Harmonic morphisms from Fefferman spaces

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Differential Geometry Workshop 2023
Iași, 8th September 2023

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Let M be a $(2n+1)-$dimensional, differentiable manifold. An **almost CR structure (of hypersurface type)** on M is a subbundle $T_{1,0}(M)$ of complex rank n of $T(M) \otimes \mathbb{C}$, such that

$$T_{1,0}(M) \cap T_{0,1}(M) = (0), \quad T_{0,1}(M) = \overline{T_{1,0}(M)}.$$

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An almost CR structure is **(formally) integrable** if for each open set $U \subset M$

$$Z, W \in C^{\infty}\left(U, T_{1,0}(M)\right) \Longrightarrow [Z, W] \in C^{\infty}\left(U, T_{1,0}(M)\right).$$

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A **CR manifold** is a differentiable manifold equipped with a **CR structure** (an integrable, almost CR structure). The integer n is the **CR dimension** of $(M, T_{1,0}(M))$.

Example

Consider a real hypersurface $M \subset \mathbb{C}^{n+1}$*. Then the complex structure of the ambient space induces on* M *the following CR structure*

$$T_{1,0}(M) := [T(M) \otimes \mathbb{C}] \cap T^{1,0}(\mathbb{C}^{n+1}),$$

where $T^{1,0}(\mathbb{C}^{n+1})$ *denotes the holomorphic tangent bundle on* $\mathbb{C}^{n+1}$ *[i.e. the span of* $\{\partial/\partial z_j \ : \ 1 \le j \le n+1\}$*].*

The **tangential Cauchy-Riemann operator** is the first order differential operator

$$\overline{\partial}_b : C^1(U, \mathbb{C}) \to C(U,\ T_{0,1}(M)^*),$$

$$(\overline{\partial}_b f)\overline{Z} = \overline{Z}(f),\ f \in C^1(U, \mathbb{C}),\ Z \in C^\infty(U,\ T_{1,0}(M)).$$

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There is an obvious analogy between holomorphic functions (solutions to $\overline{\partial} F = 0$) and CR functions (solutions to $\overline{\partial}_b f = 0$).

There is more than an analogy between the two concepts. Indeed, let $\Omega \subset \mathbb{C}^{n+1}$, an open subset, and $F : \Omega \to \mathbb{C}$ holomorphic. Then for every embedded real hypersurface $M \subset \mathbb{C}^{n+1}$ with $U = \Omega \cap M \neq \emptyset$:
$F_{|_U}$ is a CR function.

The **Levi distribution** is the distribution of rank $2n$

$$H(M)_x = \mathrm{Re}\left\{ T_{1,0}(M)_x \oplus T_{0,1}(M)_x \right\}, \quad x \in M,$$

equipped with the complex structure

$$J : H(M) \to H(M), \quad J\left(Z + \overline{Z}\right) = i\,\left(Z - \overline{Z}\right), \; Z \in T_{1,0}(M).$$

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Assume M orientable. Then there exist globally defined, differential 1-forms θ such that $H(M) = \mathrm{Ker}(\theta)$ called **pseudo-Hermitian structures**. Let $\mathcal{P}(M)$ be the set of all pseudohermitian structures.

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$$G_\theta(X, Y) = (d\theta)(X, JY), \quad X, Y \in H(M).$$

If G_θ is non degenerate (resp. positive definite) for some $\theta \in \mathcal{P}(M)$, then $(M, T_{1,0}(M))$ is said to be **non degenerate** (resp. **strictly pseudoconvex**, s.p.c. for short).

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By nondegeneracy and orientability there is a unique globally defined, nowhere vanishing vector field $T \in \mathfrak{X}(M)$ transverse to the Levi distribution, determined by

$$\theta(T) = 1, \quad T \rfloor (d\theta) = 0.$$

The $(0,2)-$tensor field g_θ given by

$$g_\theta(X, Y) = G_\theta(X, Y), \quad g_\theta(T, X) = 0, \quad g_\theta(T, T) = 1,$$

$$X, Y \in H(M),$$

[followed by linear extension relying on $T(M) = H(M) \oplus \mathbb{R}T$] is a semi-Riemannian metric on M. If M is s.p.c. then g_θ is a Riemannian metric (the **Webster metric**).

The Tanaka-Webster connection

Let M be a strictly pseudoconvex CR manifold, of CR dimension n. For every contact form $\theta \in \mathcal{P}(M)$ there is a unique linear connection ∇ [the **Tanaka-Webster connection** of (M, θ)] on M such that

i) the Levi distribution $H(M)$ is parallel, i.e. $\nabla H(M) \subset H(M)$,
ii) the complex structure J along $H(M)$, and the Webster metric g_θ, are parallel

$$\nabla J = 0, \quad \nabla g_\theta = 0,$$

iii) the torsion tensor field T_∇ obeys to

$$T_\nabla(Z, W) = 0, \quad T_\nabla(Z, \overline{W}) = 2\, i\, G_\theta(Z, \overline{W})\, T,$$

$$Z, \; W \in T_{1,0}(M),$$

$$\tau \circ J + J \circ \tau = 0,$$

where

$$\tau(X) = T_\nabla(T, X), \quad X \in \mathfrak{X}(M).$$

The Tanaka-Webster connection ∇ of (M, θ) and the Levi-Civita connection ∇^{g_θ} of (M, g_θ) are related by

$$\nabla^{g_\theta} = \nabla + \left(\Omega - A\right) \otimes T + \tau \otimes \theta + 2\left(\theta \odot J\right).$$

where

- *tau* is the pseudohermitian torsion of ∇;
- $\Omega = -d\theta$;
- $A(X, Y) = g_\theta(X, \tau Y)$ for any $X, Y \in \mathfrak{X}(M)$.

The Fefferman metric

Let $\Lambda^{p,0}(M) \to M$ be the vector bundle whose cross-sections are $(p, 0)$-forms on M, namely complex valued differential p-forms η on M satisfying $T_{0,1}(M) \rfloor \eta = 0$.

