Minimal Submanifolds of Compact Riemannian Symmetric Spaces

- The Method of Eigenfamilies -

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These lecture notes are available at

`www.matematik.lu.se/matematiklu/personal/sigma/slides/2023-09-06-Iasi.pdf`

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Co-workers in Order of Appearance

⋆ 1st wave - Martin Svensson (2006/09)

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⋆ 2nd wave - Anna Sakovich (2008/09)

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* ⋆ 3rd wave - Anna Siffert, Marko Sobak, Elsa Ghandour (2020/23)
* ⋆ 4th wave - Thomas Munn (2023)

Outline

1. Eigenfunctions - Eigenfamilies
   - The Operators τ and κ
   - The Definition and Examples

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   - The Operators τ and κ
   - The Definition and Examples
2. Minimal Submanifolds
   - The Main Result
   - Compact Examples

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The complex-linear **Laplace-Beltrami operator** τ on (M, g) acts locally on a C^2-function ϕ as

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For two C^2-functions $\phi, \psi : (M, g) \to \mathbb{C}$ we have

$$\tau(\phi\cdot\psi) = \tau(\phi)\cdot\psi + 2\cdot\kappa(\phi,\psi) + \phi\cdot\tau(\psi),$$

where the **conformality operator** κ satisfies $\kappa(\phi,\psi) = g(\nabla\phi, \nabla\psi)$.

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where the **conformality operator** κ satisfies $\kappa(\phi,\psi) = g(\nabla\phi,\nabla\psi)$.

$$\kappa(\phi,\phi) = (|\nabla u|^2 - |\nabla v|^2) + 2\,i\cdot g(\nabla u,\nabla v).$$

$\kappa(\phi,\phi) = 0$ if and only if

$$|\nabla u|^2 = |\nabla v|^2 \quad\text{and}\quad g(\nabla u,\nabla v) = 0.$$

Definition 1.1 (Eigenfunction - Eigenfamily - SG, Sakovich (2008))

Let (M, g) be a Riemennian manifold. Then a complex-valued function $\phi : M \to \mathbb{C}$ is said to be an **eigenfunction** if it is eigen both with respect to the Laplace-Beltrami operator τ and the conformality operator κ i.e. there exist complex numbers $\lambda, \mu \in \mathbb{C}$ such that

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A set $\mathcal{E} = \{\phi_i : M \to \mathbb{C} \,|\, i \in I\}$ of complex-valued functions is said to be an **eigenfamily** on M if there exist complex numbers $\lambda, \mu \in \mathbb{C}$ such that for all $\phi, \psi \in \mathcal{E}$ we have

$$\tau(\phi) = \lambda \cdot \phi \quad \text{and} \quad \kappa(\phi, \psi) = \mu \cdot \phi \cdot \psi.$$

Example 1.2 (The Euclidean $\mathbb{C} \cong \mathbb{R}^{2n}$, $(\lambda = 0,\ \mu = 0)$)

On the standard **Euclidean space** $\mathbb{C}^n \cong \mathbb{R}^{2n}$ we define the functions $\phi_1, \dots, \phi_n : \mathbb{C}^n \to \mathbb{C}$ by $\phi_j : (z_1, \dots, z_n) \mapsto z_j$. Then the tension field τ and the conformality operator κ on $\mathbb{C}^n$ satisfy

$$\tau(\phi_j) = 0 \quad \text{and} \quad \kappa(\phi_j, \phi_k) = 0.$$

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Theorem 1.3 (Fuglede (1978) + Ishihara (1979), $(\lambda = 0$ and $\mu = 0)$)

A complex-valued C^2-function $\phi : (M, g) \to \mathbb{C}$ on a Riemannian manifold is a **harmonic morphism** *if and only if it is an* **eigenfunction** *with $\lambda = 0$ and $\mu = 0$.* $[\tau(\phi) = 0 \quad \textit{and} \quad \kappa(\phi, \phi) = 0]$

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Theorem 1.4 (Baird, Eells (1981), $(\lambda = 0$ and $\mu = 0)$)

Let $\phi : (M, g) \to \mathbb{C}$ be a **horizontally conformal** $[\kappa(\phi, \phi) = 0]$ *function from a Riemannian manifold. Then ϕ is* **harmonic** $[\tau(\phi) = 0]$ *if and only if its fibres are* **minimal** *at regular points of ϕ* $[\nabla\phi \neq 0]$.

