On the stability of Killing cylinders in hyperbolic space

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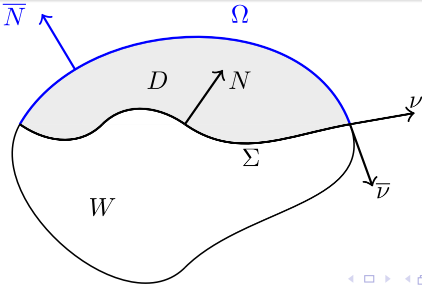
The partitioning problem in $\mathbb{H}^3$

- Let W be a bounded domain in $\mathbb{H}^3$ and Σ a compact surface with $\text{int}(\Sigma) \subset \text{int}(W)$ and $\partial\Sigma \subset \partial W$.
- Let D be one of the components of W determined by Σ and $\Omega = \partial D \cap \partial W$.
- Let N be the unit normal of Σ pointing towards D and $\overline{N}$ the unit normal of Ω pointing outwards D.
- Let ν be the exterior unit conormal to Σ in $\partial\Sigma$ and $\overline{\nu}$ the exterior unit conormal to $\partial\Sigma$ in Ω



The partitioning problem in $\mathbb{H}^3$

- Let $\Psi(p,t)$ be a variation of Σ, $\Sigma_t := p \mapsto \Psi(p,t),\ p \in \Sigma, t \in (-\epsilon,\epsilon)$.
- Let $\Omega(t)$ be the domain bounded by $\partial\Sigma_t$ in ∂W and

$$V(t) = \int_{[0,t]\times M} \Psi^* dV.$$

- Define the energy functional $E : (-\epsilon,\epsilon) \to \mathbb{R}$ by

$$E(t) = \operatorname{area}(\Sigma_t) - \cos\gamma \operatorname{area}(\Omega(t)), \quad \gamma \in (0,\pi).$$

- The first variations of $E(t)$ and $V(t)$ are

$$E'(0) = -2\int_\Sigma Hu + \int_{\partial\Sigma} \langle \nu - \cos\gamma\, \overline{\nu}, \xi \rangle, \quad V'(0) = \int_\Sigma u,$$

  where H is the mean curvature of Σ, $\xi = \frac{\partial\Psi}{\partial t}(p,0)$ and $u = \langle \xi, N \rangle$.
- Ψ preserves the volume if $\int_\Sigma u = 0$.

Definition

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- Σ is capillary $\Leftrightarrow$ H and $\langle \overline{N}, N\rangle$ are constant. Moreover, $\cos\gamma = \langle \overline{N}, N\rangle$.
- The second variation of the energy is

$$Q[u] := E''(0) = -\int_\Sigma u(\Delta u + (|A|^2 - 2)u) + \int_{\partial\Sigma} u\left(\frac{\partial u}{\partial \nu} - \mathbf{q}u\right),$$

$$\mathbf{q} = \frac{1}{\sin\gamma}\overline{A}(\overline{\nu}, \overline{\nu}) + \cot\gamma A(\nu, \nu).$$

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- The stability problem motivates to define the Jacobi operator $J = \Delta + |A|^2 - 2$ and consider the eigenvalue problem

$$\begin{cases} Ju + \lambda u = 0 & \text{in } \Sigma, \\ \dfrac{\partial u}{\partial \nu} - \mathbf{q} u = 0 & \text{in } \partial\Sigma. \end{cases}$$

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- The index of Σ, $\mathrm{index}(\Sigma)$, is the number of negative eigenvalues of J.