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There is a natural free action of $\mathbb{R}_+ = \mathrm{GL}^+(1,\mathbb{R})$ (the multiplicative positive reals) on

$$K_0(M) = \Lambda^{n+1,0}(M) \setminus \{\text{zero section}\}$$

(top degree $(p,0)$-forms) and the quotient space

$$C(M) = K_0(M)/\mathbb{R}_+$$

is the total space of a principal S^1-bundle over M, referred to as the **canonical circle bundle** over M.

By a result of J.M. Lee, $C(M)$ carries the Lorentzian metric F_θ [the **Fefferman metric** of (M, θ)] given by

$$F_\theta = \pi^* \tilde{G}_\theta + 2 \left(\pi^* \theta \right) \odot \sigma,$$

where

- $\tilde{G}_\theta$ is the (degenerate) extension of G_θ to the whole of $T(M)$ got by requiring that $\tilde{G}_\theta = G_\theta$ on $H(M) \otimes H(M)$;
- σ is the connection form given locally by

$$\sigma = \frac{1}{n+2} \left\{ d\gamma + \pi^* \left(i\, \omega_\alpha{}^\alpha - \frac{i}{2}\, g^{\alpha\overline{\beta}}\, dg_{\alpha\overline{\beta}} - \frac{\rho}{4(n+1)}\, \theta \right) \right\}$$

with γ a local fibre coordinate on $C(M)$ and ρ the pseudohermitian scalar curvature of (M, θ).

Subelliptic harmonic maps

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If $\Phi : \Omega \subset \mathbb{C}^n \to D \subset \mathbb{C}^m$ is a holomorphic map between two open subsets Ω, D and $M \subset \Omega$, $N \subset D$ are real hypersurfaces such that $\Phi(M) \subset N$, then $\phi = \Phi_{|_M} : M \to N$ is a **CR map**.

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Any stricly pseudoconvex (s.p.c.) CR manifold M equipped with a positively oriented contact form θ of vanishing pseudohermitian torsion is Sasakian; hence a pair (M, θ) consisting of a s.p.c. CR manifold and a contact form may be thought of as an *odd dimensional* analog to a Kählerian manifold.

Q: Is an (anti) CR map of s.p.c. CR manifolds $\phi : (M, \theta) \to (N, \Theta)$ harmonic with respect to the Webster metrics g_θ and g_Θ?

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A: **H. Urakawa** (1993) proved that ϕ is harmonic w.r.t g_θ and g_Θ if and only if $\phi_*(T) = T_\Theta$, where T and T_Θ are the Reeb vector fields of (M, θ) and (N, Θ).

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Lichnerowicz's result doesn't admit an immediate analog within the CR category, when the Webster metrics and the ordinary (Riemannian) concept of harmonic maps are involved.

Subelliptic harmonic maps

E. Barletta, S. Dragomir & H. Urakawa (2001) introduced **subelliptic harmonic maps** as smooth maps $\phi : (M, \theta) \to (N, h)$, with (M, θ) a s.p.c. CR manifold and (N, h) a Riemannian manifold which are critical points of the functional

$$E_{b,\Omega}(\phi) = \int_\Omega e_b(\phi)\ \theta \wedge (d\theta)^n$$

with Ω a relatively compact domain in M ($\Omega \subset\subset M$) and

$$e_b(\phi) = \frac{1}{2} \operatorname{trace}_{G_\theta} \left(\Pi_H \phi^* h \right) \ ,$$

the **pseudohermitian energy density**. [Here, if B is a bilinear form on $T(M) \otimes T(M)$, $\Pi_H B$ is the restriction of B to $H(M) \otimes H(M)$].

The **pseudohermitian second fundamental form** $\beta_b(\phi)$ is

$$\beta_b(\phi)(X, Y) = D^{\phi}_X \phi_* Y - \phi_* \nabla_X Y, \quad X, Y \in \mathfrak{X}(M),$$

where ∇ is the *Tanaka-Webster connection* of (M, θ) and $D^{\phi} = \phi^{-1}\nabla^h$ is the pullback of the Levi-Civita connection ∇^h of (N, h) [a connection in the pullback bundle $\phi^{-1}TN \to M$].

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The **pseudohermitian tension field** is

$$\tau_b(\phi) = \mathrm{trace}_{G_\theta}\left[\Pi_H \, \beta_b(\phi)\right] \in C^\infty(\phi^{-1}TN).$$

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A calculation shows that

$$\tau(\phi) = \tau_b(\phi) + D^\phi_T \, \phi_* \, T \tag{1}$$

The **Euler-Lagrange equations** of the variational principle $\delta\, E_{b,\Omega}(\phi) = 0$ are

$$-\Delta_b \phi^\alpha + \sum_{a=1}^{2n} \left(\left\{ \begin{array}{c} \alpha \\ \beta\gamma \end{array} \right\} \circ \phi \right) E_a(\phi^\beta)\, E_a(\phi^\gamma) = 0,$$

for $1 \leq \alpha \leq m$.
Here $\{E_a \,:\, 1 \leq a \leq 2n\}$ is a local G_θ-orthonormal frame for the $H(M)$, defined on some open subset $U \subset M$, and $\left\{ \begin{smallmatrix} \alpha \\ \beta\gamma \end{smallmatrix} \right\}$ are the Christoffel symbols of $h_{\alpha\beta}$.

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Consider the volume form $\Psi = \theta \wedge (d\theta)^n$.

$$\Delta_b u = -\mathrm{div}\left(\nabla^H u\right), \quad u \in C^2(M),$$

is the **sublaplacian** of (M, θ), where $\nabla^H u = \Pi_H \nabla u$ [the **horizontal gradient** of u] while the divergence is computed w.r.t. Ψ.