Example 1.5 (S^{2n-1}, $(\lambda = -(2n-1),\ \mu = -1)$)

Let S^{2n-1} be **the odd dimensional unit sphere** in the standard Euclidean $\mathbb{C}^n \cong \mathbb{R}^{2n}$ and define $\phi_1, \dots, \phi_n : S^{2n-1} \to \mathbb{C}$ by

$$\phi_j : (z_1, \dots, z_n) \mapsto \frac{z_j}{\sqrt{|z_1|^2 + \cdots + |z_n|^2}}.$$

Then the tension field τ and the conformality operator κ on S^{2n-1} satisfy

$$\tau(\phi_j) = -(2n-1) \cdot \phi_j \quad \text{and} \quad \kappa(\phi_j, \phi_k) = -1 \cdot \phi_j \cdot \phi_k.$$

Example 1.6 ($\mathbb{C}P^n$, $(\lambda = -4(n+1),\ \mu = -4)$)

Let $\mathbb{C}P^n$ be the standard n-dimensional **complex projective space**. For a fixed integer $1 \le \alpha < n+1$ and $1 \le j \le \alpha < k \le n+1$ define the function $\phi_{jk} : \mathbb{C}P^n \to \mathbb{C}$ by

$$\phi_{jk} : [z_1, \dots, z_{n+1}] \mapsto \frac{z_j \cdot \bar{z}_k}{z_1 \cdot \bar{z}_1 + \cdots + z_{n+1} \cdot \bar{z}_{n+1}}.$$

Then the tension field τ and the conformality operator κ on $\mathbb{C}P^n$ satisfy

$$\tau(\phi_{jk}) = -4(n+1) \cdot \phi_{jk} \quad \text{and} \quad \kappa(\phi_{jk}, \phi_{lm}) = -4 \cdot \phi_{jk} \cdot \phi_{lm}.$$

Note that

$$\#(\{\phi_{jk} \mid j \le \alpha < k\}) = \alpha \cdot (n+1-\alpha).$$

Theorem 1.7 (SG - Ghandour (2020))

Let (M, g) be a Riemannian manifold and the set of complex-valued functions

$$\mathcal{E} = \{\phi_i : M \to \mathbb{C} \,|\, i = 1, 2, \dots, n\}$$

be a finite **eigenfamily** *i.e. there exist complex numbers $\lambda, \mu \in \mathbb{C}$ such that for all $\phi, \psi \in \mathcal{E}$*

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Then the set $\mathcal{P}_d(\mathcal{E})$ of **complex homogeneous polynomials of degree** d

$$\mathcal{P}_d(\mathcal{E}) = \{P : M \to \mathbb{C} \,|\, P \in \mathbb{C}[\phi_1, \phi_2, \dots, \phi_n],\, P(\alpha \cdot \phi) = \alpha^d \cdot P(\phi),\, \alpha \in \mathbb{C}\}$$

is an **eigenfamily** *on M such that for all $P, Q \in \mathcal{P}_d(\mathcal{E})$ we have*

$$\tau(P) = (d\,\lambda + d(d-1)\,\mu) \cdot P \quad \text{and} \quad \kappa(P, Q) = (d^2 \mu) \cdot P\, Q.$$

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$$\tau(P) = (d\,\lambda + d(d-1)\,\mu) \cdot P \quad \text{and} \quad \kappa(P, Q) = (d^2 \mu) \cdot P\,Q.$$

$$\dim_{\mathbb{C}} \mathcal{P}_d(\mathcal{E}) = \binom{n + d - 1}{n}.$$

Theorem 2.1 (SG, Munn (2023))

Let $\phi : (M, g) \to \mathbb{C}$ *be a complex-valued* **eigenfunction** *on a Riemannian manifold, such that* $0 \in \phi(M)$ *is a regular value for* ϕ*. Then the fibre* $\mathcal{F}_0 = \phi^{-1}(\{0\})$ *is a* **minimal submanifold** *of* M *of codimension two.*

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Notice that along the fibre $\mathcal{F}_0 = \phi^{-1}(\{0\})$ we have the following important properties $\tau(\phi) = \lambda \cdot \phi = 0$ and $\kappa(\phi, \phi) = \mu \cdot \phi^2 = 0$.

