\mathbf{SU}(2) \times \mathbf{SU}(2)$	$\mathbf{SU}(2) \times \mathbf{SL}_2(\mathbb{R})$
Non-semisimple	$\mathbf{SU}(2) \times \mathbf{SO}(2)$	$\mathbf{SL}_2(\mathbb{R}) \times \mathbf{SO}(2)$

An equivalent conjecture for the semi-Riemannian case was investigated by Ghandour, Gudmundsson and Ottosson in [3].

Lie Foliations

Let (G, g) be a semi-Riemannian Lie group i.e. a Lie group G equipped with a left-invariant semi-Riemannian metric g. Further we assume that (K, h) is a subgroup of G, of codimension two, equipped with the natural induced metric h. Let $\{V_1, \dots, V_n, X, Y\}$ be an orthonormal basis for the Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, such that $X, Y \in \mathcal{H}$. Then the Koszul formula implies that the second fundamental forms $B^{\mathcal{V}}$ and $B^{\mathcal{H}}$ of the horizontal and vertical distributions satisfy

$$\begin{aligned} 2 \cdot B^{\mathcal{V}}(V_j, V_k) &= \varepsilon_X \cdot \big((g([X, V_j], V_k) + g([X, V_k], V_j)\big) \cdot X \\ &\quad + \varepsilon_Y \cdot \big(g([Y, V_j], V_k) + g([Y, V_k], V_j)\big) \cdot Y, \\ B^{\mathcal{H}}(X, Y) &= \sum_{i=1}^{n} \varepsilon_i \cdot \big(g([X, V_i], Y) + g([Y, V_i], X)\big) \cdot V_i. \end{aligned}$$

We would like to describe the Lie brackets of $\mathfrak{g}$ with its structure constants.

Structure Constants

With this set-up, we can describe the Lie brackets of $\mathfrak{g}$ by the following system

$$[V_i, V_j] = \sum_{k=1}^{n} c_{ij}^k V_k, \qquad [X, Y] = \rho X + \sum_{k=1}^{n} \theta^k V_k$$

$$[X, V_i] = \sum_{k=1}^{n} x_i^k V_k + \alpha_i X + \beta_i Y,$$

$$[Y, V_i] = \sum_{k=1}^{n} y_i^k V_k + \gamma_i X + \eta_i Y,$$

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Useful results

Proposition 1 (Ghandour, Gudmundsson, Ottosson (2022)[3])

Let (G, g) be a semi-Riemannian Lie group with a subgroup K generating a left-invariant conformal foliation $\mathcal{F}$ on G. Let $\mathcal{V}$ be the integrable distribution tangent to $\mathcal{F}$ and $\mathcal{H}$ be the orthogonal complementary distribution of dimension two. Then

$$\mathcal{H}[[\mathcal{V}, \mathcal{V}], \mathcal{H}] = 0.$$

Proposition 2 (Ghandour, Gudmundsson, Ottosson (2022)[3])

Let (G, g) be a semi-Riemannian Lie group with a semi-simple subgroup K, of codimension two, generating a left-invariant conformal foliation $\mathcal{F}$ on G. Then the foliation $\mathcal{F}$ is semi-Riemannian.

Simplified Structure Constants

Since K is a semisimple subgroup of G generating the conformal foliation $\mathcal{F}$, it follows from Proposition 1 that $\mathcal{H}[\mathcal{V}, \mathcal{H}] = 0$, and so we can simplify to

$$\mathrm{ad}_X(V_i) = [X, V_i] = \sum_{k=1}^{n} x_i^k V_k, \quad \mathrm{ad}_Y(V_i) = [Y, V_i] = \sum_{k=1}^{n} y_i^k V_k.$$

Structure constants and minimality

The structure constants of $\mathfrak{g}$ can then be used to describe when the foliation $\mathcal{F}$ is minimal or even totally geodesic. Using the above results we get that

$$2 \cdot B^{\mathcal{V}}(V_i, V_j) \quad = \quad \varepsilon_X \cdot (\varepsilon_j x_i^j + \varepsilon_i x_j^i) \cdot X + \varepsilon_Y \cdot (\varepsilon_j y_i^j + \varepsilon_i y_j^i) \cdot Y.$$