Δ_b is a positive, formally self adjoint, second order differential operator, and degenerate elliptic yet

- **subelliptic of order** $\epsilon = 1/2$ i.e. for any $x \in M$ there is an open neighborhood $U \subset M$ of x and a constant $C > 0$ such that

$$\|u\|_\epsilon^2 \leq C\Big((\Delta_b u,\, u)_{L^2} + \|u\|_{L^2}^2 \Big), \quad \forall\, u \in C_0^\infty(U),$$

$\| \cdot \|_\epsilon$ is the Sobolev norm of order ϵ i.e.

$$\|u\|_\epsilon = \left(\int \left(1 + |\xi|^2\right)^\epsilon |\hat{u}(\xi)|^2 \, d\xi \right)^{1/2}$$

$\hat{u}$ is the Fourier transform of u.

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$\hat{u}$ is the Fourier transform of u.

- **hypoelliptic** i.e. if u is a distribution on M and $\Delta_b u = f \in C^\infty(M)$ (in the distribution sense), then $u \in C^\infty$ [i.e. u is the distribution associated to some C^∞ function].

J. Jost & C-J. Xu (1998) started a program aiming to recover known properties of solutions to quasi-linear systems of PDEs, of variational origin, whose principal part is a second order linear elliptic operator, to the case where the principal part is at least hypoelliptic.

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Given a Hörmander system of vector fields $X = \{X_1, \cdots, X_p\}$ on $\mathcal{U} \subset \mathbb{R}^N$ open set, a C^∞ map $\phi : \mathcal{U} \to N$ into a Riemannian manifold (N, h) is a subelliptic harmonic map (in the sense of J. Jost & C-J. Xu) if ϕ is a critical point of the functional

$$E_X(\phi) = \frac{1}{2} \int_\Omega \left| X(\phi) \right|^2 d\mu, \quad \Omega \subset\subset \mathcal{U}$$

where $|X(\phi)|^2 = \sum_{a=1}^p X_a(\phi^\beta) X_a(\phi^\gamma) h_{\beta\gamma}(\phi)$ and μ is the Lebesgue measure on $\mathbb{R}^N$.

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Such maps were recognized as local manifestations, with respect to a given local G_θ-orthonormal frame of the Levi distribution, of subellitpic harmonic maps (in the sense of E. Barletta et al.).

For any local coordinate system $\chi : U \subset M \to \mathbb{R}^{2n+1}$, the pushforward $\{\chi_* E_a \,:\, 1 \le a \le 2n\}$ is a Hörmander system of vector fields on $\mathcal{U} = \chi(U) \subset \mathbb{R}^{2n+1}$ and

$$\Delta_b = \sum_{a=1}^{2n} E_a^* E_a \,,$$

where E_a^* is the formal adjoint of E_a with respect to $\Psi = \theta \wedge (d\theta)^n$, so that the study of the (local properties of the) solutions to the Euler-Lagrange equations is performed within subelliptic theory and more generally within the theory of Hörmander systems of vector fields and associated Hörmander "sums of squares" of vector fields.

Harmonic maps of the Fefferman Spaces

Let $\Box$ be the Laplace-Beltrami operator of the Lorentzian manifold $(C(M), F_\theta)$ (the geometric wave operator). By a result of J.M. Lee the pushforward of $\Box$ is precisely the sublaplacian Δ_b of (M, θ) i.e.

$$\pi_* \Box = \Delta_b \, .$$

By a result of E. Barletta & S. Dragomir & H. Urakawa a C^∞ map $\phi : (M, \theta) \to (N, h)$ is subelliptic harmonic if and only if its vertical lift $\Phi = \phi \circ \pi : (C(M), F_\theta) \to (N, h)$ is a harmonic map.

Harmonic morphisms from $(C(M), F_\theta)$
&
Subelliptic harmonic morphisms

Subelliptic harmonic morphisms

Let (M, θ) be a s.p.c. CR manifold of CR dimension n (i.e. $\dim M = 2n+1$) and let $(N,\ h)$ be a $m-$dimensioanal Riemannian manifold.

Definition

A continuous map ϕ of (M, θ) into (N, h) is a **subelliptic harmonic morphism** if for every open subset $V \subset N$, and every C^2 function $v : V \to \mathbb{R}$, if $\Delta_h v = 0$ in V then the pullback function $u = v \circ \phi$ is a distribution-solution to $\Delta_b u = 0$ in $U = \phi^{-1}(V)$.

Here Δ_h is the Laplace-Beltrami operator on (V, h).

Proposition

Every subelliptic harmonic morphism ϕ of the pseudohermitian manifold $(M,\ \theta)$ into the Riemannian manifold $(N,\ h)$ is smooth.

Levi conformal maps

Definition

A C^∞ map $\phi : M \to N$ is **Levi conformal** if there is a continuous map $\lambda = \lambda(\phi) : M \to [0, +\infty)$ (the θ-**dilation** of ϕ) such that λ^2 is C^∞ and

$$G_\theta\big(\nabla^H \phi^\alpha , \nabla^H \phi^\beta\big)_x = \lambda(x)^2\, \delta^{\alpha\beta}$$

for any $x \in M$ and any local normal coordinate system (V, y^α) on N with center at $\phi(x) \in V$.

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for any $x \in M$ and any local normal coordinate system $(V,\, y^\alpha)$ on N with center at $\phi(x) \in V$.

By a result of E. Barletta, a C^∞ map $\phi : M \to N$ is a subelliptic harmonic morphism of $(M,\, \theta)$ into $(N,\, h)$ if and only if ϕ is Levi conformal and a subelliptic harmonic map. Moreover

- if $m > 2n$ then every subelliptic harmonic morphism is a constant;
- if $m \leq 2n$ then for every point $x \in M$ with $\lambda(x) \neq 0$ there is an open neighborhood U of x such that $\phi : U \to N$ is a C^∞ submersion.

Let $\phi : M \to N$ be a subelliptic harmonic morphism and let us set

$$\mathscr{V}_x^\phi = \mathrm{Ker}(d_x\phi), \quad \mathscr{H}_x^\phi = \left(\mathscr{V}_x^\phi\right)^\perp, \quad x \in M,$$

where the orthogonal complement is meant with respect to the inner product $g_{\theta,\,x}$.

