Biconservative submanifolds in complex space forms

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Harmonic and biharmonic maps

Let $\varphi : (M,g) \to (N,h)$ be a smooth map.

Energy functional

$$E(\varphi) = E_1(\varphi) = \frac{1}{2}\int_M |d\varphi|^2 v_g$$

Euler-Lagrange equation

$$\begin{aligned} \tau(\varphi) = \tau_1(\varphi) &= \operatorname{trace}_g \nabla d\varphi \\ &= 0 \end{aligned}$$

Critical points of E: harmonic maps

Bienergy functional

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 v_g$$

Euler-Lagrange equation

$$\begin{aligned} \tau_2(\varphi) &= \Delta^{\varphi}\tau(\varphi) - \operatorname{trace}_g \overline{R}(d\varphi, \tau(\varphi))d\varphi \\ &= 0 \end{aligned}$$

Critical points of E_2: biharmonic maps

The biharmonic equation

G.-Y. Jiang (1986):

$$\tau_2(\varphi) = \Delta^{\varphi}\tau(\varphi) - \operatorname{trace}_g \overline{R}(d\varphi, \tau(\varphi))d\varphi = 0$$

where

$$\Delta^{\varphi} = \operatorname{trace}_g \left(\nabla^{\varphi}\nabla^{\varphi} - \nabla^{\varphi}_{\nabla}\right)$$

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is the rough Laplacian on sections of $\varphi^{-1}TN$

- it is a fourth-order non-linear elliptic equation
- any harmonic map is biharmonic
- a non-harmonic biharmonic map is proper-biharmonic
- a submanifold $i : M \to N$ is a biharmonic submanifolds if the immersion i is biharmonic

Biconservative submanifolds

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- P. Baird and J. Eells (1981); A. Sanini (1983): used the tensor

$$S = \frac{1}{2}|d\varphi|^2 g - \varphi^* h$$

that satisfies

$$\operatorname{div} S = -\langle \tau(\varphi), d\varphi \rangle,$$

to study harmonic maps, since

$$\varphi = \text{harmonic} \Rightarrow \operatorname{div} S = 0$$

Obviously

$$\varphi : M \to N \quad \text{is an isometric immersion} \Rightarrow \tau(\varphi) = \text{normal} \Rightarrow \operatorname{div} S = 0$$

- G. Y. Jiang (1987): the stress-energy tensor S_2 of the bienergy

$$S_2(X,Y) = \frac{1}{2}|\tau(\varphi)|^2\langle X,Y\rangle + \langle d\varphi, \nabla\tau(\varphi)\rangle\langle X,Y\rangle - \langle d\varphi(X), \nabla_Y\tau(\varphi)\rangle - \langle d\varphi(Y), \nabla_X\tau(\varphi)\rangle$$

that satisfies

$$\operatorname{div} S_2 = -\langle \tau_2(\varphi), d\varphi\rangle$$

If $\varphi : M \to N$ is an isometric immersion, then $\operatorname{div} S_2 = -\tau_2(\varphi)^\top$

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Remark

If $\varphi : M \to (N,h)$ is a fixed map, then E_2 can be thought as a functional on the set of all Riemannian metrics on M whose critical points are Riemannian metrics determined by $S_2 = 0$.

Definition

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Theorem (Balmuş, Montaldo, Oniciuc - 2012)

A submanifold M in a Riemannian manifold N, with second fundamental form B, mean curvature vector field H, and shape operator A, is biharmonic iff

$$\frac{m}{2}\nabla|H|^2 + 2\operatorname{trace} A_{\nabla^\perp_{\cdot} H}(\cdot) + 2\operatorname{trace}(\overline{R}(\cdot,H)\cdot)^\top = 0$$

and

$$\Delta^\perp H + \operatorname{trace} B(\cdot, A_H \cdot) + \operatorname{trace}(\overline{R}(\cdot,H)\cdot)^\perp = 0,$$

where $\Delta^\perp$ is the Laplacian in the normal bundle.

Submanifolds with parallel mean curvature

Let M^m be a submanifold of a Riemannian manifold N

$$\overline{\nabla}_X Y = \nabla_X Y + B(X,Y) \quad \text{(Eq. Gauss)}$$

$$\overline{\nabla}_X V = -A_V X + \nabla^\perp_X V \quad \text{(Eq. Weingarten)}$$

Definition

If the mean curvature vector field H is parallel in the normal bundle, i.e., $\nabla^\perp H = 0$, then M is called a PMC surface.