Theorem (Koiso, Tohoku Math. J. (2002))

- *If* $\mathrm{index}(\Sigma) = 0$ *(*$\lambda_1 \geq 0$*), then* Σ *is strongly stable.*
- *If* $\mathrm{index}(\Sigma) = 2$ *(*$\lambda_2 < 0$*), there exists* $u \in \mathcal{M}$ *with* $Q[u] < 0$*. In particular,* Σ *is unstable.*

Stability and eigenvalues

Theorem (Koiso, Tohoku Math. J. (2002))

- *If $\lambda_1 \geq 0$, then Σ is strongly stable.*
- *If $\lambda_2 < 0$, there exists $u \in \mathcal{M}$ with $Q[u] < 0$. In particular, Σ is unstable.*
- *If $\lambda_1 < 0$ and $\lambda_2 > 0$, there exists a solution to $Ju = 1$ and*
  - *If $\int_\Sigma u \geq 0$, then Σ is stable.*
  - *If $\int_\Sigma u < 0$, then Σ is unstable.*
- *If $\lambda_1 < 0$ and $\lambda_2 = 0$, then*
  - *If there exists an eigenfunction g of λ_2 such that $\int_\Sigma g \neq 0$, then Σ is unstable.*
  - *If $\int_\Sigma g = 0$ for every eigenfunction g of λ_2, then there exists a solution to $Ju = 1$ and*
    - *If $\int_\Sigma u \geq 0$, then Σ is stable.*
    - *If $\int_\Sigma u < 0$, then Σ is unstable.*

Should we study stable surfaces?

Theorem (Souam, Math Z (1997))

A capillary disk into a ball of $\mathbb{H}^3$ *must be a totally geodesic disk or a spherical cap.*

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Theorem (C. Wang, C. Xia, Math. Ann. (2019))

A capillary stable surface in a ball of $\mathbb{H}^3$ *must be totally umbilical.*

Theorem (J. Guo, G. Wang, C. Xia, Adv. Math. (2022))

A capillary stable surface in a horosphere of $\mathbb{H}^3$ *must be totally umbilical.*

- Besides umbilical surfaces, as far as we know, there are no works on explicit examples of capillary surfaces in domains of $\mathbb{H}^3$.
- This contrasts with the great literature on capillary surfaces in a ball of $\mathbb{R}^3$ (Nitsche (1985), Ros-Vergasta (1995), Fraser-Schoen (2011)).
- We emphasize the large amount of results concerning the stability of circular cylinders in different supports (R. López, Vogel).

- Objective: investigate the index of Killing cylinders supported on different umbilical surfaces.
- A Killing cylinder can be defined as:
  1. The point set obtained by the movement of a circle by hyperbolic translations.
  2. A surface of revolution.
  3. The point-set of equidistant points from a given geodesic.

The half-space model of $\mathbb{H}^3$

- We regard $\mathbb{H}^3$ as $\mathbb{R}^3_+ = \{(x,y,z) \in \mathbb{R}^3 \colon z > 0\}$ endowed with the metric
$$\langle \cdot, \cdot \rangle = \frac{dx^2 + dy^2 + dz^2}{z^2}.$$
- The Levi-Civita connections ∇ and ∇^e of $\mathbb{H}^3$ and $\mathbb{R}^3$, respectively, are related by
$$\nabla_X Y = \nabla^e_X Y - \frac{X_3}{z} Y - \frac{Y_3}{z} X + \frac{\langle X, Y \rangle_e}{z} \mathbf{e}_3, \quad \mathbf{e}_3 = (0,0,1).$$
- The Euclidean mean curvature H_e and the hyperbolic one H are related by
$$H = zH_e + (N_e)_3 = zH_e + \frac{N_3}{z}.$$

Umbilical surfaces of $\mathbb{H}^3$

- The umbilical surfaces of $\mathbb{H}^3$ are, as point sets, the intersection of the umbilical surfaces of $\mathbb{R}^3$ with $\mathbb{R}^3_+$.
- They are:
  1. Totally geodesic planes. Vertical planes $\mathsf{P}_\tau = \{y = \tau, \tau > 0\}$ and hemispheres $\mathsf{S}_\tau = \{x^2 + y^2 + z^2 = \tau^2,\ \tau > 0,\ z > 0\}$. $H = 0$.
  2. Equidistant surfaces. Slopped planes. Given $\theta \in (0, \pi/2)$, $\mathsf{E}_\theta = \{z = y \tan\theta\}$. $0 < H < 1$.
  3. Horospheres. Horizontal planes $\mathsf{H}_\tau = \{z = \tau,\ \tau > 0\}$. $H = 1$.
  4. Geodesic spheres. Spheres included in $\mathbb{R}^3_+$. For $0 < \rho < c$, $\mathsf{S}(c, \rho) = \{x^2 + y^2 + (z - c)^2 = \rho^2\}$. $H = c/\rho > 1$.