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where the orthogonal complement is meant with respect to the inner product $g_{\theta,\,x}$.
A regular point in the set

$$S(\phi) = \left\{x \in M \setminus \mathrm{Crit}(\phi) \ : \ d_x\phi \quad \text{is on-to}\right\}$$

is called a **submersive point** of the morphism ϕ.
At every submersive point $x \in S(\phi)$

$$\dim_{\mathbb{R}} \mathscr{H}_x^{\phi} = m, \quad \dim_{\mathbb{R}} \mathscr{V}_x^{\phi} = 2n - m + 1.$$

For each $x \in M$ we set

$$\mathscr{V}^{\phi}_{H,x} = H(M)_x \cap \mathscr{V}^{\phi}_x , \quad \mathscr{H}^{\phi}_{H,x} = H(M)_x \cap \mathscr{H}^{\phi}_x .$$

If $x \in \mathrm{Crit}(\phi)$ then

$$\mathscr{V}^{\phi}_{H,x} = H(M)_x , \quad \mathscr{H}^{\phi}_{H,x} = \{0\}.$$

If $x \in M \setminus \mathrm{Crit}(\phi)$ then the differential $d_x\phi : T_x(M) \to T_{\phi(x)}(N)$ may, or may not, be an epimorphism.

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For a subelliptic harmonic morphism, of θ-dilation $\sqrt{\Lambda}$. Then i)

$$M \setminus Z(\Lambda) \subset S(\phi).$$

ii) For every submersive point $x \in S(\phi)$

$$m - 1 \leq \dim_{\mathbb{R}} \mathscr{H}^{\phi}_{H,x} \leq m, \qquad 2n - m \leq \dim_{\mathbb{R}} \mathscr{V}^{\phi}_{H,x} \leq 2n - m + 1.$$

Here $Z(\Lambda) = \{x \in M \ : \ \Lambda(x) = 0\}$.

A partition

For any $x \in S(\phi)$, x can belong to one of the following sets

$$S(\phi) = \mathrm{I}_m(\phi) \cup \mathrm{II}_m(\phi) \cup \mathrm{III}_m(\phi) :$$

	$\mathrm{I}_m(\phi)$	$\mathrm{II}_m(\phi)$	$\mathrm{III}_m(\phi)$
$\dim_{\mathbb{R}} \mathscr{H}^{\phi}_{H,x}$	m	$m-1$	$m-1$
$\dim_{\mathbb{R}} \mathscr{V}^{\phi}_{H,x}$	$2n-m$	$2n-m+1$	$2n-m$
$\Lambda(x)$	$\Lambda(x) > 0$	$\Lambda(x) = 0$	$\Lambda(x) > 0$
T	$T_x \in \mathscr{V}^{\phi}_x$	$T_x \in \mathscr{H}^{\phi}_x$	transverse

Moreover

i) If $m = 1$ then

$$Z(\Lambda) = \mathrm{II}_1(\phi) \cup \mathrm{Crit}(\phi), \quad M \setminus S(\phi) = \mathrm{Crit}(\phi).$$

ii) If $m \geq 2$ then

$$\mathrm{II}_m(\phi) = \emptyset, \quad Z(\Lambda) = M \setminus S(\phi).$$

A little notation

Let $\Phi : C(M) \to N$ be a C^1 map, and let $p \in C(M)$ be a point. Φ is **horizontally weakly conformal** at p provided that

i) If $p \in C(M) \setminus \mathrm{Crit}(\Phi)$ and $\mathscr{V}_p^\Phi$ is nondegenerate, then the differential $d_p\Phi : \mathcal{H}_p^\Phi \to T_{\Phi(p)}(N)$ is on-to, and there is a unique nonzero number $L(p) \in \mathbb{R} \setminus \{0\}$ such that

$$h_{\Phi(p)}\big((d_p\Phi)X,\, (d_p\Phi)Y\big) = L(p)\, F_{\theta,\, p}(X, Y)$$

for any $X, Y \in \mathcal{H}_p^\Phi$.

ii) If $p \in C(M)$ and $\mathscr{V}_p^\Phi$ is degenerate, then

$$\mathscr{H}_p^\Phi \subset \mathscr{V}_p^\Phi$$

[i.e. $F_{\theta,\, p}(X, Y) = 0$ for any $X, Y \in \mathscr{H}_p^\Phi$]. The number $L(p)$ is the (**square**) **dilation** at p. It is customary to set $L(p) = 0$ when $p \in \mathrm{Crit}(\Phi)$ or $\mathscr{V}_p^\Phi$ is degenerate.

Main results

Theorem

Let M be a strictly pseudoconvex CR manifold, of CR dimension n, equipped with the positively oriented contact form $\theta \in \mathcal{P}_+(M)$, and let (N, h) be a m-dimensional Riemannian manifold. Let $\Phi : C(M) \to N$ be a continuous S^1 invariant map, and let $\phi : M \to N$ be the corresponding base map. The following statements are equivalent

i) Φ *is a harmonic morphism of the Lorentzian manifold* $(C(M), F_\theta)$ *into* (N, h), *of square dilation* $\Lambda(\phi) \circ \pi$.

ii) ϕ *is a subelliptic harmonic morphism of the pseudohermitian manifold* (M, θ) *into* (N, h), *of* θ*-dilation* $\sqrt{\Lambda(\phi)}$.