Definition

If the mean curvature $|H|$ is constant, then M is called a CMC surface.

PMC biconservative submanifolds

Let M^m be a submanifold of a complex space form $N^n(c)$ with mean curvature vector field H and decompose $JH = T + N$.

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Theorem (Bibi, Chen, F., Oniciuc - 2021)

Let M^m be a PMC submanifold of $N^n(c)$.

- *If $c = 0$, then M^m is biconservative, and*
- *if $c \neq 0$, then M^m is biconservative iff JT is normal to M.*

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Corollary

Let M^m be a PMC totally real submanifold of $N^n(c)$ (i.e., $JTM^m \subset NM^m$). Then M^m is biconservative.

Corollary

Any PMC real hypersurface M^{2n-1} of $N^n(c)$ is biconservative.

PMC biconservative surfaces

Theorem (Bibi, Chen, F., Oniciuc - 2021)

Let M^2 be a PMC surface in a complex space form $N^n(c)$.

- *If $c = 0$, then M^2 is biconservative.*
- *If $c \neq 0$, then M^2 is biconservative iff it is totally real.*

Proof.

- When $c \neq 0$, M is biconservative iff JT is normal.

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- the Ricci equation

$$\langle [A_H, A_V]X, Y\rangle = -\langle \overline{R}(X,Y)H, V\rangle \quad \text{implies that} \quad [A_H, A_{JT}] = 0$$

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- it follows that, at each point $p \in M$, there exists a basis $\{e_1, e_2\}$ such that
$$A_H = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad A_{JT} = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}$$

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$$A_H = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad A_{JT} = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}$$
- from the expressions of $\overline{\nabla}_X JH$ and $\overline{\nabla}_X JT$, as $\nabla^\perp H = 0$, we have, at p,
$$\begin{aligned} \mu_1 + \mu_2 &= -(\lambda_1 + \lambda_2)\langle Je_1, e_2\rangle^2 + \sum_{i=1}^{2} \langle B(e_i, T), Je_i\rangle \\ 2|T|^2 &= 2|H|^2 \langle Je_1, e_2\rangle^2 + \operatorname{trace}\langle JB(\cdot, T), \cdot\rangle \end{aligned}$$

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- Now, if $T_p \neq 0$, consider the orthonormal basis $\{X_1 = T_p/|T_p|, X_2\}$, tangent to M^2. Since JT is normal, we have

$$\langle JX_2, X_1\rangle = 0$$

- Next, assume that $T_p = 0$. It follows that

$$2|H|^2 \langle Je_1, e_2\rangle^2 = 0,$$

that is

$$\langle Je_1, e_2\rangle = 0$$

Remark (Sato - 1997)

If M^2 is a PMC surface in $N^n(c)$ and JH is normal, i.e., $T = 0$, then it is known that M^2 is totally real and $n > 2$.

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Remark

For $c = 0$, every PMC submanifold of $\mathbb{C}^n$ is biconservative, but not necessarily totally real. For instance, $\mathbb{S}^2(1) \subset \mathbb{E}^3 \subset \mathbb{C}^2$ is PMC and biconservative in $\mathbb{C}^2$ but not totally real.

Theorem (Nistor - 2017)

Let M^2 be a complete CMC biconservative surface in a Riemannian manifold N^n with Gaussian curvature $K \geq 0$ and $\mathrm{Riem}^N \leq K_0 = \mathrm{constant}$. Then $\nabla A_H = 0$ and M^2 is either flat or pseudo-umbilical.

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Corollary

Let M^2 be a complete PMC totally real surface with $K \geq 0$ in $N^n(c)$. Then $\nabla A_H = 0$ and M^2 is either flat or pseudo-umbilical.

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Remark

This corollary is similar to a result obtained by B.-Y. Chen and K. Ogiue in 1974 compact surfaces (here surfaces are only complete).