Sketch of the work

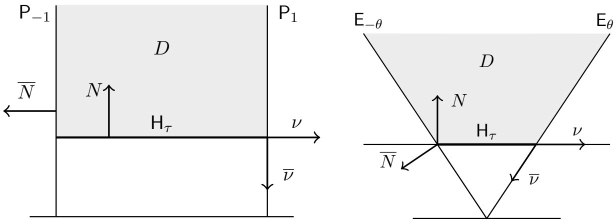
1. Fix an umbilical surface as support surface and a Killing cylinder, Σ, supported in the umbilical surface.
2. Explicitly compute the eigenvalues of the problem

$$\begin{cases} Ju + \lambda u = 0 & \text{in } \Sigma, \qquad J = \Delta + |A|^2 - 2. \\ \dfrac{\partial u}{\partial \nu} - \mathbf{q} u = 0 & \text{in } \partial\Sigma, \qquad \mathbf{q} = \frac{1}{\sin\gamma}\overline{A}(\overline{\nu}, \overline{\nu}) + \cot\gamma A(\nu, \nu). \end{cases}$$

3. Main difficulty: the computation of $\mathbf{q}$.
4. Advantage: the symmetries of the Killing cylinder simplify the geometric quantities appearing in $\mathbf{q}$.

Theorem (Strong stability of horospheres)

A horosphere H_τ *is strongly stable, when supported between two totally geodesic planes, or between two equidistant surfaces.*



Sketch of the proof

- In both cases $\mathbf{q} = 0$. Same eigenvalue problem.
- We solve explicitly the eigenvalue problem, exhibiting their positiveness, hence the strong stability of H_τ.

Killing cylinders in $\mathbb{H}^3$

- Let Γ_R be a circle of radius $R > 0$ parametrized by

$$\theta \mapsto (r\cos\theta, r\sin\theta, 1), \quad r = \sinh R.$$

- The Killing cylinder C_R is the image of Γ_R under the hyperbolic translations from **o** $= (0,0,0)$ (homotheties from **o**).
- A parametrization for C_R is

$$\psi(t,\theta) = e^t(r\cos\theta, r\sin\theta, 1), \quad t,\theta \in \mathbb{R}.$$

  We define C_R^T when $t \in [0,T]$.
- Some interesting geometric quantities are

$$N = \frac{e^t}{\sqrt{1+r^2}}(-\cos\theta, -\sin\theta, r), \qquad H = \frac{1+2r^2}{2r\sqrt{1+r^2}},$$

$$g = \begin{pmatrix} 1+r^2 & 0 \\ 0 & r^2 \end{pmatrix}, \qquad J = \frac{1}{1+r^2}\partial_t^2 + \frac{1}{r^2}\partial_\theta^2 + \frac{1}{r^2(1+r^2)}.$$

The stability problem for fixed boundary

Theorem (Plateau-Rayleigh instability of CMC cylinders in $\mathbb{R}^3$)

Let C_r^L be a circular cylinder of radius r and length L in $\mathbb{R}^3$.