Theorem (Continuation)

If this is the case then

a) Φ *is nondegenerate at* $p \Longleftrightarrow \pi(p) \in \Omega(\phi) := M \setminus Z\big[\Lambda(\phi)\big]$.

b) $p \in \mathrm{Crit}(\Phi) \Longleftrightarrow \pi(p) \in \mathrm{Crit}(\phi)$.

c) Φ *is degenerate at* $p \Longleftrightarrow$ *either* $m = 1$ *and* $\pi(p) \in \mathrm{II}_1(\phi)$, *or* $m \geq 2$ *and* $\pi(p) \in M \setminus S(\phi)$.

d) Φ *is a harmonic map of the Lorentzian manifold* $(C(M), F_\theta)$ *into the Riemannian manifold* (N, h), *while* ϕ *is a subelliptic harmonic map of the pseudohermitian manifold* (M, θ) *into* (N, h).

e) Φ *is horizontally weakly conformal, while* ϕ *is Levi conformal.*

Theorem

Let M be a strictly pseudoconvex CR manifolds, equipped with the positively oriented contact form $\theta \in \mathscr{P}_+(M)$, and let N be a Riemannian manifold. Any nonconstant S^1 invariant harmonic morphism $\Phi : C(M) \to N$ from the total space of the canonical circle bundle $S^1 \to C(M) \to M$, endowed with the Lorentzian metric F_θ is **smooth** *and an* **open map**. *Moreover, if M is compact and N is connected then N is compact and Φ is surjective.*

Foliation theory

Assume $m \geq 2$. Let $\phi : M \to N$ be a subelliptic harmonic morphism of (M, θ) into (N, h), of θ-dilation $\lambda(\phi)$, and let $\Phi = \phi \circ \pi$ be its vertical lift [a harmonic morphism of square dilation $L(\Phi) = \lambda^2(\phi) \circ \pi$]. The connected components of the fibres of $\phi : S(\phi) \to N$ are the leaves of a foliation $\mathscr{F}$ of $S(\phi)$.

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Let us set

$$S(\Phi) := \pi^{-1}\big[S(\phi)\big] \subset C(M).$$

Then $\Phi : S(\Phi) \to N$ is a submersion and the corresponding foliation of $S(\Phi)$ is the pullback of $\mathscr{F}$ by π i.e. the foliation $\pi^*\mathscr{F}$ of $C(M)$ whose tangent bundle is

$$T(\pi^*\mathscr{F}) = T(\mathscr{F})^{\uparrow} \oplus \mathrm{Ker}(d\pi).$$

The horizontal lift is meant with respect to the Graham connection σ.

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The horizontal lift is meant with respect to the Graham connection σ. $C(M)$ is foliated by $(2n - m + 2)$-dimensional **Lorentzian manifolds**, whose normal bundles are spacelike.

Theorem

Let $\phi : M \to N$ *be a subelliptic harmonic morphism of* (M, θ) *into* (N, h)*, of* θ*-dilation* $\lambda(\phi)$*, and let* $\Phi = \phi \circ \pi : C(M) \to N$ *be its vertical lift* $\big($*a harmonic morphism of square dilation* $\ell(\Phi)^2 = \big[\lambda(\phi) \circ \pi\big]^2\big)$*. The tension field of* Φ *is given by*

$$\tau_{F_\theta}(\Phi) = -(m-2)\,\Phi_*\,\nabla\,\log\,\ell(\Phi) - (2n-m+2)\,\Phi_*\,\mu^{\mathscr{V}^\Phi}$$

everywhere in $S(\Phi)$

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Theorem

Let Φ *be a harmonic morphism of the Lorentzian manifold* $(C(M), F_\theta)$ *into* (N, h)*; if* $m = 2$ *i.e.* (N, h) *is a real surface, then every leaf of the pullback foliation* $\pi^* \mathscr{F}$ *of* $S(\Phi)$ $[$*the foliation of* $S(\Phi)$ *tangent to* $\mathscr{V}^\Phi]$ *is a minimal submanifold of* $\big(C(M), F_\theta\big)$*.*

A little notation

Let $\mathscr{D}^\perp$ be the orthogonal complement of $\mathscr{D}$, and let $\pi^\perp : T(\mathfrak{M}) \to \mathscr{D}^\perp$ be the projection associated to the direct sum decomposition $T(M) = \mathscr{D} \oplus \mathscr{D}^\perp$. Let us consider the bilinear form $B_\mathscr{D} = B_\mathscr{D}(g, D)$ given by

$$B_\mathscr{D}(X, Y) = \pi^\perp \mathscr{D}_X Y, \quad X, Y \in \mathscr{D}.$$

Next, let $\mu^\mathscr{D} = \mu^\mathscr{D}(g, D)$ be given by

$$\mu^\mathscr{D} = \frac{1}{r} \operatorname{Trace}_g B_\mathscr{D} \in C^\infty(\mathscr{D}^\perp).$$

When $D = \nabla^g$ [the Levi-Civita connection of $(\mathfrak{M}, g)$] $\mu^\mathscr{D} = \mu^\mathscr{D}(g, \nabla^g)$ is the **mean curvature** vector of $\mathcal{D}$

The "subelliptic" fundamental equation

The fundamental equation of Φ projects on

$$\tau_b(\phi) = -\frac{m-2}{2}\,\phi_*\,\nabla^H \log \Lambda(\phi) - (2n-m+1)\,\phi_*\,\mu^{\mathscr{V}^\phi}(g_\theta\,,\,\nabla) +$$

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$$-\phi_*\left\{\frac{2}{\theta(\mathscr{T})}\,J\,\mathscr{T} - \nabla_{\mathscr{T}}\,\mathscr{T}\right\}.$$

Here

$$\mathscr{T} := \frac{1}{\|T^{\mathscr{V}}\|}\;T^{\mathscr{V}} \in C^\infty\big(\Omega(\phi),\,\mathscr{V}^\phi\big)\,, \quad T^{\mathscr{V}} = \pi_{\mathscr{V}^\phi}\,T\,.$$

Proposition

Let $\phi : M \to N$ *be a subelliptic harmonic morphism of the pseudohermitian manifold* (M, θ) *into the real surface* (N, h).

i) *If the Reeb foliation (the codimension* $2n$ *foliation* $\mathscr{R}$ *of* M *tangent to* T*) is a subfoliation of* $\mathscr{F}$*, then every leaf of* $\mathscr{F}$ *is a minimal submanifold of the Riemannian manifold* (M, g_θ).

ii) *If* $(d_x\phi)T_x \neq 0$ *for some* $x \in M \setminus \mathrm{Crit}(\phi)$*, then*

$$(2n-1)\,\mu^{\mathscr{V}^\phi}(g_\theta\,,\,\nabla) = \pi_{\mathscr{H}^\phi}\left\{\nabla_{\mathscr{T}}\mathscr{T} - \frac{2}{\theta(\mathscr{T})}\,J\,\mathscr{T}\right\}.$$

$\epsilon-$contractions

Let $0 < \epsilon < 1$ and let g_ϵ be the Riemannian metric

$$g_\epsilon(X, Y) = G_\theta(X, Y), \quad g_\epsilon(X, T) = 0, \quad g_\theta(T, T) = \epsilon^{-2},$$

for any $X,\ Y \in H(M)$. Equivalently

$$g_\epsilon = g_\theta + \left(\frac{1}{\epsilon^2} - 1\right)\, \theta \otimes \theta$$

(the ϵ-**contraction** of G_θ).

