Kähler identities for almost complex manifolds

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- An *almost complex structure*: bundle map $J : TM \to TM$ such that $J^2 = -I$.
- An *almost complex manifold* is a manifold with an almost complex structure.
- An almost complex structure is *integrable* if the *Nijenhuis* tensor, defined by

$$N(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y],$$

  vanishes (among many equivalent conditions).
- A *complex manifold* is a manifold M^{2n} with a holomorphic atlas.
  It carries a natural "almost" complex structure J.
  It is well known (Newlander-Nirenberg'57)

$$M \text{ complex} \iff J \text{ integrable.}$$

- For M (almost) complex, let $\langle\ ,\ \rangle$ be a compatible metric, i.e. $\langle JX, JY\rangle = \langle X, Y\rangle$.
  Define the *fundamental form* $\omega(X, Y) = \langle JX, Y\rangle$.
- An almost complex manifold is *almost Kähler* if $d\omega = 0$.
- A complex manifold is *Kähler* if $d\omega = 0$.

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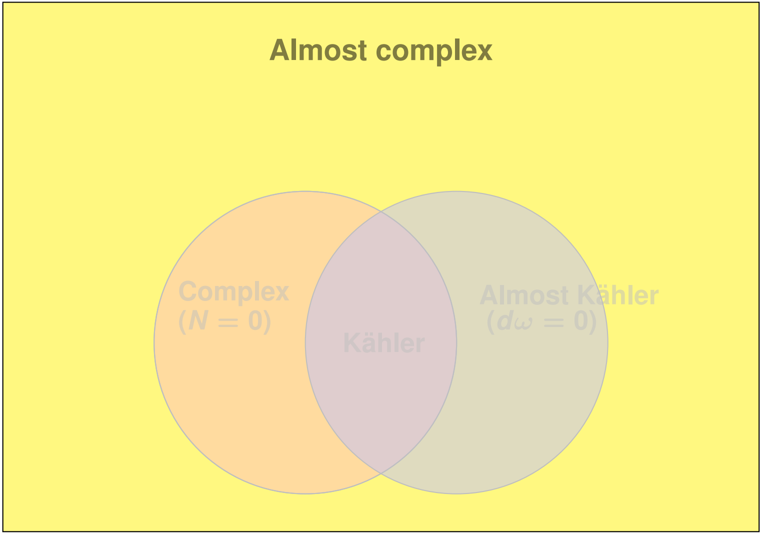

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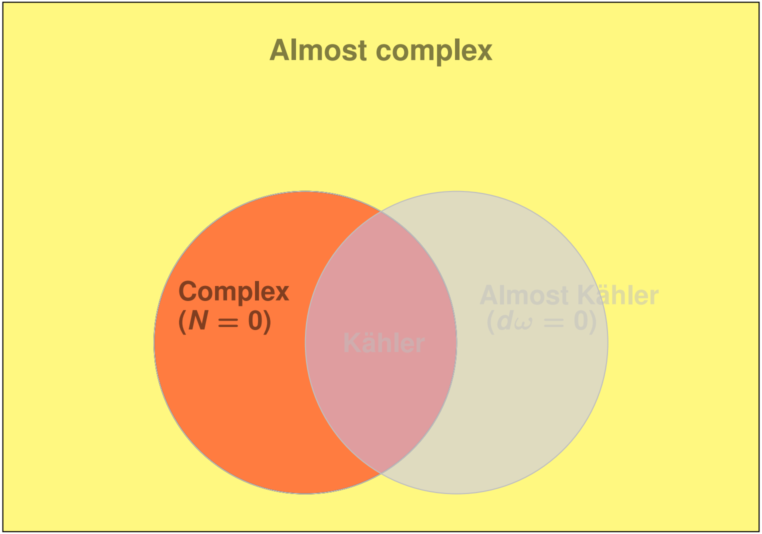

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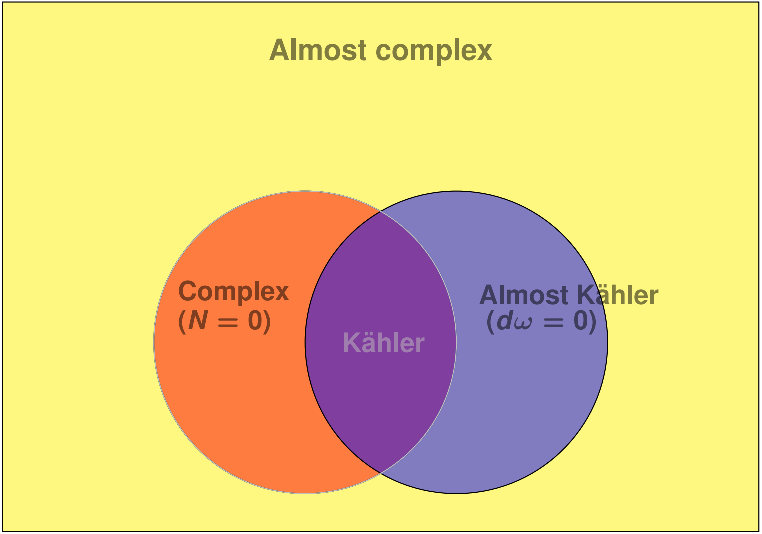