- *If $L > \pi r$, then C_r^L is not strongly stable.*
- *If $L > 2\pi r$, then C_r^L is not stable.*

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Theorem (Bueno-López)

Let $R, T > 0$ *and define*

$$\eta(T) = \text{máx}\{m \in \mathbb{N} \cup \{0\} : m < \frac{T}{\pi \sinh R}\}.$$

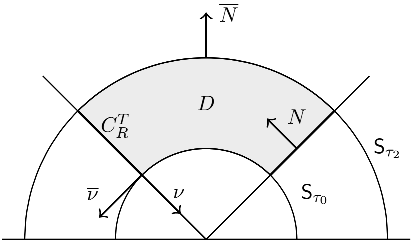
Then, $\text{index}(C_R^T) = \eta(T)$*. In consequence,*

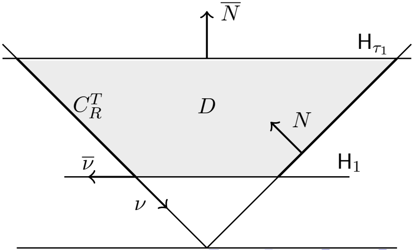
- *If* $T > \pi \sinh R$*, then* C_R^T *is not strongly stable.*
- *It* $T > 2\pi \sinh R$*, then* C_R^T *is not stable.*

Stability of Killing cylinders in bounded domains

- We consider C_R^T supported on either $\mathsf{S}_{\tau_0} \cup \mathsf{S}_{\tau_2}$ or $\mathsf{H}_1 \cup \mathsf{H}_{\tau_1}$.
- We can consider C_R^T supported on $\mathsf{S}_{\tau_0} \cup \mathsf{H}_{\tau_1}$ or $\mathsf{H}_1 \cup \mathsf{S}_{\tau_2}$. Not addressed in this paper.
- The boundary of C_R^T is included in two totally geodesic planes or in two horospheres.

$$\Gamma_R \subset \mathsf{H}_1 \cap \mathsf{S}_{\tau_0}, \quad \tau_0 = \sqrt{1+r^2},$$
$$\Gamma_R' \subset \mathsf{H}_{\tau_1} \cap \mathsf{S}_{\tau_2}, \quad \tau_1 = e^T, \; \tau_2 = \tau_0 e^T.$$





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Consider C_R^T *supported in either* $\mathsf{S}_{\tau_0} \cup \mathsf{S}_{\tau_2}$ *or* $\mathsf{H}_1 \cup \mathsf{H}_{\tau_1}$. *Then,*

$$\text{index}(C_R^T) = \begin{cases} \eta(T) & \text{in} \quad \mathsf{S}_{\tau_0} \cup \mathsf{S}_{\tau_2}, \\ \eta(T) + 1 & \text{in} \quad \mathsf{H}_1 \cup \mathsf{H}_{\tau_1}. \end{cases}$$

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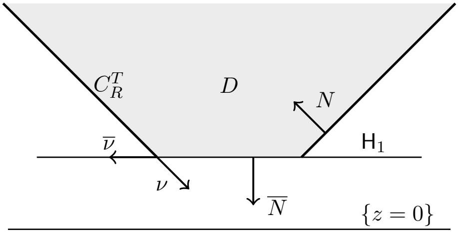
In consequence, if C_R^T *is supported between geodesic spheres, then*

- *if* $T > \pi \sinh R$*, then* C_R^T *is not strongly stable;*
- *if* $T > 2\pi \sinh R$*, then* C_R^T *is not stable.*

If C_R^T *is supported between horospheres, then is not strongly stable regardless of* T*, and if* $T > \pi \sinh R$ *then is not stable.*

Stability of Killing cylinders in unbounded domains

- Next, we investigate the index of a Killing cylinder in unbounded domains of $\mathbb{H}^3$ determined by horospheres, equidistant surfaces and geodesic planes.
- First, assume that the support is the horosphere H_1 and W is the domain over H_1. This domain is not isometric to the one below H_1.