From this we see that the foliation $\mathcal{F}$ is *minimal* if and only if

$$\sum_{i=1}^{n} \varepsilon_i \, x_i^i = 0 = \sum_{i=1}^{n} \varepsilon_i \, y_i^i..$$

Furthermore, $\mathcal{F}$ is *totally geodesic* if and only if for all $1 \leq i, j \leq n$ we have

$$\varepsilon_j \, x_i^j + \varepsilon_i \, x_j^i = 0 = \varepsilon_j \, y_i^j + \varepsilon_i \, y_j^i.$$

Killing Form

Definition 2

The *Killing form* of a Lie algebra $\mathfrak{g}$ is the function $B : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}$ given by $B(X, Y) = \mathrm{trace}(\mathrm{ad}_X \circ \mathrm{ad}_Y)$.

Lemma 3

The Killing form B of $\mathfrak{g}$ is a symmetric bilinear form that is invariant under all automorphism of $\mathfrak{g}$ and satisfies

$$B(\mathrm{ad}_Z X, Y) = -B(X, \mathrm{ad}_Z Y).$$

Cartan-Killing Metric

A Lie group is semisimple if and only if its Killing form B is nondegenerate. In particular this means that the Killing form induces a left-invariant semi-Riemannian metric on the semisimple subgroup K.

Proposition 3

Let $\mathfrak{k}$ be a semisimple Lie algebra over $\mathbb{R}$. Then $\mathfrak{k}$ is compact if and only if the Killing form of $\mathfrak{k}$ is strictly negative definite.

We can define a left-invariant metric of any index using the Killing form B.

Lemma 4

Let K be a semisimple Lie group, then there exists a so-called Cartan involution θ of $\mathfrak{k}$ such that $-B(\cdot, \theta(\cdot))$ is a positive definite bilinear form on $\mathfrak{k}$. $-B(\cdot, \theta(\cdot))$ is called the Cartan-Killing metric

Let $\{V_1, \ldots, V_n\}$ be an orthonormal basis, with respect to the Cartan-Killing metric g_θ, then for any $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{\pm 1\}^n$ We define the semi-Riemannian metric g_ε on K by

$$g_\varepsilon(V_i, V_j) = \varepsilon_i \cdot B(V_i, \theta(V_j)).$$

Idea of Proof

Theorem 5 (Gudmundsson, TM)

Let G be a Lie group equipped with left-invariant metric g and let K be a subgroup of (G, g) generating a left-invariant conformal foliation $\mathcal{F}$ of G of codimension two. If K is semisimple then $\mathcal{F}$ is minimal.

The structure of the proof is quite simple:

1. Show $\mathcal{F}$ is minimal when the metric on K is the previously defined g_ε.
2. Show that the minimality of $\mathcal{F}$ is independent of choice of left-invariant metric.

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Proposition 4 (Gudmundsson, TM)

Let K be a subgroup of the Lie group (G, g) generating a left-invariant conformal foliation $\mathcal{F}$ of G of codimension two. If K is semisimple and $g|_{\mathfrak{k}\times\mathfrak{k}} = g_\varepsilon$, then $\mathcal{F}$ is minimal.

Proof.

Since K is equipped with a Cartan-Killing metric, it follows that $B(V_i, V_j) = \varepsilon_i \delta_{ij} \theta_j$ for a orthonormal basis $\{V_1, \dots V_n\}$. Then

$$\begin{aligned} B([V_i, V_j], X) &= B(V_i, [V_j, X]) \\ &= -B(V_i, \sum_{k=1}^{n} x_j^k V_k) \\ &= \varepsilon_i \theta_i x_j^i. \end{aligned}$$

□

Proof (continued).

$$\begin{aligned} B([V_i, V_j], X) &= B(V_i, [V_j, X]) \\ &= -B(V_i, \sum_{k=1}^{n} x_j^k V_k) \\ &= \varepsilon_i \theta_i x_j^i. \end{aligned}$$