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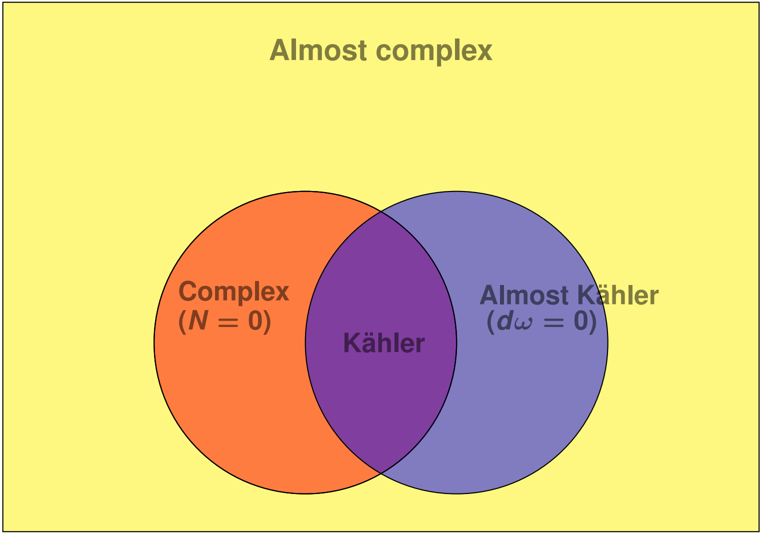

Perhaps the main question in almost complex manifolds is the following:

Existence of an almost complex structure depends only on the topology (it is a section of a bundle, so one can use obstruction theory).

Is integrability also a topological condition? If not, what is it?

In fact, there are no examples of almost complex manifolds of real dimension 6 or higher that do not also admit an integrable complex structure.

The most famous example is S^6.

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In an almost complex manifold M with a compatible metric, we have several operators:

- d, the usual differential operator.
- L, the *Lefschetz* operator, $L(\alpha) = \omega \wedge \alpha$.
- Λ, the adjoint of L with respect to the metric.
- $d^* = - * \circ d \circ *$, the *formal adjoint* of d.
- $d^c = J_a^{-1} \circ d \circ J_a$
  (J_a = extension of J to the exterior algebra as an algebra map: $J_a(\alpha \wedge \beta) = J\alpha \wedge J\beta$).

There is also a descomposition of the complexified exterior algebra as follows:

Consider the dual of J, $\check{J} : A^1 M \to A^1 M$.
Extend it complex-linearly to $\check{J} : A^1 M \otimes \mathbb{C} \to A^1 M \otimes \mathbb{C}$.

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Denote the corresponding eigenspaces by $A^{1,0}M$ and $A^{0,1}M$, respectively.

This decomposition propagates: let $A^{p,q}M = \bigwedge_{r=0}^{p} A^{1,0}M \bigotimes \bigwedge_{s=0}^{q} A^{0,1}M$. Then