Stability of Killing cylinders in unbounded domains

Theorem (Bueno-López)

Let be $R, T > 0$ *and*

$$\eta(T) = \max\{m \in \mathbb{N} \cup \{0\} \colon \delta_m < \frac{1}{\sinh R}\},$$

where δ_m *is the only solution of the equation* $x = \tan(Tx)$ *in each* $I_m = \left(\frac{(2m-1)\pi}{2T}, \frac{(2m+1)\pi}{2T}\right), m \geq 1$. *Here,* δ_0 *is the solution of such equation in* $(0, \frac{\pi}{2T})$ *in the case that* $T > 1$, *and* $\delta_0 = 0$ *if* $T \leq 1$. *Then,*

$$\operatorname{index}(C_R^T) = \begin{cases} \eta(T) & T < 1, \\ 1 + \eta(T) & T \geq 1. \end{cases}$$

In particular, C_R *is not stable.*

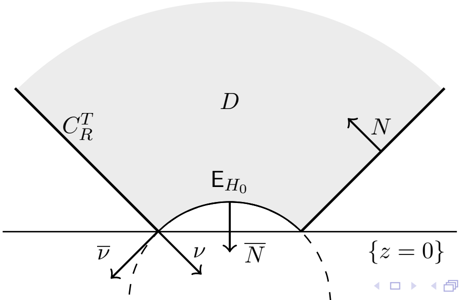
Stability of Killing cylinders in unbounded domains

- Next, we assume C_R supported on an equidistant surface, regarded as a spherical cap

$$\mathsf{E}_{H_0} = \{(x,y,z) \in \mathbb{R}^3 \colon x^2 + y^2 + (z+c)^2 = \rho^2\}, \ 0 < c < \rho,$$

whose mean curvature is $H_0 = c/\rho \in (0,1)$.

- We define $\Theta = \frac{H_0}{\sqrt{1+r^2(1-H_0^2)}}$.



Stability of Killing cylinders in unbounded domains

Theorem

Let be $R, T > 0$ *and define*

$$\eta(T) = \text{máx}\{m \in \mathbb{N} \cup \{0\} : \delta_m < \frac{1}{\sinh R}\},$$

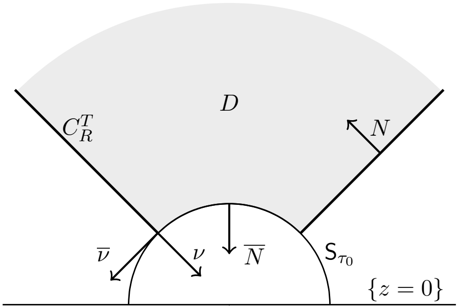
where δ_m *is the only solution of the equation* $x/\Theta = \tan(Tx)$ *in each interval* $I_m, m \geq 1$. *Here* δ_0 *is the root of the equation in the interval* $(0, \frac{\pi}{2T})$ *in case that* $T\Theta > 1$, *and* $\delta_0 = 0$ *otherwise. Consider* C_R^T *as a capillary surface supported in* E_{H_0}. *Then*

$$\text{index}(C_R^T) = \begin{cases} \eta(T) & T\Theta < 1, \\ 1 + \eta(T) & T\Theta \geq 1. \end{cases}$$

In consequence, C_R *is not stable.*

Stability of Killing cylinders in unbounded domains

- Finally, we assume C_R supported on the totally geodesic plane S_{τ_0}, where $\tau_0 = \sqrt{1 + r^2}$.
- This time, W is the domain above S_{τ_0} and it is isometric to the domain inside S_{τ_0}.



Stability of Killing cylinders in unbounded domains

Theorem (Bueno-López)

Let be $R, T > 0$ *and*

$$\eta(T) = \text{máx}\{m \in \mathbb{N} \cup \{0\} : 2m - 1 < \frac{2T}{\pi \sinh R}\}.$$

Consider C_R^T *as a capillary surface supported in* S_{τ_0}*. Then,* $\text{index}(C_R^T) = \eta(T)$*. Moreover,*

- *if* $T > \pi \sinh R/2$*, then* C_R^T *is not stable;*
- *if* $T > 3\pi \sinh R/2$*, then* C_R^T *is not stable.*

Stability of Killing cylinders in a ball

Theorem (Souam, Math Z (1997))