Since $B([V_i, V_j], X) = -B([V_j, V_i], X)$, the above steps show us that

$$\varepsilon_i \theta_i x_j^i = -\varepsilon_j \theta_j x_i^j.$$

Then for $i = j$ we get that $x_i^i = -x_i^i = 0$. By an identical argument replacing X with Y it follows that $y_i^i = -y_i^i$ and so $\mathcal{F}$ is minimal. □

Corollary 6

Furthermore, if the metric on K is the Killing form, then $\mathcal{F}$ is totally geodesic.

Proof.

In the case, we have the same argument, except where $\theta_i \equiv 1$, so we have that $\varepsilon_i x_i^j = -\varepsilon_j x_j^i$ and $\varepsilon_i y_i^j = -\varepsilon_j y_i^j$ so $\mathcal{F}$ is totally geodesic. □

Remark 1

The above corollary can be extended to the case when K is compact and equipped with a bi-invariant *Riemannian* metric, as these are always proportional to the Killing form.

Main Result

Theorem 7 (Gudmundsson, TM)

Let G be a Lie group equipped with left-invariant metric g and let K be a subgroup of (G, g) generating a left-invariant conformal foliation $\mathcal{F}$ of G of codimension two. If K is semisimple then $\mathcal{F}$ is minimal.

Proof.

Let $\{V_1, ..., V_n, X, Y\}$ be an orthonormal basis for $\mathfrak{g}$ with respect to g such that $\{V_1, ..., V_n\}$ generate K. Notice

$$trace(\mathrm{ad}_X) = \sum_{i=1}^{n} \varepsilon_i x_i^i + g(\mathrm{ad}_X(Y), Y),$$

$$\mathrm{trace}(\mathrm{ad}_Y) = \sum_{i=1}^{n} \varepsilon_i y_i^i + g(\mathrm{ad}_Y(X), X).$$

Proof. (continued).

Now equip G with an additional left-invariant metric $\hat{g}$, which we can fully describe by its action on $\mathfrak{g}$. First define $\hat{g}|_{\mathcal{V}\times\mathcal{V}} = g_\varepsilon$ and then let

$$\hat{g}|_{\mathcal{H}\times\mathcal{H}} = g|_{\mathcal{H}\times\mathcal{H}}, \;\; \hat{g}|_{\mathcal{V}\times\mathcal{H}} = g|_{\mathcal{V}\times\mathcal{H}}.$$

Then we can use the Gram-Schmidt process to obtain an orthonormal basis $\{B_1, ..., B_n, X, Y\}$ with respect to $\hat{g}$. Since both metrics are left-invariant and of the same signature, changing metrics simply amounts to a change of basis. Then it follows from Proposition 4 that the structure constants with respect to $\hat{g}$,

$$\hat{x}_i^i = \hat{g}(B_i, [X, B_i])$$

are equal to zero. ☐

Proof. (continued).

Then since the trace of a linear operator is invariant under change of basis, and the fact that X and Y are orthonormal for both g and $\hat{g}$, we have that

$$\begin{aligned}\sum_{i=1}^{n} \varepsilon_i x_i^i &= \operatorname{trace}(\operatorname{ad}_X) - g(\operatorname{ad}_X(Y), Y) \\ &= \operatorname{trace}(\operatorname{ad}_X) - \hat{g}(\operatorname{ad}_X(Y), Y) \\ &= \sum_{i=1}^{n} \varepsilon_i \hat{x}_i^i \\ &= 0.\end{aligned}$$

Similarly we obtain

$$\sum_{i=1}^{n} \varepsilon_i y_i^i = \sum_{i=1}^{n} \varepsilon_i \hat{y}_i^i = 0,$$

so $\mathcal{F}$ is minimal. □

Recall that we proved the corollary

Corollary 8

If K is compact and equipped with a bi-invariant Riemannian metric, then $\mathcal{F}$ is totally geodesic.