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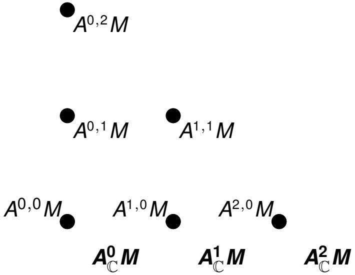
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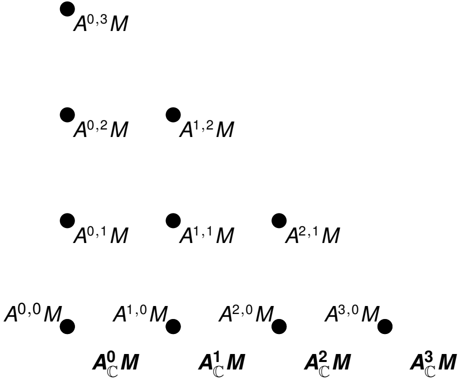
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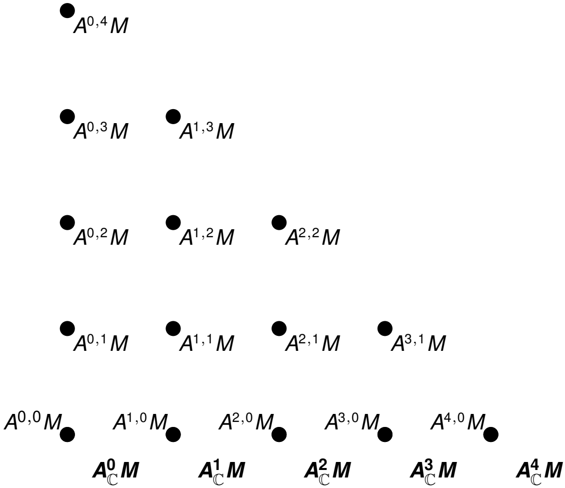
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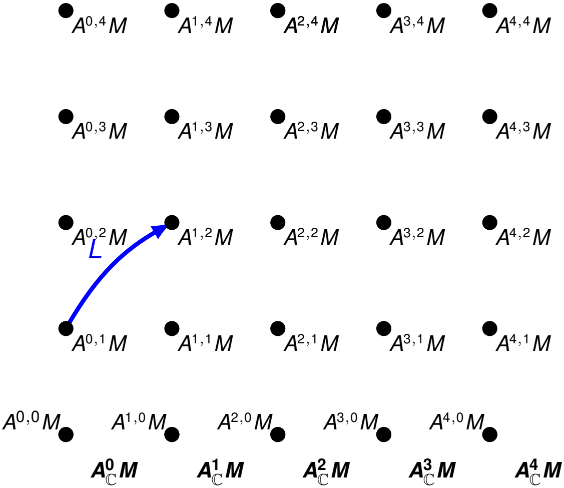

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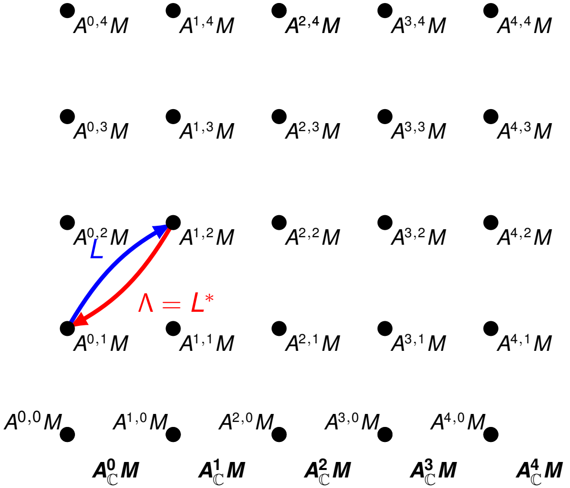
Since ω is a $(1,1)$-form,
$L : A^k M \to A^{k+2} M$



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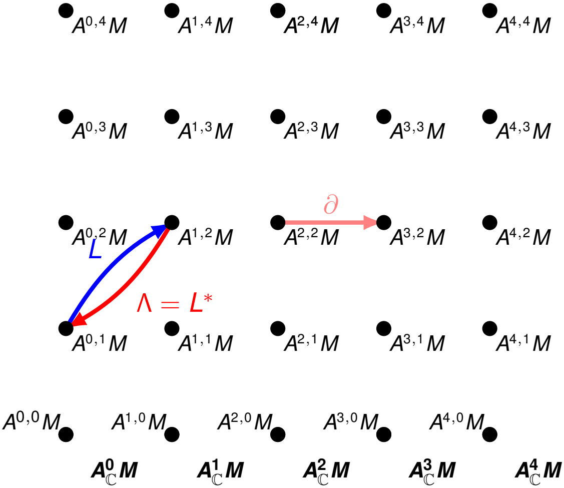

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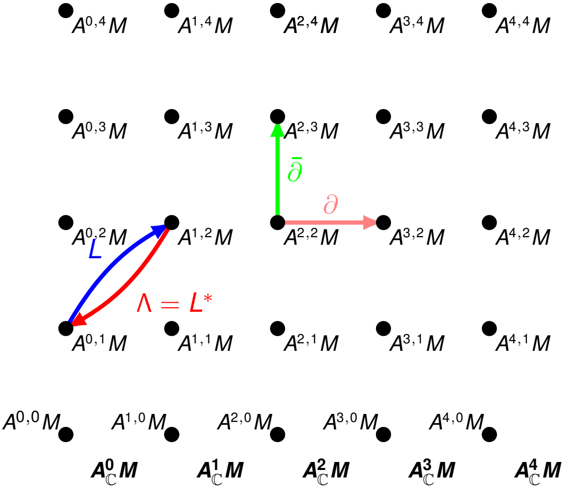