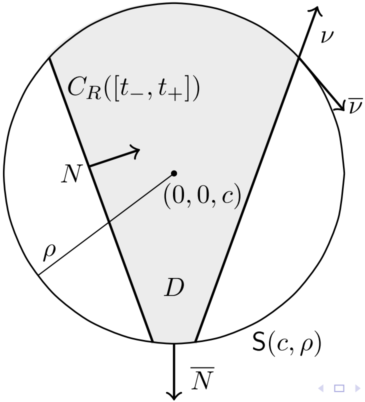
A capillary stable surface of genus zero into a ball of $\mathbb{H}^3$ *must be totally umbilical.*

- It is natural to ask by the index of a Killing cylinder in a ball, since it is not umbilical and there are no explicit computations of the index of a capillary surface in a ball of $\mathbb{H}^3$.
- The index of the critical catenoid or of circular cylinders in an Euclidean ball has been computed:
  - P. Sternberg, K. Zumbrun (J. Reine Angew. Math., 1998).
  - B. Debyver (Geom. Dedicata, 2019).
  - H. Tran (Comm. Anal. Geom., 2020).

Stability of Killing cylinders in a ball

- The Killing cylinder is given by $C_r([t_-, t_+]) = \psi(t, \theta)$, where $r \in (0, 1/\sqrt{H_0^2 - 1})$ and $t_\pm$ are such that

$$e^{t_\pm} = \frac{\rho}{1 + r^2}\left(H_0 \pm \sqrt{1 - r^2(H_0^2 - 1)}\right), \quad H_0 = c/\rho > 1.$$



Stability of Killing cylinders in a ball

Theorem (Bueno-López)

Consider the Killing cylinder $C_r([t_-, t_+])$ in $\mathsf{S}(c, \rho)$. Then, $\text{index}(C_r[t_-, t_+]) \geq 1$. *Moreover,*

- *The index grows to ∞ as $r \to 0$.*
- *There exists r_0 close to $\frac{1}{\sqrt{H_0^2 - 1}}$ such that $\text{index}(C_r[t_-, t_+]) = 1$ for all $r \in (r_0, \frac{1}{\sqrt{H_0^2 - 1}})$.*
- *The function $r \mapsto \text{index}(C_r([t_-, t_+])$ is decreasing in the interval $(0, \frac{1}{\sqrt{H_0^2 - 1}})$.*

Stability of Killing cylinders in a ball

Theorem (Bueno-López)

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- *The index grows to ∞ as $r \to 0$.*
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- *The function $r \mapsto \mathrm{index}(C_r([t_-, t_+])$ is decreasing in the interval $(0, \frac{1}{\sqrt{H_0^2-1}})$.*

- If r approaches to zero, the Killing cylinder is not stable.
- If r is close to $\frac{1}{\sqrt{H_0^2-1}}$, the index is 1. From the work of Souam we know that the Killing cylinder is not stable.

Sketch of the proof

- We apply the method of separation of variables $u(t,\theta) = f(t)g(\theta)$. After some effort, the eigenvalue problem takes the form

$$\begin{cases} \frac{1}{1+r^2} f''g + \frac{1}{r^2} fg'' + (\varpi + \lambda) fg = 0 \quad \text{in } [t_-, t_+] \times [0, 2\pi], \\ f'(t_-) + \frac{H_0}{\alpha} f(t_-) = 0, \\ f'(t_+) - \frac{H_0}{\alpha} f(t_+) = 0, \end{cases}$$

where $\varpi = \frac{1}{r^2(1+r^2)}$ and $\alpha = \sqrt{1 - r^2(H_0^2 - 1)}$.

- Hence,

$$\frac{g''}{r^2 g} = \mu = -\frac{f''}{(1+r^2)f} - (\varpi + \lambda), \quad \mu \in \mathbb{R}.$$

Then, $g(\theta) = c_1 \cos(n\theta) + c_2 \sin(n\theta)$, $\mu = -n^2/r^2$, $n \in \mathbb{N}$, and f satisfies

$$f'' + (1+r^2)(\varpi - \frac{n^2}{r^2} + \lambda) f = 0.$$

Sketch of the proof

- **Case** $\varpi - \frac{n^2}{r^2} + \lambda = 0$. Then, $f(t) = at + b$ and the boundary condition yields to a contradiction (after some effort!).
- **Case** $\varpi - \frac{n^2}{r^2} + \lambda = -\delta^2, \ \delta > 0$. Then, $f(t) = ae^{\delta t} + be^{-\delta t}$ and the boundary condition is equivalent to the existence of a solution to