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The main ingredient for our proof is the following **OLD result** in the special case when $n = 2$. For this we have a **NEW proof** "avoiding" the **stress-energy tensor.** ☺ ☺ ☺

Theorem 2.2 (Baird, SG (1992))

Let $\phi : (M, g) \to (N^n, h)$ be a submersion and $\mathcal{F}_y = \phi^{-1}(\{y\})$ for some $y \in N$. If ϕ is **horizontally conformal up to first order along** *$\mathcal{F}_y$, then the following two conditions are equivalent*

(i) *ϕ is n-**harmonic** along $\mathcal{F}_y$ i.e. $\tau_n(\phi)(x) = 0$, for all $x \in \mathcal{F}_y$,*

(ii) *$\mathcal{F}_y$ is a* **minimal submanifold** *of (M, g).*

Definition 2.3 (Baird, SG (1992))

Let $\phi : (M, g) \to (N^n, h)$ be a smooth submersion and let the functions $\lambda_1^2, \dots, \lambda_n^2$ denote the non-zero eigenvalues of the first fundamental form $\phi^* h$ with respect to the metric g. Further, let $\mathcal{F}$ be a submanifold of M. Then ϕ is said to be **horizontally conformal up to first order along** $\mathcal{F}$ if

(i) $\lambda_1^2(x) = \cdots = \lambda_n^2(x)$,

(ii) $\nabla(\lambda_1^2(x)) = \cdots = \nabla(\lambda_n^2(x))$, for every $x \in \mathcal{F}$.

Example 2.4 (SG, Munn (2023))

Let S^{2n-1} be the odd-dimensional **unit sphere** in the standard Euclidean $\mathbb{C}^n \cong \mathbb{R}^{2n}$ and $A \in \mathbb{C}^{n\times n}$ be a complex matrix which is invertible i.e. $\det A \neq 0$. Further define the function $\Phi : S^{2n-1} \to \mathbb{C}$ with

$$\Phi : z = (z_1, z_2, \dots, z_n) \mapsto \frac{1}{|z|^2} \cdot \Big(\sum_{k\neq l}^{n} a_{kl}\, z_k z_l + \tfrac{1}{2} \cdot \sum_{k=1}^{n} a_{kk}\, z_k^2 \Big).$$

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Then Φ is a **submersive eigenfunction** and its compact fibres form a foliation of the unit sphere of codimension two. The fibre $\mathcal{F}_0 = \Phi^{-1}(\{0\})$ is a **minimal submanifold** of S^{2n-1}.

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This provides a complex n^2-dimensional family of compact minimal submanifolds of S^{2n-1}.

Example 2.5 (SG, Munn (2023) ($d \in \mathbb{Z}^+$))

Let S^{2n-1} be the odd-dimensional **unit sphere** in the Euclidean $\mathbb{C}^n \cong \mathbb{R}^{2n}$, $d \in \mathbb{Z}^+$ and $(a_1, \dots, a_n) \in \mathbb{C}^n$ such that $a_1, \dots, a_n \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Further define the function $\Phi : S^{2n-1} \to \mathbb{C}$ with

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For each $d \in \mathbb{Z}^+$, this provides a complex n-dimensional family of compact minimal submanifolds of S^{2n-1}.

Example 2.6 (SG, Munn (2023))

Let $\mathbb{C}P^{2n-1}$ be the odd-dimensional **complex projective space** and $(a_1, \dots, a_n) \in \mathbb{C}^n$ such that $a_1, \dots, a_n \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Then define the function $\Phi : \mathbb{C}P^{2n-1} \to \mathbb{C}$ with

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Definition 3.1 (Riemannian Symmetric Space)

A Riemannian manifold (M,g) is said to be a **symmetric space** if to each point $p \in M$ there exists a global isometry $\sigma_p : (M,g) \to (M,g)$ such that $\sigma_p(p) = p$ and the differential $d\sigma_p : T_pM \to T_pM$ at p satisfies

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The irreducible Riemannian symmetric spaces were **classified by Élie Cartan** in 1926. They constitute 20 countably infinite families and 24 exceptional single cases. They are quotents of Lie groups and come in **pairs** $(U/K, G/K)$, where U/K is **compact** and G/K is **non-compact**.