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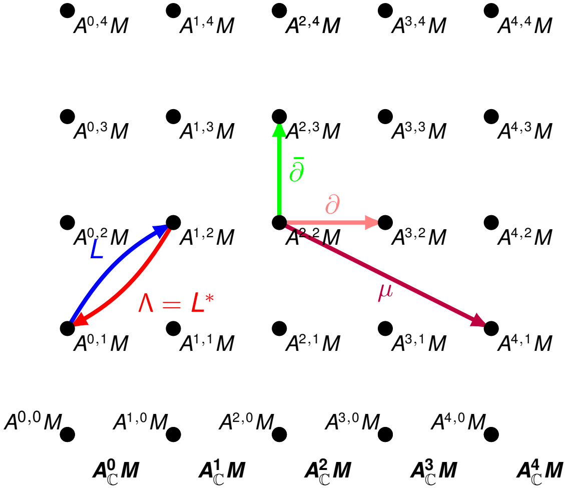

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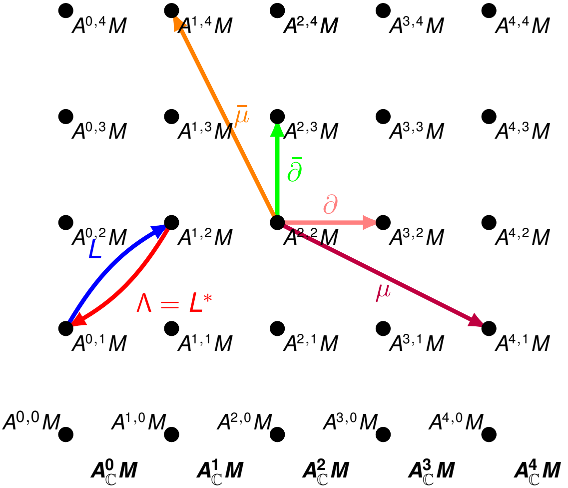

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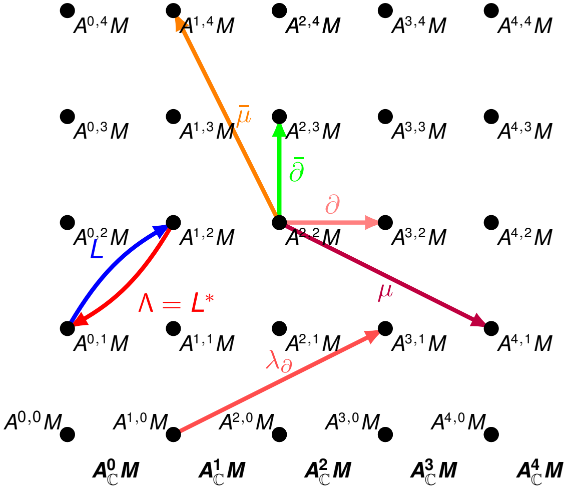

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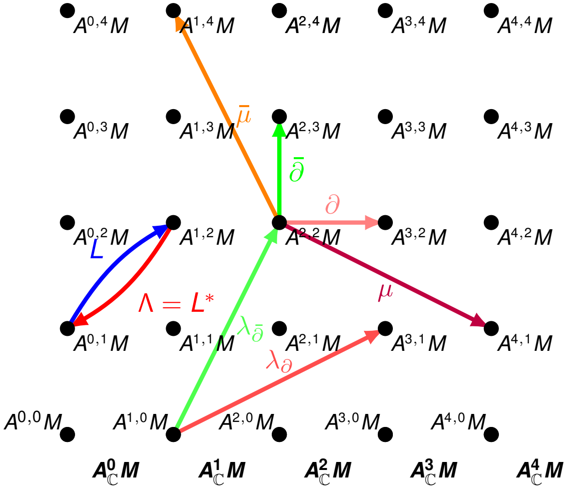