A Simons type formula

Theorem (Bibi, Chen, F., Oniciuc - 2021)

Let M^2 be a PMC totally real surface in the complex space form $N^n(c)$ with Gaussian curvature K. Then $\nabla T = A_N$ and

$$-\frac{1}{2}\Delta|T|^2 = K|T|^2 + |A_N|^2.$$

Theorem (Bibi, Chen, F., Oniciuc - 2021)

If M^2 is a complete PMC totally real surface with $K \geq 0$ in $N^n(c)$, then $\nabla T = A_N = 0$ and either $K = 0$ everywhere, or $K > 0$ at some point on the surface and $T = 0$ on M^2.

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Proof.

- $|T|^2 \leq |JH|^2 = |H|^2$ and M^2 is CMC $\Rightarrow |T|^2$ is a bounded function on M^2
- $\Delta |T|^2 \leq 0$, i.e., $|T|^2$ is a subharmonic function $\Rightarrow |T|^2 = \text{constant}$ (since M^2 is parabolic ($K \geq 0$), see A. Huber (1957))
- It follows that $K|T|^2 + |A_N|^2 = 0$

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Corollary

Let M^2 be a complete PMC totally real surface with $K \geq 0$ in $N^n(c)$. Then

$$\nabla A_H = \nabla T = A_N = 0$$

and either M^2 is flat or pseudo-umbilical with $T = 0$. In the latter case $n > 2$.

CMC biconservative surfaces in $N^2(c)$ (the case $c \neq 0$)

Theorem (Bibi, Chen, F., Oniciuc - 2021)

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Remark

If M^2 is a pseudo-umbilical CMC biconservative surface in $N^2(c)$, with $c \neq 0$, then M^2 is PMC and JT is normal.

Biconservative surfaces in $\mathbb{C}^2$

A non-PMC biconservative surface in $\mathbb{C}^2$ has the following parametric equation.

Proposition (Montaldo, Oniciuc, Ratto - 2016)

Let M^2 be a non-PMC biconservative surface with constant mean curvature in $\mathbb{C}^2$. Then, locally, the surface is given by

$$X(u,v) = (\gamma^1(u), \gamma^2(u), \gamma^3(u), v),$$

where $\gamma : I \to \mathbb{E}^3$ is a curve parametrized by arc-length with constant non-zero curvature and non-zero torsion.

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Proposition (Bibi, Chen, F., Oniciuc - 2021)

Let M^2 be a non-PMC biconservative surface with constant mean curvature in $\mathbb{C}^2$. Then $(JT)^\top \neq 0$.

A geometric property of CMC biconservative surfaces

Theorem (Bibi, Chen, F., Oniciuc - 2021)

Let M^2 be a CMC biconservative surface in a complex space form $N^2(c)$. If JT is normal, then M^2 is PMC and totally real.

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Remark

Complete PMC surfaces in a complex space form $N^2(c)$ were classified by K. Kenmotsu (two papers in 2016 and 2018). When $c > 0$, these surfaces are totally real flat tori.

Reduction of codimension

Theorem (Bibi, Chen, F., Oniciuc - 2021)

Let M^2 *be a non-pseudo-umbilical PMC totally real surface in a complex space form* $N^n(c)$, $c \neq 0$, $n \geq 4$. *Then there exists a totally geodesic complex submanifold* $N^4(c) \subset N^n(c)$ *such that* $M^2 \subset N^4(c)$.

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Proof.

- Consider $L = \operatorname{span}\{\operatorname{Im} B \cup (J \operatorname{Im} B)^{\perp} \cup JTM^2\} \subset NM^m$
  where $(J \operatorname{Im} B)^{\perp} = \{(JB(X,Y))^{\perp} : X,\, Y$ tangent vector fields to $M^2\}$
- $\nabla^{\perp} L \subseteq L, \qquad \operatorname{Im} B \subseteq L, \qquad \dim L \leq 6$
- consider $\widetilde{L} = L \oplus TM^2$
- $\overline{\nabla}\widetilde{L} \subset \widetilde{L}$ and $J\widetilde{L} = \widetilde{L}$, which implies that $\widetilde{L}$ is invariant by the curvature tensor $\overline{R}$
- we conclude by using a result of J. H. Eschenburg and R. Tribuzy (1997).

□

Remark

When the surface M^2 is pseudo-umbilical and a topological sphere the situation is quite different:

- *S. Montaldo, C. Oniciuc, and A. Ratto (2016) showed that if M^2 is a CMC biconservative sphere in an arbitrary Riemannian manifold, then it is pseudo-umbilical.*
- *B. Opozda (1988) proved that if M^2 is a PMC totally real sphere in a complex space form $N^n(c)$, $c \neq 0$, then there exists a totally geodesic totally real submanifold N' such that $M^2 \subset N'$.*

Note that the technique used here is a completely different one.