One may ask if the result requires the stronger condition of a bi-invariant metric. The following example shows that K being compact and equipped with a left-invariant metric is not sufficient to ensure that $\mathcal{F}$ is totally geodesic.

Berger Sphere

Consider $\mathfrak{k} = \mathfrak{su}(2)$ equipped with the Berger metric, which is given by

$$g = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

for some $\lambda > 0$. So an orthonormal basis with respect to g will be written

$$\{A, B, C\}.$$

Then the Lie brackets are given by

$$[A, B] = \frac{2}{\sqrt{\lambda}} C, \quad [C, A] = \frac{2}{\sqrt{\lambda}} B, \quad [B, C] = 2\sqrt{\lambda} A.$$

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Proposition 5

Let (G, g) be a five dimensional Riemannian Lie group with a subgroup $\mathbf{SU}(2)$ generating a left-invariant conformal foliation $\mathcal{F}$ on G. Let $g|_{\mathfrak{su}(2)}$ be the standard Berger metric on $\mathbf{SU}(2)$ with $\lambda \neq 0$. Let $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{m}$ be an orthogonal decomposition of the Lie algebra $\mathfrak{g}$ of G and $\{A, B, C, X, Y\}$ be the orthonormal basis for $\mathfrak{g}$ such that A, B and C generate the subalgebra $\mathfrak{su}(2)$. Then $\mathcal{F}$ need not be totally geodesic.

Proof.

Then since **SU**(2) is a simple Lie subgroup generating a left-invariant conformal foliation, it follows from Proposition 1 that the remaining Lie bracket relations for $\mathfrak{g}$ are given by

$$[X, A] = a_{11}A + a_{12}B + a_{13}C, \quad [Y, A] = a_{21}A + a_{22}B + a_{23}C,$$
$$[X, B] = b_{11}A + b_{12}B + b_{13}C, \quad [Y, B] = b_{21}A + b_{22}B + b_{23}C,$$
$$[X, C] = c_{11}A + c_{12}B + c_{13}C, \quad [Y, C] = c_{21}A + c_{22}B + c_{23}C,$$

$$[X, Y] = \rho X + \theta_1 A + \theta_2 B + \theta_3 C,$$

for some constant coefficients. Since $\mathfrak{g}$ is a Lie algebra, the Lie brackets must satisfy the Jacobi identity, which allows us to simplify the above coefficients. □

Proof (continued).

After simplifying with the Jacobi identity we obtain:

$$\begin{aligned} [X, A] &= a_{12}B + a_{13}C, & [Y, A] &= a_{22}B + a_{23}C, \\ [X, B] &= -a_{12}\lambda A - c_{12}C, & [Y, B] &= -a_{22}\lambda A - c_{22}C, \\ [X, C] &= -a_{13}\lambda A + c_{12}B, & [Y, C] &= -a_{23}\lambda A + c_{22}B. \end{aligned}$$

Then notice that

$$g(C, [X, A]) = a_{13}$$

and

$$g(A, [X, C]) = -\lambda \cdot a_{13}.$$

So $\mathcal{F}$ is not totally geodesic whenever $\lambda \neq 1$ and $a_{13} \neq 0$ since $B^{\mathcal{H}}(A, C) \neq 0$.

□

When we have the additional condition that K is a *closed* subgroup of G, the leaf space $L^2 = G/K$ has a unique structure such that the natural projection is a Riemannian submersion.
Since the 2-dimensional Riemannian space G/K is homogeneous it is of constant Gaussian curvature K_L. For this special situation, we will now employ O'Neill's famous curvature formula

$$K_L = K(X, Y) + \tfrac{3}{4} \cdot |\mathcal{V}[X, Y]|^2.$$

Using structure constants, we can write

$$K(X, Y) \quad = \quad -\rho^2 - \tfrac{3}{4} \sum_{k=1}^{n} (\theta^k)^2,$$

and obtain that the leaf space L has constant non-positive Gaussian curvature $K_L = -\rho^2$.

References

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Thank you for your attention!