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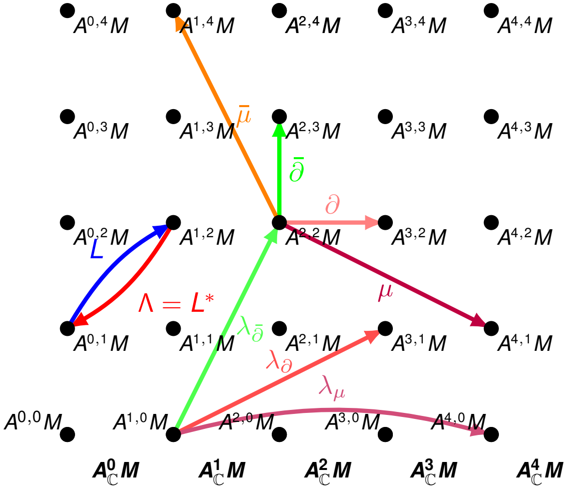

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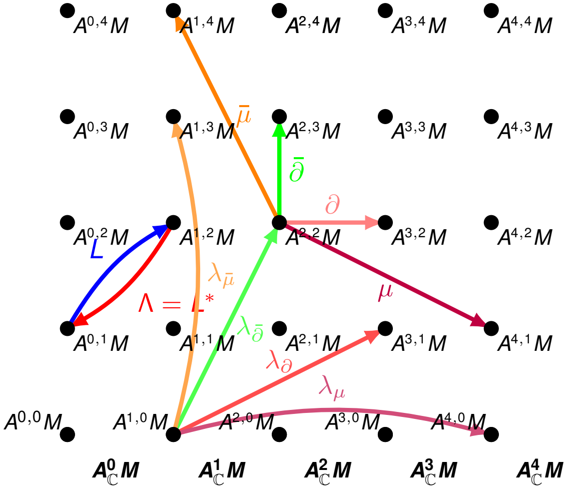

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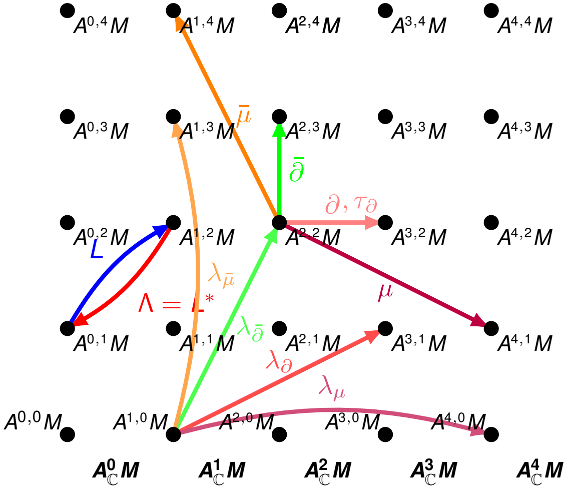

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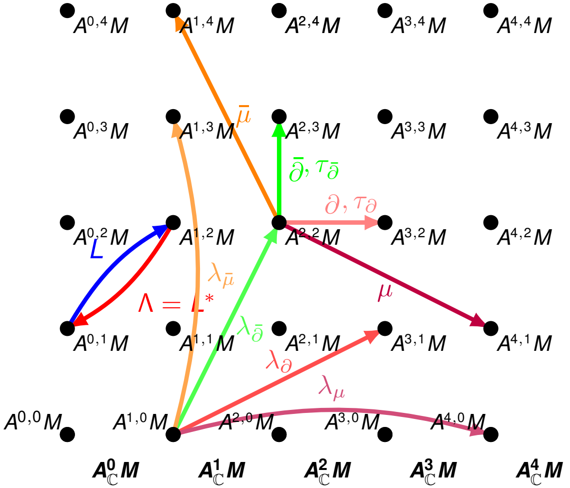

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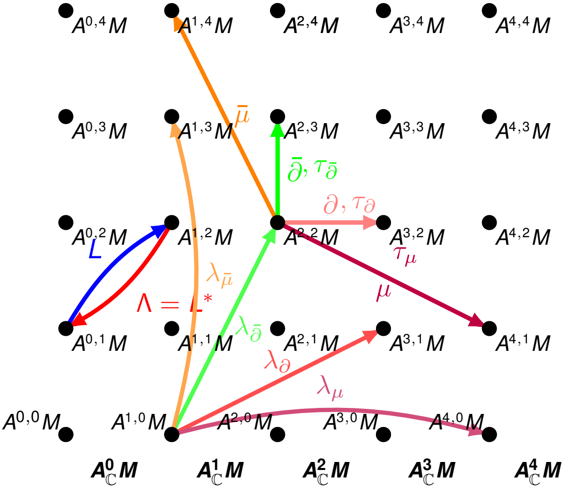