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$$\begin{array}{rcl} S^m = \mathbf{SO}(1+m)/\mathbf{SO}(m) & * & \mathbb{R}H^m = \mathbf{SO}_o(1,m)/\mathbf{SO}(m) \\ \mathbb{C}P^m = \mathbf{U}(1+m)/\mathbf{U}(1)\times\mathbf{U}(m) & * & \mathbb{C}H^m = \mathbf{U}(1,m)/\mathbf{U}(1)\times\mathbf{U}(m) \\ \mathbb{H}P^m = \mathbf{Sp}(1+m)/\mathbf{Sp}(1)\times\mathbf{Sp}(m) & * & \mathbb{H}H^m = \mathbf{Sp}(1,m)/\mathbf{Sp}(1)\times\mathbf{Sp}(m) \end{array}$$

Theorem 3.2 (SG, Svensson (2009), ($\lambda = 0$ and $\mu = 0$))

Let (M, g) *be an irreducible Riemannian* **symmetric space** *other than the* **compact** $G_2/\mathbf{SO}(4)$ *or its* **non-compact** *dual* $G_2^*/\mathbf{SO}(4)$.

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Example 3.4 (SG, Sakovich (2008) ** $\lambda \cdot \mu \neq 0$)

Eigenfamilies on the **compact** symmetric spaces

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Eigenfunctions on the **compact** symmetric spaces

$$\mathbf{SU}(n)/\mathbf{SO}(n),\ \mathbf{Sp}(n)/\mathbf{U}(n),\ \mathbf{SO}(2n)/\mathbf{U}(n),\ \mathbf{SU}(2n)/\mathbf{Sp}(n),$$

Example 3.6 (SG, Ghandour (2022) ** $\lambda = -(m+n)$, $\mu = -2$)

Eigenfamilies on the **compact** real **Grassmannians**

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Example 3.8 (SG, Ghandour (2023) ** $\lambda = -2(m+n)$, $\mu = -1/2$)

Eigenfamilies on the **compact** quaternionic **Grassmannians**

$$\mathbf{Sp}(m+n)/\mathbf{Sp}(m) \times \mathbf{Sp}(n)$$

Theorem 4.1 (SG, Munn (2023))

Let $\phi : (M, g) \to \mathbb{C}$ be a complex-valued **eigenfunction** *on a Riemannian manifold, such that $0 \in \phi(M)$ is a regular value for ϕ. Then the fibre $\mathcal{F}_0 = \phi^{-1}(\{0\})$ is a* **minimal submanifold** *of M.*

Definition 4.2 (Baird, SG (1992))

Let $\phi : (M, g) \to (N^n, h)$ be a smooth submersion and let the functions $\lambda_1^2, \dots, \lambda_n^2$ denote the non-zero eigenvalues of the first fundamental form $\phi^* h$ with respect to the metric g. Further, let $\mathcal{F}$ be a submanifold of M. Then ϕ is said to be **horizontally conformal up to first order along** $\mathcal{F}$ if

(i) $\lambda_1^2(x) = \dots = \lambda_n^2(x)$,

(ii) $\nabla(\lambda_1^2(x)) = \dots = \nabla(\lambda_n^2(x))$, for every $x \in \mathcal{F}$.

Lemma 4.3

Let $\phi = u + iv : (M, g) \to (\mathbb{C}, h)$ *be a smooth submersion that is eigen with respect to the horizontal conformality operator* κ *i.e.* $\kappa(\phi, \phi) = \mu \cdot \phi^2$*. Then the eigenvalues of the first fundamental form* $\phi^* h$ *are given by*

$$\lambda_{1,2} = \frac{1}{2} \cdot \big((|\nabla u|^2 + |\nabla v|^2) \pm \mu \cdot (u^2 + v^2) \big).$$

Lemma 4.3

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Proof (of Main Result).

For a point p in the fibre $\mathcal{F}_0$, over $(u, v) = (0, 0) \in \mathbb{C}$, we have $\lambda_1^2 = \lambda_2^2$ and

$$X(\lambda_1^2 - \lambda_2^2)(p) = X\Big(\mu \cdot (|\nabla u|^2 + |\nabla v|^2) \cdot (u^2 + v^2)\Big)(p)$$

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$$\begin{aligned} X(\lambda_1^2 - \lambda_2^2)(p) &= X\Big(\mu \cdot (|\nabla u|^2 + |\nabla v|^2) \cdot (u^2 + v^2)\Big)(p) \\ &= \mu \cdot X(|\nabla u|^2 + |\nabla v|^2)(p) \cdot (u^2 + v^2)(p) \\ &\quad + \mu \cdot (|\nabla u|^2 + |\nabla v|^2)(p) \cdot X(u^2 + v^2)(p) \end{aligned}$$

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□

Thank you for your attention !!