Reduction of codimension

With the additional hypothesis $H \in C(JTM^2)$ one obtains that $L = JTM^2$ and the codimension reduces even more.

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Theorem (Bibi, Chen, F., Oniciuc - 2021)

Let M^2 be a non-pseudo-umbilical PMC totally real surface in a complex space form $N^n(c)$, $c \neq 0$. If $H \in C(JTM^2)$, then there exists a totally geodesic complex submanifold $N^2(c) \subset N^n(c)$ such that $M^2 \subset N^2(c)$.

The Segre embedding (definition)

Consider the isometric and holomorphic embedding introduced by C. Segre (1891)

$$j = S_{pq} : \mathbb{C}P^p(4) \times \mathbb{C}P^q(4) \to \mathbb{C}P^{p+q+pq}(4)$$

given by

$$S_{pq}([(z_0, \cdots, z_p)], [(w_0, \cdots, w_q)]) = [(z_j w_t)_{0 \le j \le p, 0 \le t \le q}],$$

where $(z_0, \cdots, z_p)$ and $(w_0, \cdots, w_q)$ are the homogeneous coordinates in $\mathbb{C}P^p(4)$ and $\mathbb{C}P^q(4)$, respectively.

The Segre embedding

(properties, B.-Y. Chen (1981, 2002); H. Nakagawa and R. Takagi (1976))

Let B^j be the second fundamental form of $j = S_{pq}$.

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- $B^j(X_1, X_2) = B^j(Y_1, Y_2) = 0$, X_1, X_2 tangent to $\mathbb{C}P^p(4)$; Y_1, Y_2 tangent to $\mathbb{C}P^q(4)$

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- $(\nabla^\perp B^j)(X, Y, Z) = 0$

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- $(\nabla^\perp B^j)(X, Y, Z) = 0$
- Let M^p be a Lagrangian submanifold of $\mathbb{C}P^p(4)$ (i.e., $JTM = NM$). Then
  - $\left\{B^j(E_a, \bar{E}_\alpha)\right\}$ are orthonormal vector fields;
  - $\left\{B^j(JE_a, \bar{E}_\alpha)\right\}$ are orthonormal vector fields,

  where $\{E_a\}_{i=1}^{p}$ is a local orthonormal frame field on M^p and $\{\bar{E}_\alpha\}_{\alpha=1}^{2q}$ is a local orthonormal frame field on $\mathbb{C}P^q(4)$.

Biharmonic submanifolds via Segre embedding

Theorem (Bibi, F., Oniciuc - 2023)

If M^p is a Lagrangian submanifold in $\mathbb{C}P^p$, then

1. *via the Segre embedding of $\mathbb{C}P^p \times \mathbb{C}P^q$ into $\mathbb{C}P^{p+q+pq}$, the product $\Sigma^{p+2q} = M^p \times \mathbb{C}P^q$ is a biconservative submanifold of $\mathbb{C}P^{p+q+pq}$ iff M^p is a biconservative submanifold in $\mathbb{C}P^p$;*
2. *Σ^{p+2q} is a proper-biharmonic submanifold in $\mathbb{C}P^{p+q+pq}$ iff M^p is a proper-biharmonic submanifold in $\mathbb{C}P^p$.*

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2. *Σ^{p+2q} is a proper-biharmonic submanifold in $\mathbb{C}P^{p+q+pq}$ iff M^p is a proper-biharmonic submanifold in $\mathbb{C}P^p$.*

Remark

Σ^{p+2q} is a non-PMC submanifold of $\mathbb{C}P^{p+q+pq}$ provided that M is not minimal in $\mathbb{C}P^p$. On the other hand, Σ^{p+2q} is a CMC submanifold in $\mathbb{C}P^{p+q+pq}$ iff M is a CMC submanifold of $\mathbb{C}P^p$.