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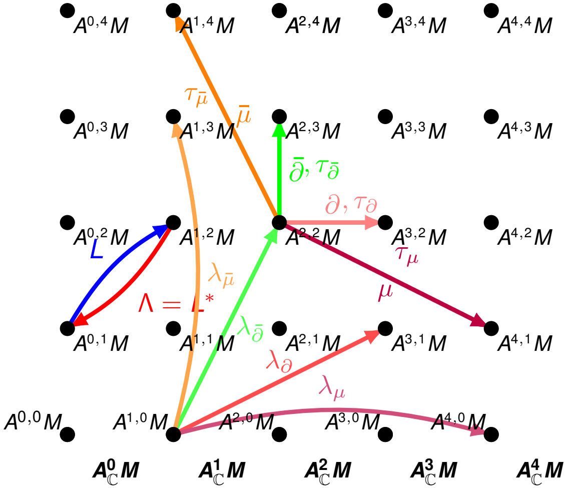

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For M a Kähler manifold, $\mu = \bar{\mu} = 0$, and $d\omega = 0$, so $\lambda = \tau = 0$.
The *Kähler identities* (or Hodge identities) read:

$$[d, L] = 0 \qquad [d, \Lambda] = d^{c*},$$

or decomposing into type,

$$[\partial, L] = 0, \qquad [\bar{\partial}, L] = 0 \qquad [\partial, \Lambda] = -i\bar{\partial}^* \qquad [\bar{\partial}, \Lambda] = i\partial^*.$$

- $[P, Q] := P \circ Q - Q \circ P$.
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- $d^* = - * \circ d \circ *$ is the *formal adjoint* of d.
- $d^c = -i(\partial - \bar{\partial})$ $(= J_a^{-1} \circ d \circ J_a)$

The Kähler identities are a fundamental tool to show, for example, the Lefschetz decomposition of complex DeRham cohomology or the fact that, on a Kähler manifold, the notions of d-harmonic, ∂-harmonic, and $\bar{\partial}$-harmonic forms coincide, which has important topological implications.

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Demailly, 1986: Define the following operators on $A_{\mathbb{C}}(M)$ $(= A(M) \otimes \mathbb{C})$:

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Then the Kähler identities generalize to

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For almost Kähler manifolds ($\lambda = \tau = 0$): d decomposes as $d = \bar{\mu} + \bar{\partial} + \partial + \mu$.

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For 6-dimensional nearly Kähler manifolds ($\partial\omega = \bar{\partial}\omega = 0$, $\tau_\mu = \mu$):

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We show that Demailly's identities hold, in greatest generality, for almost complex manifolds:

For almost complex manifolds:

Theorem (F., Hosmer):

Let M be an *almost complex manifold* with a compatible metric. Then

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(Plus the adjoints and conjugates of these identities.)

These identities include all the generalizations above.

The approach is to formulate the problem in the *Clifford bundle* where the algebraic structure is somewhat richer, and then translate the result to the exterior bundle.

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The approach is to formulate the problem in the *Clifford bundle* where the algebraic structure is somewhat richer, and then translate the result to the exterior bundle.

Let V be a vector space with a metric, and $\{e_j\}_{j=1}^n$ an orthonormal basis.

The *Clifford algebra* is the free algebra $\mathrm{Cl}(V)$ generated by the e_j, subject to the conditions

$$e_j \cdot e_k + e_k \cdot e_j = 0 \text{ if } j \neq k, \qquad e_j^2 = -1.$$

On a manifold we can construct the *Clifford bundle* $\mathrm{Cl}(M) := \mathrm{Cl}(TM)$.

It is well known that $\mathrm{Cl}(M)$ and the exterior bundle $A(M)$ are *canonically isomorphic* as vector bundles, as are their complexifications $\mathbb{C}\mathrm{l}(M)$ and $A_{\mathbb{C}}(M)$.

The ac structure $J : TM \to TM$ and its dual $\check{J} : A^1M \to A^1M$
can be extended to $\mathbb{C}\mathrm{l}(M)$ and $A_{\mathbb{C}}(M)$, respectively, in *two different ways*:

→ As a *derivation*: $J_d(u \cdot v) = J_d(u) \cdot v + u \cdot J_d(v)$ (similarly for $\check{J}_d$).

→ As an *algebra map*: $J_a(u \cdot v) = J_a(u) \cdot J_a(v)$ (similarly for $\check{J}_a$).

Following Michelsohn'80, in $\mathbb{C}\mathrm{l}(M)$ we define

- If $\varphi \in \Gamma(Cl(M))$, the *Dirac operator* is defined by $D\varphi := \sum_{B=1}^{2n} e_B \cdot \nabla_{e_B}\varphi$.
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Through the isomorphism, we have the following **conversion table**:

$\mathbb{C}l(M)$		$A_{\mathbb{C}}(M)$
D	$\cong$	$d + d^*$
ω	$\cong$	ω
$\mathcal{H}$	$\cong$	$i(\Lambda - L)$ (Michelsohn'80)
D^c	$\cong$	$-(d^c + d^{c*})$

(For an operator P, $P^c := J_a^{-1} \circ P \circ J_a$.)