The family of Riemannian metrics $\{g_\epsilon\}_{0<\epsilon<1}$ is devised such that $\left(M,\, d_\epsilon\right) \to \left(M,\, d_H\right)$ as $\epsilon \to 0^+$, in the Gromov-Hausdorff distance. Here d_ϵ and d_H are respectively the distance function of the Riemannian manifold $\left(M,\, g_\epsilon\right)$, and the Carnot-Carathéodory distance function associated to the sub-Riemannian structure $\left(H(M),\, G_\theta\right)$

Our strategy

Let us assume that, for every $0 < \epsilon < 1$, the map $\phi : (M,\, g_\epsilon) \to (N,\, h)$ is horizontally weakly conformal, with square dilation Λ_ϵ i.e. for any $x_0 \in M \setminus \mathrm{Crit}(\phi)$ and any local coordinate system $\left(V,\, y^\alpha\right)$ on N with $\phi(x_0) \in V$

$$m\,\Lambda_\epsilon = \left(h_{\alpha\beta} \circ \phi\right) g_\epsilon\left(\nabla^\epsilon \phi^\alpha\,,\, \nabla^\epsilon \phi^\beta\right).$$

Here ∇^ϵ is the gradient with respect to g_ϵ. Choose $V \subset N$ such that $U = \phi^{-1}(V) \subset M$ is a relatively compact domain.
The one can prove that

$$\Lambda_\epsilon \to \frac{1}{m}\, G_\theta\left(\nabla^H \phi^\alpha\,,\, \nabla^H \phi^\beta\right)\, h_{\alpha\beta} \circ \phi\,, \quad \epsilon \to 0^+\,,$$

uniformly on U relatively compact domain, and the Levi conformality condition is got, in the limit as $\epsilon \to 0^+$, from the horizontal weak conformality condition on $\phi : (M,\, g_\epsilon) \to (N,\, h)$.

Let $\mu_\epsilon^{\mathscr{V}^\phi}$ be the mean curvature vector of the vertical distribution $\mathscr{V}^\phi$ on the Riemannian manifold (M, g_ϵ).
Let us set by definition

$$\mu_{\rm hor}^{\mathscr{V}^\phi} := \pi_{\mathscr{H}^\phi} H(\mathscr{V}^\phi), \tag{2}$$

$$(2n - m + 1) H(\mathscr{V}^\phi) := (2n - m + 1) \mu^{\mathscr{V}^\phi}(g_\theta, \nabla) +$$
$$+ \frac{2}{\theta(\mathscr{T})} \pi_{\mathscr{H}^\phi} J \mathscr{T} - \pi_{\mathscr{H}^\phi} \nabla_{\mathscr{T}} \mathscr{T} +$$
$$- \left\{ {\rm div}_{\mathscr{F}}(\mathscr{T}) + \theta(\mathscr{T}) \left[A(\mathscr{T}, \mathscr{T}) - {\rm Trace}_{g_\theta} \Pi_{\mathscr{V}^\phi} A \right] \right\} \mathscr{T} .$$

One can prove that

$$\lim_{\epsilon \to 0^+} \mu_\epsilon^{\mathscr{V}^\phi} = \mu_{\rm hor}^{\mathscr{V}^\phi} .$$

On the other hand, by the "subelliptic" fundamental equation becomes

$$\tau_b(\phi) = -\frac{m-2}{2}\,\phi_*\,\log\Lambda(\phi) - (2n-m+1)\,\phi_*\,\mu^{\mathscr{V}^\phi}_{\mathrm{hor}}$$

so that $\tau_b(\phi) = 0$ and $m = 2$ yield $\mu^{\mathscr{V}^\phi}_{\mathrm{hor}} = 0$.

Theorem

Let $\phi : M^{2n+1} \to N^2$ be a non-constant subelliptic harmonic morphism, of the pseudohermitian manifold (M, θ) into the real surface (N, h). Let $\mu^{\mathscr{V}^\phi}_\epsilon$ be the mean curvature vector of $\mathscr{V}^\phi$, as a distribution on the Riemannian manifold (M, g_ϵ). Then $\pi_{\mathscr{H}^\phi}\,\mu^{\mathscr{V}^\phi}_\epsilon \to 0$ as $\epsilon \to 0^+$, uniformly on any relatively compact domain $U \subset M$.