Biharmonic submanifolds via Segre embedding

Theorem (Bibi, Chen, F., Oniciuc - 2021)

Let γ be a curve of nowhere vanishing curvature κ in $\mathbb{C}P^1(4)$. Then, we have:

- *the product $\Sigma^{1+2q} = \gamma \times \mathbb{C}P^q(4)$ is a biconservative submanifold of $\mathbb{C}P^{1+2q}(4)$ via the Segre embedding iff $\kappa = constant$.*
  In this case, Σ^{1+2q} is CMC non-PMC and it is not totally real.
- *Moreover, Σ^{1+2q} is a proper-biharmonic submanifold of $\mathbb{C}P^{1+2q}(4)$ iff $\kappa^2 = 4$, i.e., γ is proper-biharmonic in $\mathbb{C}P^1(4)$.*

Biharmonic submanifolds via Segre embedding

Theorem (Bibi, F., Oniciuc - 2023)

If M_1^p and M_2^q are Lagrangian submanifolds in $\mathbb{C}P^p$ and $\mathbb{C}P^q$, respectively, then

1. *via the Segre embedding of $\mathbb{C}P^p \times \mathbb{C}P^q$ into $\mathbb{C}P^{p+q+pq}$, the product $\Sigma^{p+q} = M_1^p \times M_2^q$ is a biconservative submanifold of $\mathbb{C}P^{p+q+pq}$ iff M_1^p and M_2^q are biconservative submanifolds in $\mathbb{C}P^p$ and $\mathbb{C}P^q$, respectively;*
2. *Σ^{p+q} is a proper-biharmonic submanifold in $\mathbb{C}P^{p+q+pq}$ iff one of the submanifolds M_1^p or M_2^q is minimal and the other is proper-biharmonic in $\mathbb{C}P^p$ or $\mathbb{C}P^q$, respectively.*

Biharmonic submanifolds via Segre embedding

Theorem (Bibi, F., Oniciuc - 2023)

If M_1^p and M_2^q are Lagrangian submanifolds in $\mathbb{C}P^p$ and $\mathbb{C}P^q$, respectively, then

1. *via the Segre embedding of $\mathbb{C}P^p \times \mathbb{C}P^q$ into $\mathbb{C}P^{p+q+pq}$, the product $\Sigma^{p+q} = M_1^p \times M_2^q$ is a biconservative submanifold of $\mathbb{C}P^{p+q+pq}$ iff M_1^p and M_2^q are biconservative submanifolds in $\mathbb{C}P^p$ and $\mathbb{C}P^q$, respectively;*
2. *Σ^{p+q} is a proper-biharmonic submanifold in $\mathbb{C}P^{p+q+pq}$ iff one of the submanifolds M_1^p or M_2^q is minimal and the other is proper-biharmonic in $\mathbb{C}P^p$ or $\mathbb{C}P^q$, respectively.*

Remark

Σ^{p+q} is a non-PMC submanifold of $\mathbb{C}P^{p+q+pq}$ if at least one of M_1 or M_2 is not minimal in its ambient space. If M_1 and M_2 are CMC submanifolds in $\mathbb{C}P^p$ and $\mathbb{C}P^q$, respectively, so is Σ^{p+q} in $\mathbb{C}P^{p+q+pq}$, while the converse is not true in general.

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Remark

- *Consider two curves γ_1 and γ_2 in $\mathbb{C}P^1$ with constant curvatures κ_1 and κ_2, respectively. Then, they are biconservative and Lagrangian. It follows that $\Sigma^2 = \gamma_1 \times \gamma_2$ is a CMC (non-PMC) biconservative surface in $\mathbb{C}P^3$.*
- *On the other hand, a CMC biconservative surface in $\mathbb{C}P^2$ with $J((JH)^\top)$ normal to the surface is PMC. As Σ^2 satisfies all these conditions but it is not PMC when either κ_1 or κ_2 is positive, one sees that this result does not hold in $\mathbb{C}P^3$.*
- *Moreover, since a PMC biconservative surface in $\mathbb{C}P^n$ with JH tangent to the surface lies in $\mathbb{C}P^2$, this example shows that this only works for PMC surfaces, the CMC condition not being sufficient.*
- *The surface $\Sigma^2 = \gamma_1 \times \gamma_2$ is proper-biharmonic in $\mathbb{C}P^3$ iff γ_1 is a geodesic of $\mathbb{C}P^1$, i.e., it is a great circle of Euclidean sphere $\mathbb{S}^2$ of radius $1/2$, and γ_2 is a circle of radius $1/(2\sqrt{2})$ of the same sphere.*

References

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Thank you!