Observe that in the Kähler identities,

- The left hand side is $[d, L]$ or $[d, \Lambda]$.
- The right hand side has a term d^{c*}.

To find a formulation in the Clifford bundle, it is natural to consider the commutator

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After a very simple computation using an adapted basis, one obtains:

Theorem (F., Hosmer) *(Kähler identities on the Clifford bundle)*:

$$[D, \mathcal{H}] = -iD^c + iD_\sigma - iL_{D\omega}$$

where, for $\varphi \in \Gamma(\mathbb{C}\mathrm{l}(M))$,

$$L_{D\omega}(\varphi) := D\omega \cdot \varphi, \qquad D_\sigma(\varphi) := \sum_{A=1}^{2n} e_A \cdot \sigma_{e_A}\varphi,$$

with

$$\sigma_X := \nabla_X J_d + J_a^{-1} \circ \nabla_{JX} J_a.$$

To obtain the Kähler identities in the exterior bundle, it only remains to *convert this identity*.

Via relatively simple *linear algebraic* computations, one obtains

$$D_\sigma - L_{D\omega} \cong \tau^c - \lambda + \tau^{c*} - \lambda^*.$$

Thus we have

$$[D, \mathcal{H}] = -iD^c + iD_\sigma - iL_{D\omega},$$

and

$$\begin{aligned} [D, \mathcal{H}] &\cong [d + d^*, i(\Lambda - L)] \\ D^c &\cong -(d^c + d^{c*}) \\ D_\sigma - L_{D\omega} &\cong \tau^c - \lambda + \tau^{c*} - \lambda^* \end{aligned}$$

Putting all together, we get

$$[d + d^*, \Lambda - L] = d^c + \tau^c - \lambda + d^{c*} + \tau^{c*} - \lambda^*,$$

which gives the Kähler identities by separating by degrees.

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Here is the *full table of commutators* of the common operators with L and Λ.

	Λ	L
d	$d^{c*} + \tau^{c*}$	λ
μ	$i(\bar{\mu}^* + \tau^*_{\bar{\mu}})$	λ_μ
τ_μ	$-2i\tau^*_{\bar{\mu}}$	$-3\lambda_\mu$
$\bar{\mu}$	$-i(\mu^* + \tau^*_\mu)$	$\lambda_{\bar{\mu}}$
$\tau_{\bar{\mu}}$	$2i\tau^*_\mu$	$-3\lambda_{\bar{\mu}}$
∂	$-i(\bar{\partial}^* + \tau^*_{\bar{\partial}})$	λ_∂
τ_∂	$2i\tau^*_{\bar{\partial}}$	$-3\lambda_\partial$
ρ_∂	$-i\rho^*_{\bar{\partial}} + \tau^*_{\bar{\partial}}$	$i\lambda_\partial$
$\bar{\partial}$	$i(\partial^* + \tau^*_\partial)$	$\lambda_{\bar{\partial}}$
$\tau_{\bar{\partial}}$	$-2i\tau^*_\partial$	$-3\lambda_{\bar{\partial}}$
$\rho_{\bar{\partial}}$	$i\rho^*_\partial + \tau^*_\partial$	$-i\lambda_{\bar{\partial}}$
$\lambda_{\bar{\mu}}$	$-\tau_{\bar{\mu}}$	0
$\lambda_{\bar{\partial}}$	$-\tau_{\bar{\partial}}$	0
λ_∂	$-\tau_\partial$	0
λ_μ	$-\tau_\mu$	0