Scalar valued subelliptic harmonic morphisms

Let $\phi : M \to N$ be a subelliptic harmonic morphism, of the pseudohermitian manifold (M, θ) into the Riemannian manifold (N, h). Let $\mathscr{F}$ be the foliation of $S(\phi)$ by maximal integral manifolds of $\mathscr{V}^{\phi}$. A point $x \in S(\phi)$ is a **characteristic point** of $\mathscr{F}$ if

$$H(M)_x \subset \mathscr{V}^{\phi}_x \,. \tag{3}$$

Let $\Sigma(\mathscr{F})$ be the set of all characteristic points of $\mathscr{F}$. If $x \in \Sigma(\mathscr{F})$ and $L \in S(\phi)/\mathscr{F}$ is the leaf of $\mathscr{F}$ passing through x, then x is a characteristic point of L, e.g. in the sense of L. Capogna & G. Citti . It can be proved that

$$\Sigma(\mathscr{F}) \neq \emptyset \Longrightarrow m = 1.$$

Let $\{g_\epsilon\}_{0<\epsilon<1}$ be the family of $\epsilon-$contractions of the Levi form G_θ , and let $\mathbf{n}^\epsilon \in C^\infty\big(S(\phi),\, \mathscr{H}_\epsilon^\phi\big)$ such that $g_\epsilon\big(\mathbf{n}^\epsilon\,,\, \mathbf{n}^\epsilon\big) = 1$. Next, let

$$\nu^\epsilon := \Pi_H\, \mathbf{n}^\epsilon = \mathbf{n}^\epsilon - \theta\big(\mathbf{n}^\epsilon\big)\; T$$

For every $x \in S(\phi)$,

$$x \in \Sigma(\mathscr{F}) \quad \Longleftrightarrow \quad \nu_x^\epsilon = 0 \text{ for any } 0 < \epsilon < 1\,.$$

Let $\{g_\epsilon\}_{0<\epsilon<1}$ be the family of $\epsilon-$contractions of the Levi form G_θ , and let $\mathbf{n}^\epsilon \in C^\infty\big(S(\phi),\, \mathscr{H}_\epsilon^\phi\big)$ such that $g_\epsilon\big(\mathbf{n}^\epsilon\,,\, \mathbf{n}^\epsilon\big) = 1$. Next, let

$$\nu^\epsilon := \Pi_H\, \mathbf{n}^\epsilon = \mathbf{n}^\epsilon - \theta\big(\mathbf{n}^\epsilon\big)\; T$$

For every $x \in S(\phi)$,

$$x \in \Sigma(\mathscr{F}) \quad \Longleftrightarrow \quad \nu_x^\epsilon = 0 \text{ for any } 0 < \epsilon < 1\,.$$

Let us set

$$\mathbf{n}^0(x) := \frac{1}{\sqrt{f_\epsilon(x)}}\, \nu_x^\epsilon\,, \quad x \in \Omega \setminus \Sigma(\mathscr{F}),$$

$$f_\epsilon := g_\epsilon\big(\nu^\epsilon\,,\, \nu^\epsilon\big) \in C^\infty\big(\Omega,\, \mathbb{R}_+\big),$$

with $\mathbb{R}_+ = [0,\, +\infty)$. According to the terminology by L. Capogna et al. $\mathbf{n}^0$ is the **horizontal normal** (on the leaves of $\mathscr{F}$).
One can prove that $\mathbf{n}^0(x)$ doesn't depend on $0 < \epsilon < 1$.

The **horizontal mean curvature** of $\mathcal{F}$ is

$$K_0 = \mathrm{div}\left(\mathbf{n}^0\right) \in C^\infty(\Omega).$$

Theorem

Let $\phi : M \to N^1$ be a subelliptic harmonic morphism, of square dilation Λ. Then

i) *For every local coordinate system (V, y^1) on N such that $U = \phi^{-1}(V) \subset \Omega$*

$$\mathbf{n}^0 = \frac{1}{\sqrt{\Lambda_0}} \nabla^H \phi^1, \quad \Lambda_0 = \frac{\Lambda}{h_{11} \circ \phi}, \quad \phi^1 = y^1 \circ \phi,$$

so that

$$K_0 = \mathrm{div}\left(\frac{1}{\sqrt{\Lambda_0}} \nabla^H \phi^1\right) =$$

$$= -\frac{1}{\sqrt{\Lambda_0}} \left\{\Delta_b \phi^1 + \left(\nabla^H \phi^1\right) \log \sqrt{\Lambda_0}\right\}$$

everywhere in U.

Theorem (Continuation)

ii) *The vector field* $\mu_{\rm hor}^{\mathscr{V}\phi}$ *and the mean curvature* K_0 *are related by*

$$2n\, g_\theta\big(\mu_{\rm hor}^{\mathscr{V}\phi}\,,\,\mathbf{n}^0\big) = \big\{\varphi\, T(\phi^1) - 1\big\}\; K_0\,,$$

$$\varphi^2\big\{\Lambda_0 + T(\phi^1)^2\big\} = 1 - \theta(\mathscr{T})^2\,, \quad \mathscr{T} = \|\,T^{\mathscr{V}}\|^{-1}\,T^{\mathscr{V}}.$$

Consequently

$$2n\,\mu_{\rm hor}^{\mathscr{V}\phi} = \alpha\,\nabla\phi^1\,, \quad \alpha := -\frac{\Delta_b\phi^1 + \sqrt{\Lambda_0}\,K_0}{\Lambda_0 + T(\phi^1)^2}\,.$$

In particular, for any local harmonic coordinate system (V, y^1) *on* N [*i.e.* $\Delta_h y^1 = 0$ *in* V] *with* $U = \phi^{-1}(V) \subset \Omega$

$$2n\,\Pi_H\,\mu_{\rm hor}^{\mathscr{V}\phi} = -\frac{\Lambda_0}{\Lambda_0 + T(\phi^1)^2}\,K_0\,\mathbf{n}^0 \qquad {\rm on}\, U\,.$$

Example: s.h.m. from the Heisenberg group

Let $\mathbb{H}_n$ be the **Heisenberg group** i.e. the noncommutative Lie group $\mathbb{H}_n = \mathbb{C}^n \times \mathbb{R}$ with the group law

$$(z,\ t) \cdot (w,\ s) = \big(z + w,\ \ t + s + 2\,\mathrm{Im}(z \cdot \overline{w})\big),$$

$$t,\ s \in \mathbb{R}, \quad z,\ w \in \mathbb{C}^n\,, \quad z \cdot \overline{w} = \delta^{\alpha\beta}\, z_\alpha\, \overline{w}_\beta\,,$$

equipped with the strictly pseudoconvex, left invariant, CR structure $T_{1,0}(\mathbb{H}_n)$ spanned by

$$L_\alpha \equiv \frac{\partial}{\partial z_\alpha} + i\,\overline{z}_\alpha \frac{\partial}{\partial t}\,, \quad 1 \leq \alpha \leq n,$$

[so that $\overline{L}_\alpha$ are the *Lewy operators*] and with the contact form

$$\theta_0 = dt + i \sum_{\alpha=1}^{n} \big(z_\alpha\, d\overline{z}_\alpha - \overline{z}_\alpha\, dz_\alpha\big) \in \mathscr{P}_+(\mathbb{H}_n).$$

Let us set $f(z,t) = |z|^2 - i\,t$, so that f is a CR function on $\mathbb{H}_n$ i.e. $\overline{L}_\alpha f = 0$ for any $1 \leq \alpha \leq n$.