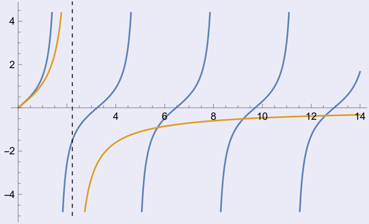
$$\left(\frac{H_0 + \alpha}{H_0 - \alpha}\right)^{\delta} \left(\frac{H_0 - \alpha\delta}{H_0 + \alpha\delta}\right) = 1.$$

  Again, after some more effort we see that $\delta = 1$ is the unique solution.
- The eigenvalues are $\lambda_n = \frac{n^2 - 1}{r^2}$ and $n = 0$ gives the only negative eigenvalue in this case.
- This proves the first statement of the theorem (the index is at least 1).

- **Case** $\varpi - \frac{n^2}{r^2} + \lambda = \delta^2,\ \delta > 0$. Then, $f(t) = a\cos(\delta t) + b\sin(\delta t)$.
- The boundary condition is equivalent to

$$\tan(\delta T) = \frac{2H_0\alpha\delta}{H_0^2 - \alpha^2\delta^2}, \quad T = t_+ - t_- = \log\frac{H_0+\alpha}{H_0-\alpha}.$$

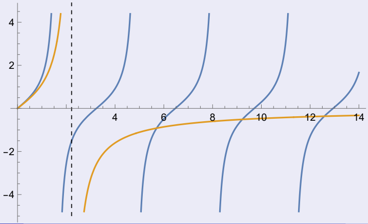
- Let $p(\delta) = \tan(\delta T)$ and $q(\delta) = \frac{2H_0\alpha\delta}{H_0^2-\alpha^2\delta^2}$. Denote to $I_0 = (0, \frac{\pi}{2T})$ and $I_m = \left(\frac{(2m-1)\pi}{2T}, \frac{(2m+1)\pi}{2T}\right)$.



- The equation $p(\delta) = q(\delta)$ has at most one solution δ_m at each $I_m,\ m \geq 0$. After some (a little more) effort, we prove that no solution exists at I_0.

Sketch of the proof

- The eigenvalues are $\lambda_{m,n} = \frac{\delta_m^2}{1+r^2} + \frac{n^2}{r^2} - \frac{1}{r^2(1+r^2)}$.The negative ones appear if and only if $n = 0$, that is if and only if $\delta_m < 1/r$.
- If $r \to 0$, then $1/r \to \infty$ and the number of $\delta_m < 1/r$ increases to ∞.
- If $r \to 1/\sqrt{H_0^2 - 1}$, after a final effort, the solutions $\delta_m < 1/r$ would correspond to a intersection between both positive branches, a contradiction.
- The infinite roots δ_m form a discrete set going to ∞. Since the negative ones correspond to $\delta_m < 1/r$, this number is a decreasing function of r varying from ∞ to 1.



- **A. Bueno. R. López, On the stability of Killing cylinders in hyperbolic space, submitted.**
- J. Nitsche, Stationary partitioning of convex bodies. Arch. Ration. Mech. Anal. 89 (1985), 1–19.
- A. Ros, E. Vergasta, Stability for hypersurfaces of constant mean curvature with free boundary. Geom. Dedic. 56 (1995), 19–33.
- R. Souam, On stability of stationary hypersurfaces for the partitioning problem for balls in space forms. Math. Z. 224 (1997), 195–208.
- T. I. Vogel, Liquid bridges between contacting balls. J. Math. Fluid Mech. 16 (2014), 737–744.
- T. I. Vogel, Capillary surfaces in circular cylinders. J. Math. Fluid Mech. 23 (2021), Paper No. 68, 13 pp.

Thank you for your attention!