	L	Λ
d^*	$-(d^c + \tau^c)$	$-\lambda^*$
μ^*	$i(\bar{\mu} + \tau_{\bar{\mu}})$	$-\lambda^*_\mu$
τ^*_μ	$-2i\tau_{\bar{\mu}}$	$3\lambda^*_\mu$
$\bar{\mu}^*$	$-i(\mu + \tau_\mu)$	$-\lambda^*_{\bar{\mu}}$
$\tau^*_{\bar{\mu}}$	$2i\tau_\mu$	$3\lambda^*_{\bar{\mu}}$
∂^*	$-i(\bar{\partial} + \tau_{\bar{\partial}})$	$-\lambda^*_\partial$
τ^*_∂	$2i\tau_{\bar{\partial}}$	$3\lambda^*_\partial$
ρ^*_∂	$-i\rho_{\bar{\partial}} - \tau_{\bar{\partial}}$	$i\lambda^*_\partial$
$\bar{\partial}^*$	$i(\partial + \tau_\partial)$	$-\lambda^*_{\bar{\partial}}$
$\tau^*_{\bar{\partial}}$	$-2i\tau_\partial$	$3\lambda^*_{\bar{\partial}}$
$\rho^*_{\bar{\partial}}$	$i\rho_\partial - \tau_\partial$	$-i\lambda^*_\partial$
$\lambda^*_{\bar{\mu}}$	$\tau^*_{\bar{\mu}}$	0
$\lambda^*_{\bar{\partial}}$	$\tau^*_{\bar{\partial}}$	0
λ^*_∂	τ^*_∂	0
λ^*_μ	$-i\tau^*_\mu$	0

- *What I like:*
  - → The proof is *completely algebraic* once it is formulated in the Clifford bundle.
  - → The operator σ has a *"companion"* ν defined as

$$\nu_X = \nabla_X J_d - J_a^{-1} \circ \nabla_{JX} J_a.$$

Recall that $D_\sigma = \sum_{A=1}^{2n} e_A \cdot \sigma_{e_A}$, and $D_\nu = \sum_{A=1}^{2n} e_A \cdot \nu_{e_A}$. It is not hard to prove that

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Thus we get a simple characterization of all the "sides of the coin".

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Thank you

It is a quick computation to show that $J_a(\varphi) = \frac{1}{2}(\omega \cdot \varphi - \varphi \cdot \omega)$.

Then $\mathcal{H} = iJ_d - iL_\omega$.

We first compute the commutator with each of the two terms and compare the result with D^c.

$$
\begin{aligned}
[D, J_d](X) &= \sum_{A=1}^{2n} (e_A \cdot \nabla_{e_A}(J_d X) - J_d(e_A \cdot \nabla_{e_A} X)) \\
&= \sum_{A=1}^{2n} (e_A \cdot (\nabla_{e_A} J_d) X + e_A \cdot J_d \nabla_{e_A} X - J e_A \cdot \nabla_{e_A} X - e_A \cdot J_d \nabla_{e_A} X) \\
&= \sum_{A=1}^{2n} e_A \cdot (\nabla_{e_A} J_d) X - \sum_{A=1}^{2n} J e_A \cdot \nabla_{e_A} X.
\end{aligned}
$$

Also,

$$
\begin{aligned}
[D, L_\omega](X) &= \sum_{A=1}^{2n} (e_A \cdot \nabla_{e_A}(\omega \cdot X) - \omega \cdot e_A \cdot \nabla_{e_A} X) \\
&= \sum_{A=1}^{2n} (e_A \cdot \nabla_{e_A} \omega \cdot X + e_A \cdot \omega \cdot \nabla_{e_A} X - \omega \cdot e_A \cdot \nabla_{e_A} X) \\
&= L_{D\omega}(X) - 2 \sum_{A=1}^{2n} J e_A \cdot \nabla_{e_A} X
\end{aligned}
$$

Thus, since $\mathcal{H} = iJ_d - iL_\omega$,

$$[D, \mathcal{H}] + iL_{D\omega}(X) = i\sum_{A=1}^{2n} e_A \cdot (\nabla_{e_A} J_d)X + i\sum_{A=1}^{2n} Je_A \cdot \nabla_{e_A} X.$$

On the other hand,

$$\begin{aligned} D^c(X) &= J_a^{-1}\left(\sum_{A=1}^{2n} e_A \cdot \nabla_{e_A} J_a X\right) \\ &= \sum_{A=1}^{2n}\left(-Je_A \cdot J_a^{-1}((\nabla_{e_A} J_a)X + J_a \nabla_{e_A} X)\right) \\ &= \sum_{A=1}^{2n} e_A \cdot J_a^{-1}(\nabla_{Je_A} J_a)X - \sum_{A=1}^{2n} Je_A \cdot \nabla_{e_A} X, \end{aligned}$$

where the first term in the last equality comes from the fact that if $1 \leq A \leq n$, then $Je_A = e_{A+n}$, and if $n + 1 \leq A \leq 2n$, then $Je_A = e_{A-n}$. Therefore,

$$[D, \mathcal{H}] + iL_{D\omega} + iD^c = i\sum_{A=1}^{2n} e_A \cdot (\nabla_{e_a} J_d + J_a^{-1} \circ \nabla_{Je_A} J_a),$$

which proves the claim.

BACK