Example 4.4 (SG, Munn (2023))

Let us consider the real-valued functions $x_{11}, x_{12} : \mathbf{SO}(n) \to \mathbb{C}$ with $x_{j\alpha} : x \mapsto e_j \cdot x \cdot e_\alpha^t$. For a tangent vector $Y_{rs} \in B_{\mathfrak{so}(n)}$ we have

$$Y_{12}(x_{11}) = -\frac{x_{12}}{\sqrt{2}},\ Y_{13}(x_{11}) = -\frac{x_{13}}{\sqrt{2}},\ \dots\ ,\ Y_{1n}(x_{11}) = -\frac{x_{1n}}{\sqrt{2}},$$

$$Y_{12}(x_{12}) = \frac{x_{11}}{\sqrt{2}},\ Y_{23}(x_{12}) = -\frac{x_{13}}{\sqrt{2}},\ \dots\ ,\ Y_{2n}(x_{12}) = -\frac{x_{1n}}{\sqrt{2}}.$$

Then define the complex-valued function $\Phi : \mathbf{SO}(n) \to \mathbb{C}$ with $\Phi(x) = (x_{11} + i\, x_{12})$. The above derivatives show that the gradient

$$\nabla\Phi = \nabla x_{11} + i\, \nabla x_{12}$$

never vanishes along $\mathbf{SO}(n)$, so Φ induces a foliation on $\mathbf{SO}(n)$ of codimension two. Here Φ is an eigenfunction. Hence the fibre $\Phi^{-1}(\{0\})$ is a **compact minimal submanifold** of $\mathbf{SO}(n)$.

Example 4.5 (SG, Munn (2023))

Let the complex-valued function $\phi : \mathbf{U}(n) \to \mathbb{C}$ be defined by $\Phi : z \mapsto z_{11}$. Then the following coefficients of the gradient $\nabla\Phi$ satisfy

$$iD_1(\Phi) = i\, z_{11},\ \ Y_{12}(\Phi) = -\frac{z_{12}}{\sqrt{2}},\ \ Y_{13}(\Phi) = -\frac{z_{13}}{\sqrt{2}},\ \ \dots\ ,\ \ Y_{1n}(\Phi) = -\frac{z_{1n}}{\sqrt{2}}.$$

Since the first row $(z_{11}, z_{12}, \dots, z_{1n})$ can not vanish, at least one of these derivatives, and hence the gradient $\nabla\Phi$ of Φ is non-zero along the unitary group $\mathbf{U}(n)$. This means that Φ induces a foliation of $\mathbf{U}(n)$ of codimension two. Here Φ is an eigenfunction. This implies that the fibre $\Phi^{-1}(\{0\})$ is a **compact minimal submanifold** of $\mathbf{U}(n)$.

Example 4.6 (SG, Munn (2023))

For $n \geq 3$, we define the complex-valued function $\Phi : \mathbf{U}(n) \to \mathbb{C}$ by

$$\Phi : z \mapsto (z_{11} \cdot z_{22} - z_{12} \cdot z_{21}) = \det \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}.$$

Here Φ is an eigenfunction on $\mathbf{U}(n)$ and the gradient $\nabla\Phi$ is non-vanishing along $\mathbf{U}(n)$. This implies that the compact fibres of $\Phi : \mathbf{U}(n) \to \mathbb{C}$ form a foliation on $\mathbf{U}(n)$ of codimension two. The fibre $\Phi^{-1}(\{0\})$ is a **compact minimal submanifold** of $\mathbf{U}(n)$.

Example 4.7 (SG, Munn (2023))

On the quaternionic unitary group $\mathbf{Sp}(n)$ we define the complex-valued eigenfunction $\Phi : \mathbf{Sp}(n) \to \mathbb{C}$ with

$$\Phi : q = z + jw \mapsto z_{11}.$$

The function $\Phi : \mathbf{Sp}(n) \to \mathbb{C}$ is submersive along $\mathbf{Sp}(n)$. Hence we obtain a foliation on $\mathbf{Sp}(n)$ of codimension two. The leaves are **compact** and the fibre $\Phi^{-1}(\{0\})$ over $0 \in \mathbb{C}$ is **minimal**.

Thank you for your attention !!