Theorem

Let $\phi : \mathbb{H}_n \setminus \{0\} \to \mathbb{R}$ *be the* C^∞ *map given by*

$$\phi = 1/\left(f\,\overline{f}\right)^{n/2}.$$

Then

i) ϕ *is a subelliptic harmonic morphism of the pseudohermitian manifold* $\left(\mathbb{H}_n \setminus \{0\}, \theta_0\right)$ *into the Riemannian manifold* $\left(\mathbb{R}, dy^1 \otimes dy^1\right)$.

ii) $\mathrm{Crit}(\phi) = \emptyset$ *and* $S(\phi) = \mathbb{H}_n \setminus \{0\}$.

iii) $\mathrm{I}_1(\phi) = \mathbb{C}^* \times \{0\}$ *where* $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

iv) ϕ *is a subelliptic harmonic map of* $(\mathbb{H}_n \setminus \{0\}, \theta_0)$ *into* $(\mathbb{R}, dy^1 \otimes dy^1)$, *and a Levi conformal map of square dilation*

$$\Lambda(x) = \frac{2n^2\,|z|^2}{|x|^{2Q}}, \quad x = (z,t) \in \mathbb{H}_n, \ \ x \neq 0.$$

Theorem (Continuation)

Consequently

$$\mathrm{II}_1(\phi) = \{0\} \times \mathbb{R}^*, \quad \mathrm{III}_1(\phi) = \mathbb{C}^* \times \mathbb{R}^*, \quad \mathbb{R}^* = \mathbb{R} \setminus \{0\}.$$

v) *The horizontal mean curvature of the leaves of* $\mathscr{F}$ *is*

$$K_0 = \frac{1}{2\sqrt{2}\,|z|} \left(f\,\overline{f}\right)^{-1/2} \left[f + \overline{f} - 2\,Q\,|z|^2\right] = -\frac{(Q-1)\,|z|}{\sqrt{2}\,|x|^2}\,.$$

Here $Q = 2n + 2$ (the *homogeneous dimension* of $\mathbb{H}_n$) and $|x| = \left(|z|^4 + t^2\right)^{1/4}$ [the *Heisenberg norm* of $x = (z, t) \in \mathbb{H}_n$].

Example: the Hopf fibration

Let $S^2 = \left\{ \left(y^1 \, , \, y^2 \, , \, y^3 \right) \in \mathbb{R}^3 \; : \; \sum_{j=1}^3 (y^j)^2 = 1 \right\}$ and $S^3 = \left\{ (z, \, w) \in \mathbb{C}^2 \; : \; |z|^2 + |w|^2 = 1 \right\}$, and let $\pi : S^3 \to S^2$ be the Hopf fibration i.e. $\pi(z, \, w) = \left(y^1 \, , \, y^2 \, , \, y^3 \right)$

$$\begin{cases} y^1 = |z|^2 - |w|^2 \, , \\ y^2 = z \, \overline{w} + \overline{z} \, w, \\ y^3 = -i \, \left(z\overline{w} - \overline{z} \, w \right) . \end{cases}$$

Let $h_{S^N} = \mathbf{j}^* g_0$ be the first fundamental form of $\mathbf{j} : S^N \hookrightarrow \mathbb{R}^{N+1}$, where g_0 is the Euclidean metric on $\mathbb{R}^{N+1}$. Let S^3 be equipped with the standard CR structure $T_{1,0}(S^3)$ [induced by the complex structure of $\mathbb{C}^2$], and with the canonical contact form

$$\theta = \frac{i}{2} \left\{ -\overline{z} \, dz + z \, d\overline{z} - \overline{w} \, dw + w \, d\overline{w} \right\} \in \mathscr{P}_+(S^3).$$

$T_{1,0}(S^3)$ is the span of $L = \overline{w}\left(\partial/\partial z\right) - \overline{z}\left(\partial/\partial w\right)$. Let us set

$$L_t = L + t\,\overline{L}, \quad |t| < 1,$$

and let H_t be CR structure on S^3 spanned by L_t [$\{(S^3\,,\,H_t)\}_{|t|<1}$ are the **Rossi spheres**]. By a result due to H. Rossi, the CR manifold $(S^3\,,\,H_t)$ is globally embeddable in $\mathbb{C}^2$ if and only if $t = 0$.

$T_{1,0}(S^3)$ is the span of $L = \overline{w}\left(\partial/\partial z\right) - \overline{z}\left(\partial/\partial w\right)$. Let us set

$$L_t = L + t\,\overline{L}, \quad |t| < 1,$$

and let H_t be CR structure on S^3 spanned by L_t [$\{(S^3\,,\,H_t)\}_{|t|<1}$ are the **Rossi spheres**]. By a result due to H. Rossi, the CR manifold $(S^3\,,\,H_t)$ is globally embeddable in $\mathbb{C}^2$ if and only if $t = 0$.

Theorem

i) *The Hopf map* $\pi : S^3 \to S^2$ *is a subelliptic harmonic morphism of* $\left(S^3\,,\ T_{1,0}(S^3),\ \theta\right)$ *into* $\left(S^2\,,\ h_{S^2}\right)$.

ii) π *is a subelliptic harmonic morphism of* $\left(S^3\,,\ H_t\,,\ \theta\right)$ *into* $\left(S^2\,,\ h_{S^2}\right)$ *if and only if* $t = 0$.